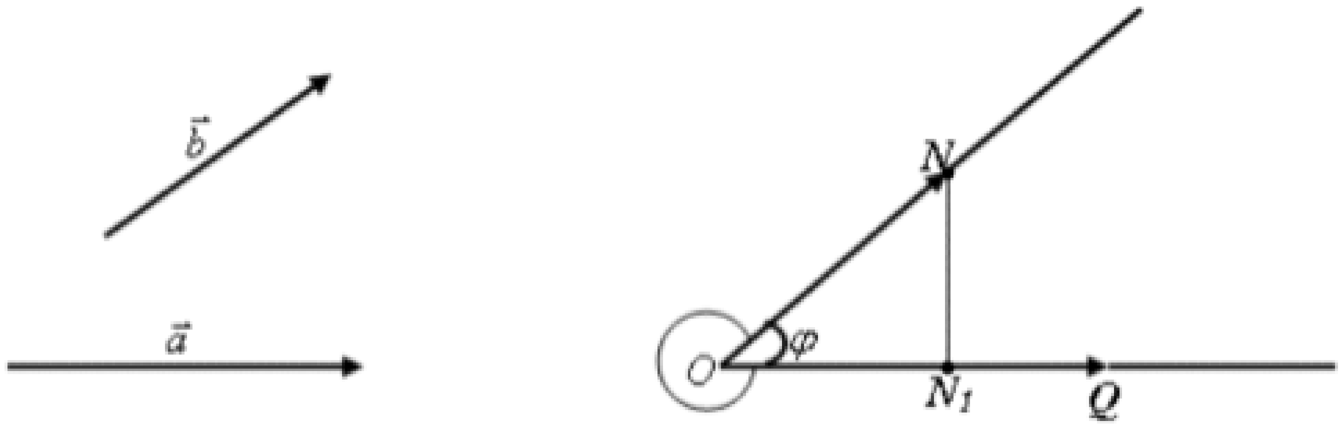
8-mavzu: Vektorlarning skalyar va vektor ko’paytmasi

Topic-8. Scalar and vector multiplication of vectors

Adabiyot: Herbert Gintis , Mathematical Literacy for Humanists, p.p11-12,14-15, 19-22, 27

Yuqorida, vektorlar ustidagi chiziqli amallar: vektorni qo'shish va ayirish, vektorni songa ko'paytirish amallari bilan tanishdik. Endi chiziqli bo'lmagan yangi amal, vektorni skalyar ko'paytirish amali bilan tanishaylik.

Fazoda (yoki tekislikda) $\vec{a}$ va $\vec{b}$ vektorlar berilgan bo'lsin. O nuqtaga $\vec{a}=\overline{OQ}, \vec{b}=\overline{ON}$ vektorlarni qo'yamiz (11-chizma)[1].



11-chizma

O,Q,N nuqtalar orqali aniqlangan tekislikda, OQ va ON nurlar yordamida ikkita burchak aniqlanadi, bulardan biri φ ikkinchisi $2\pi-\varphi$.

Bu burchaklarning eng kichigini $\vec{a}$ va $\vec{b}$ vektorlar orasidagi burchak deb aytiladi va $(\vec{a}\ \vec{b})=\varphi$ ko'rinishda belgilaymiz.

1-tarif. $\vec{A}$ va $\vec{B}$ vektorlarning uzunliklari bilan ular orasidagi burchak kosinusini ko'paytirishdan hosil bo'lgan son bu vektorlarning skalyar ko'paytmasi deb aytiladi.

Vektorlarning skalyar ko'paytmasi $\vec{A}\cdot\vec{B}$ yoki $\vec{A}\vec{B}$ ko'rinishida yoziladi.

Ta'rifga ko'ra

$$\vec{A}\bullet\vec{B}=|\vec{A}||\vec{B}|\cos\theta, \qquad (5.1)$$

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[1] Introduction to Calculus Volume II. p 7

Misol. $|\vec{a}|=3,\ |\vec{b}|=4$ bo'lib, $\varphi=60^0$ bo'lsa, $\vec{a}\cdot\vec{b}$ ni toping.

Echish: $\vec{a}\cdot\vec{b}=|\vec{a}|\cdot|\vec{b}|\cos\varphi=3\cdot4\cos60^0=12\cdot\frac{1}{2}=6.$

Natija. Nol vektorning har qanday vektorga skalyar ko'paytmasi nolga teng.

Skalyar ko'paytma xossalari

1^0. Ixtiyoriy ikkita vektor uchun $\vec{A}\cdot\vec{B}=\vec{B}\cdot\vec{A}$; (komutativ)

2^0. Ixtiyoriy uchta $\vec{A}$, $\vec{B}$ va $\vec{C}$ vektorlar uchun $\vec{A}\cdot(\vec{B}+\vec{C})=\vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C}$;

3^0. Ixtiyoriy $\vec{A}$ vektor uchun $\vec{A}\cdot\vec{A}=A^2=|\vec{A}|^2$

4^0. $\vec{a}\ \vec{a}$ coni $\vec{a}$ vektorning skalyar kvadrati deyiladi va $\vec{a}^2$ kabi belgilanadi. $\sqrt{\vec{a}^2}$ soni $\vec{a}$ vektorning uzunligi deyiladi va $|\vec{a}|$ bilan belgilanadi. ^{2}

5^0. Agar $\vec{a}=0$ bo'lsa, $\vec{a}^2=0$.

Isbot. 1^0-xossani isbotlaylik.

Ta'rifga ko'ra $\begin{array}{l}\vec{a}\cdot\vec{b}=|\vec{a}|\cdot|\vec{b}|\cos(\vec{a}\wedge\vec{b})\\ \vec{b}\cdot\vec{a}=|\vec{a}|\cdot|\vec{b}|\cos(\vec{b}\wedge\vec{a})\end{array}$.

Kosinus juft funksiya ekanini e'tiborga olsak, u holda $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$.

3^0-xossa, skalyar ko'paytma ta'rifiga ko'ra $(\lambda\vec{a})\cdot\vec{b}=|\lambda\vec{a}|\cdot|\vec{b}|\cos(\lambda\vec{a},\vec{b})$, lekin $|\lambda\vec{a}|=|\lambda|\cdot|\vec{a}|$ va $\cos(\lambda\vec{a},\vec{b})=\cos(\vec{a}\ \vec{b})$. Shuning uchun $(\lambda\vec{a})\cdot\vec{b}=\lambda(\vec{a}\cdot\vec{b})$.

4^0-xossa skalyar ko'paytma ta'rifidan

$$\vec{a}\ \vec{a}=|\vec{a}|\cdot|\vec{a}|\cos(\vec{a}\wedge\vec{a})=|\vec{a}|^2\cos 0^0=|\vec{a}| \qquad |\vec{a}|=\sqrt{\vec{a}^2}.$$

[2] Introduction to Calculus Volume II. pp 7-8

Agar $\vec{a}$ va $\vec{b}$ vektorlar perpendikulyar bo'lsa, skalyar ko'paytma nolga teng:

$$\vec{a} \perp \vec{b} \Rightarrow \vec{a}\vec{b} = 0 \qquad (5.2)$$

Buning isboti ta'rifdan kelib chiqadi.

Ortanormallangan $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ bazis uchun

$$\vec{e}_i \vec{e}_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \quad i, j = 1, 2, 3 \qquad (5.3)$$

Haqiqatan skalyar ko'paytma ta'rifidan

$$\vec{e}_i \vec{e}_j = |\vec{e}_i||\vec{e}_j| \cos(\vec{e}_i \wedge \vec{e}_j) = 1 \cdot 1 \cos\frac{\pi}{2} = 0$$

Xususiy holda

$$\vec{e}_i \cdot \vec{e}_i = |\vec{e}_i|^2 = 1 \qquad (5.4)$$

Koordinatalari bilan berilgan vektorlarning skalyar ko'paytmasi.

Uch o'lchovli vektor fazoda ortonormal bazis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ berilgan bo'lsin, bu bazisga nisbatan $\vec{A}(A_1, A_2, A_3)$, $\vec{B}(B_1, B_2, B_3)$ koordinatalarga ega:

$$\vec{A} = A_1 e_1 + A_2 e_2 + A_3 e_3$$
$$\vec{B} = B_1 e_1 + B_2 e_2 + B_3 e_3$$

$\vec{A}$ va $\vec{B}$ vektorlarning skalyar ko'paytmasini hisoblashda (5.2) va (5.4) larni e'tiborga olsak, quyidagilarga ega bo'lamiz.

$$\vec{A} \cdot \vec{B} = (A_1 e_1 + A_2 e_2 + A_3 e_3) \cdot (B_1 e_1 + B_2 e_2 + B_3 e_3)$$

Demak, koordinatalari bilan berilgan ikkita vektorning skalyar ko'paytmasi bu vektorlarning mos koordinatalari ko'paytmasining yig'indisiga teng. Ya'ni:

$$\vec{A}\cdot\vec{B} = A_1B_1e_1 + A_1B_2e_1e_2 + A_1B_3e_1e_3 + \\ + A_2B_1e_2e_1 + A_2B_2e_2e_2 + A_2B_3e_2e_3 + \\ + A_3B_1e_3e_1 + A_3B_2e_3e_2 + A_3B_3e_3e_3. \qquad (6.1)$$

bu tenglikdan $\vec{A}\cdot\vec{B} = A_1B_1 + A_2B_2 + A_3B_3$

Natijalar. 1. $\vec{A}(A_1, A_2, A_3)$ vektor uzunligi

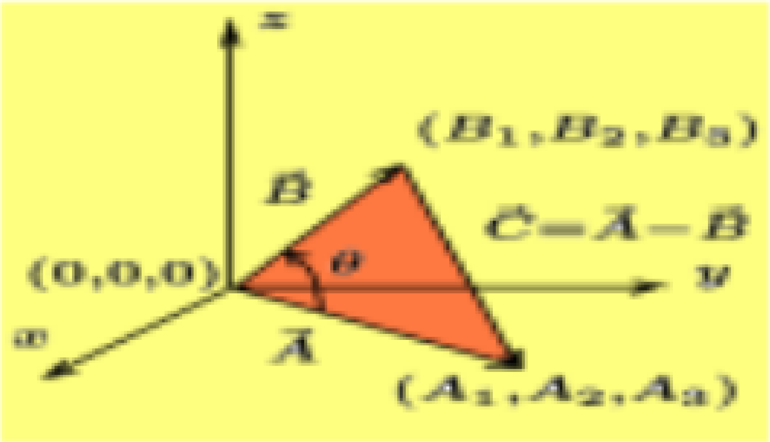
$$|\vec{A}| = \sqrt{\vec{A}\cdot\vec{A}} = \sqrt{A_1^2 + A_2^2 + A_3^3} \qquad (6.2)$$

2. Ikki $\vec{A}$, $\vec{B}$ vektorlar orasidagi burchak (5.1) ga ko'ra

$$\cos(\vec{a}\wedge\vec{b}) = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|} \qquad (6.3)$$

Agar $\vec{A}$ va $\vec{B}$ vektor koordinatalar bilan berilgan bo'lsa, bu vektorlar orasidagi burchak ushbu formula bilan aniqlanadi[3].

$$\cos(\vec{a}\wedge\vec{b}) = \frac{x_1x_2 + y_1y_2 + z_1z_2}{\sqrt{x_1^2 + y_1^2 + z_1^2}\cdot\sqrt{x_2^2 + y_2^2 + z_2^2}} \qquad (6.4)$$



1-misol. $\vec{a}(2,2,3), \vec{b}(2,-2,0), \vec{c}(5,-1,4)$ vektorlarning qaysi jufti perpendikulyar?

Yechish $\vec{a}\cdot\vec{b}$, $\vec{a}\cdot\vec{c}$, $\vec{b}\cdot\vec{c}$ skalyar ko'paytmalarini tekshiramiz:

$\vec{a}\cdot\vec{b} = 2\cdot 2 + 2\cdot(-2) + 3\cdot 0 = 4 - 4 + 0 = 0;$ $\vec{a}\cdot\vec{c} = 10 - 2 + 12 = 20$; $\vec{b}\cdot\vec{c} = 10 + 2 + 0 = 12;$

Bundan $\vec{a}\perp\vec{b}$.

2-misol. $\vec{a}(1, -1, 0)$ $\vec{b}(1, -2, 2)$ vektorlar orasidagi burchakni toping.

Yechish (6.3) formuladan foydalanamiz.

$$\cos(\vec{a}\wedge\vec{b}) = \frac{1+2+0}{\sqrt{1+1+0}\cdot\sqrt{1+4+4}} = \frac{\sqrt{2}}{2}$$. Bundan $(\vec{a}\wedge\vec{b}) = \varphi = 45^0$.

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[3] Introduction to Calculus Volume II. pp 10-11