Ma’ruza 1

Sonli ketma-ketlik va uning limiti

Darsning rejasi va maqsadi

- Reja:

- **Sonli ketma-ketlik.**
- **Sonli kyetma-ketlikning limiti.**
- **Yakinlashuvchi ketma ketlik ketma-ketliklar va ularning xossalari.**
- **Yig'indi, ko'paytma va bo'linmaning limiti**

1. Sonli ketma-ketlik.

1-ta'rif. Aniqlanish sohasi natural sonlar to'plami N dan iborat bo'lgan funksiya sonli ketma-ketlik deyiladi.

Boshqachi aytganda, har bir natural n songa biror qoida yoki qonunga binoan aniq bitta x_n son mos qo'yilagan bo'lsa, u holda sonli ketma-ketlik berilgan deyiladi.

$f(1), f(2), \ldots, f(n), \ldots$.

$f(1)=x_1,\ f(2)=x_2,\ \ldots,\ f(n)=x_n,\ \ldots$ desak, $x_1,\ x_2,\ \ldots,\ x_n,\ \ldots$ sonli ketma-ketlikka ega bo'lamiz.

x_1- ketma-ketlikning 1-hadi, x_2-2-hadi, ... , x_n-ketma-ketlikning n -hadi yoki umumiy hadi deyiladi. Ketma-ketlik (x_n) orqali belgilaylik. Ba'zi adabiyotlarda $\{x_n\}$ orqali belgilanadi.

1-misol.

1. $1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots, x_n = \frac{1}{n}$
2. $2, 4, \ldots, 2n, \ldots, x_n = 2n$
3. $-1, 1, -1, 1, \ldots, \quad x_n = (-1)^n$

2. Sonli kyetma-ketlikning limiti.

Aytaylik, biror $X \subset R$ to'plam va $x_0 \in R$ nuqta berilgan bo'lsin.

1-ta'rif. Agar x_0 nuqtaning ixtiyoriy

$$U_\varepsilon(x_0) = (x_0 - \varepsilon,\ x_0 + \varepsilon),\ (\forall \varepsilon > 0)$$

atrofida X to'plamning x_0 nuqtadan farqli kamida bitta nuqtasi bo'lsa, ya'ni

$$\forall \varepsilon > 0,\ \exists x \in X,\ \ x \neq x_0 :\ |x - x_0| < \varepsilon$$

bo'lsa, x_0 nuqta X to'plamning **limit nuqtasi** deyiladi.

3. $X = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$ to'plamning limit nuqtasi $x_0 = 0$ bo'ladi.

4. $X = N = \{1, 2, 3, \ldots\}$ to'plam limit nuqtaga ega emas.

2-ta'rif. Agar x_0 nuqtaning ixtiyoriy

$$U_\varepsilon^+(x_0) = (x_0,\ x_0 + \varepsilon)\ \ (U_\varepsilon^-(x_0) = (x_0 - \varepsilon,\ \ x_0))\ \ (\forall \varepsilon > 0)$$

o'ng atrofida (chap atrofida) X to'plamning kamida bitta nuqtasi bo'lsa, x_0 nuqta X to'plamning **o'ng** (**chap**) **limit nuqtasi** deyiladi.

3-ta'rif. Agar ixtiyoriy $r > 0$ uchun

$$I_r^+(x_0) = [x_0, x_0 + r) = \{x \in \mathbb{R}\ :\ 0 \leq x - x_0 < r\}.$$

to'plamda X to'plamning kamida bitta nuqtasi bo'lsa, "$+\infty$" X to'plamning limit "nuqta"si deyiladi.

to‘plamda X to‘plamning kamida bitta nuqtasi bo‘lsa, "$+\infty$" X to‘plamning limit “nuqta”si deyiladi.

Agar ixtiyoriy $r > 0$ uchun

$$I_r^-(x_0) = (x_0 - r, x_0] = \{x \in \mathbb{R} : 0 \leq x_0 - x < r\}.$$

to‘plamda X to‘plamning kamida bitta nuqtasi bo‘lsa, "$-\infty$" X to‘plamning limit «nuqta»si deyiladi.

The notions of *right-hand limit* and *left-hand limit* (or simply *right limit* and *left limit*) arise from the need to understand these cases. For that, we define **right neighbourhood of** x_0 **of radius** $r > 0$ the bounded half-open interval

$$I_r^+(x_0) = [x_0, x_0 + r) = \{x \in \mathbb{R} : 0 \leq x - x_0 < r\}.$$

The **left neighbourhood of** x_0 **of radius** $r > 0$ will be, similarly,

$$I_r^-(x_0) = (x_0 - r, x_0] = \{x \in \mathbb{R} : 0 \leq x_0 - x < r\}.$$

Canuto, C., Tabacco, A. Mathematical Analysis I, p.82

Keltirilgan ta'rif va misollardan ko'rinadiki, to'plamning limit nuqtasi shu to'plamga tegishli bo'lishi ham, bo'lmasligi ham mumkin ekan.

2-teorema. Agar x_0 nuqta $X \subset R$ to'plamning limit nuqta-si bo'lsa, u holda shunday sonlar ketma-ketligi $\{x_n\}$ topiladiki,

1) $\forall n \in N$ da $x_n \in X$, $x_n \neq x_0$;

2) $n \to \infty$ da $x_n \to x_0$;

bo'ladi.

Bizga $x_1, x_2, \ldots, x_n$ … ketma-ketlik berilgan bo'lsin.

2-ta'rif. Agar har bir $\varepsilon > 0$ son uchun shunday $n_0 \in N$ mavjud bo'lib, $n > n_0$ tengsizlikni qanoatlantiruvchi barcha n larda $|a_n - l| < \varepsilon$ tengsizlik o'rinli bo'lsa, u holda l son (a_n) ketma-ketlikning limiti deyiladi.

Limit $\lim\limits_{n \to \infty} a_n = l$, ko'rinishlarda belgilanadi.

Definition 3.5 *A sequence $a : n \mapsto a_n$* **converges to the limit** $\ell \in \mathbb{R}$ *(or* **converges to** ℓ *or* **has limit** ℓ*), in symbols*

$$\lim_{n \to \infty} a_n = \ell,$$

if for any real number $\varepsilon > 0$ there exists an integer n_ε such that

$$\forall n \geq n_0, \qquad n > n_\varepsilon \quad \Rightarrow \quad |a_n - \ell| < \varepsilon.$$

Canuto, C., Tabacco, A. Mathematical Analysis I, p.68

3-ta'rif. Limitga ega bo'lgan ketma-ketlik yaqinlashuvchi ketma-ketlik deyiladi. Limitga ega bo'lmagan ketma-ketlik uzoqlashuvchi ketma-ketlik deyiladi. Yaqinlashuvchi ketma-ketlikka misol keltiraylik:

2-misol. a) $x_n = \frac{n}{n+1}$ ketma-ketlikning limiti 1 bo'lishini ko'rsatamiz.

$$|x_n - 1| < \varepsilon, \left|\frac{n}{n+1} - 1\right| < \varepsilon, \frac{1}{n+1} < \varepsilon, n > \frac{1}{\varepsilon} - 1 \quad n_0 \geq \left[\frac{1}{\varepsilon}\right] - 1$$

deb olsak, n>n_0 larda $|x_n - 1| < \varepsilon$ tengsizlik o'rinli bo'ladi. Demak, $\lim \frac{n}{n+1} = 1$ ekan.

.

$$\lim_{n \to \infty} \frac{n}{n+1} = 1.$$

Examples 3.4

i) Let $a_n = \dfrac{n}{n+1}$. The first terms of this sequence are presented in Table 3.1. We see that the values approach 1 as n increases. More precisely, the real number 1 can be approximated as well as we like by a_n for n sufficiently large. This clause is to be understood in the following sense: however small we fix $\varepsilon > 0$, from a certain point n_ε onwards all values a_n approximate 1 with a margin smaller that ε.

The condition $|a_n - 1| < \varepsilon$, in fact, is tantamount to $\dfrac{1}{n+1} < \varepsilon$, i.e., $n+1 > \dfrac{1}{\varepsilon}$; thus defining $n_\varepsilon = \left[\dfrac{1}{\varepsilon}\right]$ and taking any natural number $n > n_\varepsilon$, we have $n+1 >$

$\left[\frac{1}{\varepsilon}\right]+1>\frac{1}{\varepsilon}$, hence $|a_n-1|<\varepsilon$. In other words, for every $\varepsilon>0$, there exists an n_ε such that

$$n>n_\varepsilon \quad \Rightarrow \quad |a_n-1|<\varepsilon.$$

Looking at the graph of the sequence (Fig. 3.3), one can say that for all $n>n_\varepsilon$ the points (n,a_n) of the graph lie between the horizontal lines $y=1-\varepsilon$ and $y=1+\varepsilon$.

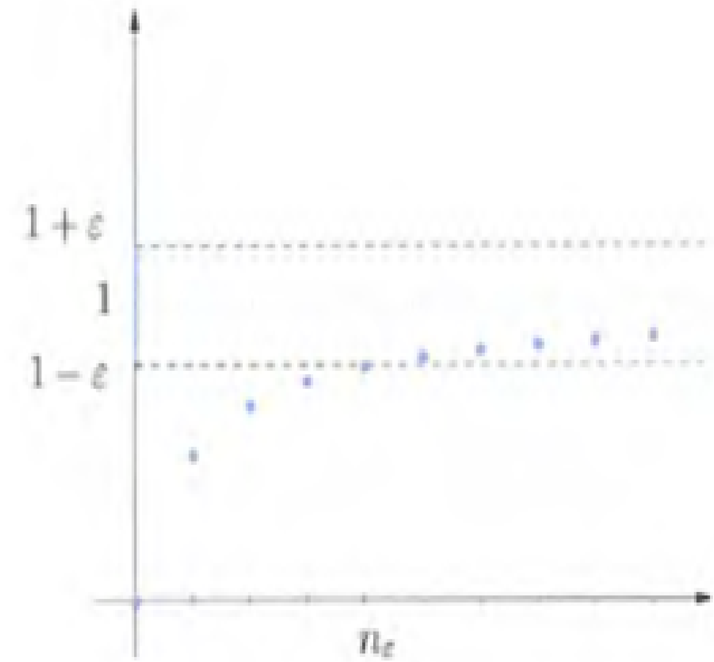
n	a_n
0	0.00000000000000
1	0.50000000000000
2	0.66666666666667
3	0.75000000000000
4	0.80000000000000
5	0.83333333333333
6	0.85714285714286
7	0.87500000000000
8	0.88888888888889
9	0.90000000000000
10	0.90909090909090
100	0.99009900990099
1000	0.99900099900100
10000	0.99990000999900

n	a_n
1	2.00000000000000
2	2.25000000000000
3	2.37037037037037
4	2.44140625000000
5	2.48832000000000
6	2.52162637174211
7	2.54649969704071
8	2.56578451395035
9	2.58117479171320
10	2.59374246010000
100	2.70481382942153
1000	2.71692393223559
10000	2.71814592682443
100000	2.7182682371975

Table 3.1. Values, estimated to the 14th digit, of the sequences $a_n = \frac{n}{n+1}$ (left) and $a_n = \left(1+\frac{1}{n}\right)^n$ (right)

b) $x_n = \left(1+\frac{1}{n}\right)^n$ ketma-ketlikning limiti e bo'lishini ko'rsatamiz.

Bu ketma-ketliklarning qiymatilari quyidagi jadvalda ko'rsatilgan.



n	a_n
0	0.00000000000000
1	0.50000000000000
2	0.66666666666667
3	0.75000000000000
4	0.80000000000000
5	0.83333333333333
6	0.85714285714286
7	0.87500000000000
8	0.88888888888889
9	0.90000000000000
10	0.90909090909090
100	0.99009900990099
1000	0.99900099900100
10000	0.99990000999900
100000	0.99999000010000
1000000	0.99999900000100
10000000	0.99999990000001
100000000	0.99999999000000

n	a_n
1	2.0000000000000
2	2.2500000000000
3	2.3703703703704
4	2.4414062500000
5	2.4883200000000
6	2.5216263717421
7	2.5464996970407
8	2.5657845139503
9	2.5811747917132
10	2.5937424601000
100	2.7048138294215
1000	2.7169239322355
10000	2.7181459268244
100000	2.7182682371975
1000000	2.7182804691564
10000000	2.7182816939804
100000000	2.7182817863958

ii) The first values of the sequence $a_n = \left(1+\frac{1}{n}\right)^n$ are shown in Table 3.1. One could imagine, even expect, that as n increases the values a_n get closer to a certain real number, whose decimal expansion starts as 2.718... This is actually the case, and we shall return to this important example later. □

2.Canuto, C., Tabacco, A. Mathematical Analysis I, 67-68p.

3. Yakinlashuvchi ketma ketlik ketma-ketliklar va ularning xossalari.

1-ta'rif. Agar biror M son mavjud bo'lib, barcha $n_0 \in N$ lar uchun $|x_n| \leq M$ tengsizlik o'rinli bo'lsa, u holda (x_n) ketma-ketlik chegaralangan deyiladi.

1^0. Agar $\lim x_n = a$ va $a > r$ ($a < q$) bo'lsa, y holda biror nomerdan boshlab $x_n > r$ ($x_n < q$) bo'ladi.

2^0. Yaqinlashuvchi ketma-ketlik chegaralangan bo'ladi.

3^0. Yaqinlashuvchi ketma-ketlik yagona limitga ega.

Tenglik va tengsizlikda limitga o'tish.

1. Agar barcha $n \in N$ lar uchun $x_n = y_n$ bo'lib, $\lim x_n = a$, $\lim y_n = b$ bo'lsa, u holda $a = b$ bo'ladi.

2. Agar barcha $n \in N$ lar uchun $x_n \le y_n$ bo'lib, $\lim x_n = a$, $\lim y_n = b$ bo'lsa, u holda $a \le b$ bo'ladi.

3. Agar barcha n lar uchun $x_n \le y_n \le z_n$ bo'lib, $\lim x_n = a$ $\lim z_n = a$ bo'lsa, u holda $\lim y_n = a$ bo'ladi.

4. Yig'indi, ko'paytma va bo'linmaning limiti.

1-teorema. Agar (x_n) va (u_n) ketma-ketliklar yaqinlashuvchi bo'lsa, u holda $(x_n \pm y_n)$ ketma-ketliklar yaqinlashuvchi bo'lib,

$\lim(x_n \pm y_n) = \lim x_n \pm \lim y_n$ tenglik o'rinli bo'ladi.

2-teorema. Agar (x_n) va (y_n) ketma-ketliklar yaqinlashuvchi bo'lsa, $(x_n\, y_n)$ ketma-ketlik ham yaqinlashuvchi bo'lib, $\lim (x_n\, y_n) = \lim x_n \, \lim y_n$ tenglik o'rinli bo'ladi.

3-teorema. Agar (x_n) va (y_n) ketma-ketliklar yaqinlashuvchi va $\lim y_n \neq 0$ bo'lsa $(\frac{x_n}{y_n})$ ketma-ketlik ham yaqinlashuvchi bo'lib, $\lim \frac{x_n}{y_n} = \frac{\lim x_n}{\lim y_n}$ tenglik o'rinli bo'ladi. (isbotlang)

Aniqmasliklar va ularni yechish.

Yuqorida (x_n) va (y_n) ketma-ketliklar yaqinlashuvchi bo'lsa, $(x_n \pm y_n)$, $(x_n y_n)$, $(\frac{x_n}{y_n})(\lim y_n \neq 0)$ ketma-ketliklarnin har biri yaqinlashuvchi bo'lishligini ko'rib o'tdik.

Endi ba'zi hollarni ko'rib o'taylik.

1. $\frac{0}{0}$ ko'rinishdagi aniqmaslik. $\lim x_n = 0$, $\lim y_n = 0$ bo'lgan holda x_n va y_n larning xarakteriga qarab, $\lim \frac{x_n}{y_n}$ turlicha bo'lishi mumkin.

1-misol.

1. $x_n = \frac{1}{n}$, $\quad y_n = \frac{1}{n^2}$, $\quad \lim \frac{1}{n} = 0$, $\quad \lim \frac{1}{n^2} = 0 \quad \lim \frac{x_n}{y_n} = \lim \frac{\frac{1}{n}}{\frac{1}{n^2}} = \lim n = +\infty$

2. $x_n = \frac{1}{n^3}, y_n = \frac{1}{n^2}, \lim \frac{1}{n^3} = 0, \lim \frac{1}{n^2} = 0, \lim \frac{x_n}{y_n} = \lim \frac{\frac{1}{n^3}}{\frac{1}{n^2}} = \lim \frac{1}{n} = 0$

2. $\frac{\infty}{\infty}$ ko'rinishdagi aniqmaslik. $\lim x_n = \infty$ va $\lim y_n = \infty$ bo'lsin. Bu holda ham x_n va y_n larning xarakteriga qarab, $\lim \frac{x_n}{y_n}$ turlicha bo'lishi mumkin.

$$x_n = n^2 + 1, \qquad y_n = 2n^2 - n, \qquad \lim(n^2+1) = +\infty,$$

2-misol. $\lim(2n^2 - n) = +\infty$

$$\lim \frac{x_n}{y_n} = \lim \frac{n^2+1}{2n^2-n} = \lim \frac{n^2(1+\frac{1}{n^2})}{n^2(2-\frac{1}{n})} = \lim_{n\to\infty} \frac{1+\frac{1}{n^2}}{2-\frac{1}{n}} = \frac{1+0}{2-1} = \frac{1}{2}$$

3. $0\cdot\infty$ ko'rinishdagi aniqmaslik. $\lim x_n = 0$, $\lim y_n = \infty$ bo'lgan holda, $0\cdot\infty$ ko'rinishdagi aniqmaslik $\frac{0}{0}$ yoki $\frac{\infty}{\infty}$ ko'rinishga keltirib yechiladi.

3-misol. $x_n = \frac{1}{n^3+1}$, $y_n = n^3 + 2n$, $\lim \frac{1}{n^3+1} = 0$, $\lim(n^3+2n) = +\infty$

$$\lim(x_n y_n) = \lim \frac{n^3+2n}{n^3+1} = \lim \frac{1+\frac{2}{n^2}}{1+\frac{1}{n^3}} = 1$$

Bulardan tashqari $\infty-\infty$, 0^0, ∞^0, 1^∞ ko'rinishdagi aniqmas-liklar mavjud, bu aniqmasliklarni $\frac{0}{0}$ va $\frac{\infty}{\infty}$ ko'rinishdagi aniqmasliklarga keltirib xal qilamiz.

e soni. $a_n = (1+\frac{1}{n})^n$ o'zgaruvchinig limiti mavjudligini ko'rsatamiz.

Buning uchun $y_n = (1+\frac{1}{n})^{n+1}$ o'zgaruvchini tekshiraylik. Bu ketma-ketlikni kamayuvchi ekanligini ko'rsatamiz:

$$\frac{y_n}{y_{n+1}} = \frac{(1+\frac{1}{n})^{n+1}}{(1+\frac{1}{n+1})^{n+2}} = \left(\frac{(n+1)^2}{n(n+2)}\right)^{n+2} \cdot \frac{n}{n+1} = \left(1+\frac{1}{n(n+2)}\right)^{n+2} \cdot \frac{n}{n+1}$$

Bernulli tengsizligiga asosan $(1+\frac{1}{n(n+1)})^{n+2} \geq 1+\frac{n+2}{n(n+2)} = 1+\frac{1}{n} = \frac{n+1}{n}$ ekanligini hisobga olsak, $\frac{y_n}{y_{n+1}} \geq \frac{n+1}{n} \cdot \frac{n}{n+1} = 1$ ya'ni $u_n \geq u_{n+1}$ tengsizlik kelib chiqadi. Ikkinchi tomondan barcha $n \in N$ lar uchun $y_n = (1+\frac{1}{n})^{n+1} > 0$ bo'lganligidan u quyidan chegaralangan. Shunday qilib, u kamayuvchi va quyidan chegaralanganligi uchun u chekli limitga ega.

$y_n=(1+{}^1/_n)^{n+1}=(1+{}^1/_n)^n \bullet (1+{}^1/_n)= =x_n \bullet (1+{}^1/_n)$ tengsizlikdan chekli lim

$$a_n = \lim(1+\frac{1}{n})^n$$

mavjud ekanligi kelib chiqadi. Bu limit e orqali belgilanadi.

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

$$e = 2.71828182845905\cdots$$

The number e

Consider the sequence $a_n = \left(1+\frac{1}{n}\right)^n$ introduced in Example 3.4 ii). It is possible to prove that it is a strictly increasing sequence (hence in particular $a_n > 2 = a_1$ for any $n > 1$) and that it is bounded from above ($a_n < 3$ for all n). Thus Theorem 3.9 ensures that the sequence converges to a limit between 2 and 3, which traditionally is indicated by the symbol e:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \mathrm{e}. \tag{3.3}$$

This number, sometimes called **Napier's number** or **Euler's number**, plays a role of the foremost importance in Mathematics. It is an irrational number, whose first decimal digits are

$$\mathrm{e} = 2.71828182845905\cdots$$

For proofs ⇝ The number e.