3-MAVZU. FUNKSIYANING LIMITI VA UNING XOSSALARI.

REJA

- **FUNKSIYANING LIMITINING TA’RIFLARI.**
- **LIMITGA EGA BO‘LGAN FUNKSIYALARNING XOSSALARI.**

1.Funksiyaning limitining ta'riflari

Faraz qilaylik, $f(x)$ funksiya $X \subset R$ to'plamda berilgan bo'lib, x_0 nuqta X to'plam-ning limit nuqtasi bo'lsin. x_0 nuqtaga intiluvchi ixtiyoriy $\{x_n\}$:

$$x_1, x_2, ..., x_n, ... \ (x_n \in X, \ x_n \neq x_0)$$

ketma-ketlikni olib, funksiya qiymatlaridan iborat $\{f(x_n)\}$:

$$f(x_1), f(x_2), ..., f(x_n), ...$$

ketma-ketlikni hosil qilamiz.

Funksiya limitining ta'riflari. Bizga X to'plam berilgan bo'lsin.

1-ta'rif. a nuqtaning ixtiyoriy $(a-\varepsilon; a+\varepsilon)$ atrofida X to'plamning a dan farqli kamida bitta nuqtasi mavjud bo'lsa, u holda a nuqta X to'plamning limit nuqtasi deyiladi.

2-ta'rif. Agar a nuqtaning ixtiyoriy $(a-\varepsilon; a+\varepsilon)$ atrofida X to'plamning cheksiz ko'p nuqtalari mavjud bo'lsa, a nuqta X to'plamning limit nuqtasi deyiladi.

Bu tahriflar o'zaro ekvivalentdir.

1-misol. a) $[0;5]$ to'plamning har bir nuqtasi uning limit nuqtasi bo'ladi, boshqa limit nuqtalari yo'q.

b) $(0;5)$ interval uchun $[0;5]$ segmentning barcha nuqtalari limit nuqta bo'ladi.

Bu misollardan ko'rinadiki to'plamning limit nuqtasi uning elementi bo'lishi ham bo'lmasligi ham mumkin.

c) $N = \{1, 2, 3, \dots, n, \dots\}$ to'plam limit nuqtaga ega emas.

Agar a X to'plamning limit nuqtasi bo'lsa, u holda X to'plamdan a ga yaqinlashuvchi $\{x_n\}$ ketma-ketlik ajratib olish mumkinligini ko'rsatamiz, bunda $x_n \in X, x_n \neq a$. a nuqta X to'plamning limit nuqtasi bo'lganligi uchun a nuqtaning har bir $(a-\frac{1}{n}; a+\frac{1}{n})$ atrofida X to'plamning a dan farqli kamida bitta x_n nuqtasi mavjud. Ya'ni $|x_n - a| < \frac{1}{n}$, $n = 1, 2, \dots$. Ixtiyoriy $\varepsilon > 0$ uchun shunday $n_0 \in N$ topilib, barcha $n > n_0$ larda $\frac{1}{n} < \varepsilon$ bo'ladi.Demak, har bir $\varepsilon > 0$ uchun $n_0 \in N$ son topilib,barcha $n > n_0$ larda $|x_n - a| < \frac{1}{n}$ tengsizlik o'rinli bo'ladi. Bundan $\lim\limits_{n \to \infty} x_n = a$ kelib chiqadi.

3-ta'rif (Geyne). Agar X to'plamdan olingan a ga intiluvchi $\{x_n\}$ ketma-ketlik qanday bo'lmasin, funksiya qiymatlaridan tuzilgan $(f(x_n))$ ketma-ketlik hamma vaqt yagona b (chekli yoki cheksiz) limitga intilsa, b son $f(x)$ funksiyaning a nuqtadagi limiti deb ataladi.

Funksiya limiti $\lim_{x \to a} f(x) = b$ kabi belgilanadi, ba'zan $x \to a$ da $f(x) \to b$ ko'rinishda yoziladi.

2-misol. a) $F(x)=3-x^2$ funksiyaning $x\to 1$ dagi limiti 2 ekanligini ko'rsating. $x_n\neq 1$ va $\lim x_n=1$ bo'ladigan (x_n) ketma-ketlik olaylik. U holda $\lim f(x_n)=\lim(3-x_n^2)=\lim 3-\lim x_n^2=3-1=2$. Demak,ta'rifga ko'ra $\lim\limits_{x\to 1}(3-x^2)=2$

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b) $f(x)=\sin x, x\to+\infty$ da limitga ega emas. Haqiqatan limiti $+\infty$ bo'lgan turli $x_n=\pi n$, $x_n^{/}=(2n+\frac{1}{2})\pi$ ketma-ketliklarni olaylik.

Bunda $f(x_n)=\sin\pi n=0$ $f(x_{0_n})=\sin(2n+\frac{1}{2})\pi=1$ bo'lib, $\lim f(x_n)=0, \lim f(x_n^{/})=1$. Bu esa $x\to+\infty$ da $f(x)=\sin x$ funksiyaning limiti mavjud emasligini ko'rsatadi.

4-ta'rif (Koshi). Agar har bir $\varepsilon>0$ son uchun shunday $\delta>0$ son topilib, x ning $0<|x-a|<\delta$ tengsizlikni qanoatlantiruvchi barcha qiymatlarida $|f(x)-b|<\varepsilon$ tengsizlik o'rinli bo'lsa, b son $f(x)$ funksiyaning a nuqtadagi limiti deb ataladi (bu yerda $x \in X$ deb qaraymiz)

Definition 3.11 *The function* f **tends to the limit** $\ell \in \mathbb{R}$ **for** x **going to** $+\infty$, *in symbols*

$$\lim_{x \to +\infty} f(x) = \ell,$$

if for any real number $\varepsilon > 0$ *there is a real* $B \geq 0$ *such that*

$$\forall x \in \operatorname{dom} f, \qquad x > B \quad \Rightarrow \quad |f(x) - \ell| < \varepsilon. \qquad (3.4)$$

Funksiya limitining Koshi va Geyne ta'riflari o'zaro ekvivalent.

5-ta'rif. Agar har bir $M>0$ son uchun shunday $\delta>0$ son topilib, x ning $0<|x-a|<\delta$ tengsizlikni qanoatlantiruvchi barcha qiymatlarida $|f(x)|>M$ tengsizlik o'rinli bo'lsa, $f(x)$ funksiyaning a nuqtadagi limiti ∞ deyiladi. Agar x ning a ga yetarlicha yaqin qiymatlarida $f(x)>0,(f(x)<0)$ bo'lsa, u holda $\lim_{x\to a} f(x)=+\infty$ $(\lim_{x\to a} f(x)=-\infty)$ bo'ladi.

Definition 3.12 *The function* f **tends to** $+\infty$ **for** x **going to** $+\infty$, *in symbols*

$$\lim_{x\to+\infty} f(x)=+\infty,$$

if for each real $A>0$ *there is a real* $B\geq 0$ *such that*

$$\forall x\in \operatorname{dom} f,\qquad x>B \quad\Rightarrow\quad f(x)>A. \qquad (3.5)$$

2.Canuto, C., Tabacco, A. Mathematical Analysis I,71-72p.

Shunga o'xshash $\lim_{x \to \infty} f(x) = b$, $\lim_{x \to \infty} f(x) = \infty$ larni ham ta'riflash mumkin.

$$\lim_{x\to+\infty} x^\alpha = +\infty, \qquad \lim_{x\to 0^+} x^\alpha = 0 \qquad \alpha > 0$$

$$\lim_{x\to+\infty} x^\alpha = 0, \qquad \lim_{x\to 0^+} x^\alpha = +\infty \qquad \alpha < 0$$

$$\lim_{x\to\pm\infty} \frac{a_n x^n + \ldots + a_1 x + a_0}{b_m x^m + \ldots + b_1 x + b_0} = \frac{a_n}{b_m} \lim_{x\to\pm\infty} x^{n-m}$$

$$\lim_{x\to+\infty} a^x = +\infty, \qquad \lim_{x\to-\infty} a^x = 0 \qquad a > 1$$

$$\lim_{x\to+\infty} a^x = 0, \qquad \lim_{x\to-\infty} a^x = +\infty \qquad a < 1$$

$$\lim_{x\to+\infty} \log_a x = +\infty, \qquad \lim_{x\to 0^+} \log_a x = -\infty \quad a > 1$$

$$\lim_{x\to+\infty} \log_a x = -\infty, \qquad \lim_{x\to 0^+} \log_a x = +\infty \quad a < 1$$

2.Canuto, C., Tabacco, A. Mathematical Analysis I,101p.

1-misol. Ushbu

$$f(x)=\frac{x^2-16}{x^2-4x}$$

funksiyaning $x_0=4$ nuqtadagi limiti topilsin.

◀ Quyidagi $\{x_n\}$:

$$\lim_{n\to\infty} x_n = 4 \quad (x_n \neq 4, \quad n=1, 2, \ldots)$$

ketma-ketlikni olaylik. Unda

$$f(x_n)=\frac{x_n^2-16}{x_n^2-4x_n}=\frac{x_n+4}{x_n}$$

bo'lib, $n\to\infty$ da $f(x_n)\to 2$ bo'ladi. Demak,

$$\lim_{n\to\infty}\frac{x^2-16}{x^2-4x}=2. \text{ ▶}$$

2-misol. Ushbu

$$f(x) = \sin\frac{1}{x}$$

funksiyaning $x \to 0$ dagi limiti mavjud bo‘lmasligi ko‘rsatilsin.

◀ Ravshanki, $n \to \infty$ da

$$x_n' = \frac{2}{(4n-1)\pi} \to 0, \ \ x_n'' = \frac{2}{(4n+1)\pi} \to 0$$

bo‘ladi.

Bu ketma-ketliklar uchun

$$f(x_n') = \frac{4n-1}{2}\pi = -1, \ \ f(x_n'') = \frac{4n+1}{2}\pi = 1$$

bo‘lib, $n \to \infty$ da

$$f(x_n') \to -1, \ \ f(x_n'') \to 1$$

bo‘ladi. Demak, berilgan funksiya $x_0 = 0$ nuqtada limitga ega emas. ▶

3-misol. $f(x) = C = \text{const}$ bo‘lsin. Bu funksiya uchun

$$\lim_{x \to x_0} f(x) = C$$

bo‘ladi.

5-misol. Faraz qilaylik, $X = R \setminus \{0\}$ da $f(x) = \frac{\sin x}{x}$ bo‘lsin. Bu funksiya uchun

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

bo‘ladi.

◀Ma’lumki, $x \in \left(0, \frac{\pi}{2}\right)$ uchun

$$\frac{1}{2}\sin x < \frac{1}{2}x < \frac{1}{2}tgx$$

bo‘ladi. Bu tengsizliklardan, funksiyalarning juftligini hisobga olib, $0 < |x| < \frac{\pi}{2}$ da

$$\cos x < \frac{\sin x}{x} < 1$$

bo‘lishini topamiz. Keyingi tengsizliklardan esa

$$0 < 1 - \frac{\sin x}{x} < 1 - \cos x = 2\sin^2 \frac{x}{2} < 2 \cdot \frac{x^2}{4} = \frac{x^2}{2}$$

bo‘lishi kelib chiqadi.

Endi $\forall \varepsilon > 0$ ni olib, $\delta = \min\{\varepsilon; 1\}$ deyilsa, unda $\forall x$, $|x| < \delta$, $x \neq 0$ uchun

$$0 < 1 - \frac{\sin x}{x} < \varepsilon$$

bo‘ladi. Demak,

$$\lim_{x\to 0} \frac{\sin x}{x} = 1. \blacktriangleright$$

Examples 4.6

i) Let us prove the fundamental limit

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1.} \qquad (4.5)$$

Observe first that $y = \dfrac{\sin x}{x}$ is even, for $\dfrac{\sin(-x)}{-x} = \dfrac{-\sin x}{-x} = \dfrac{\sin x}{x}$. It is thus sufficient to consider a positive x tending to 0, i.e., prove that $\lim\limits_{x\to 0^+} \dfrac{\sin x}{x} = 1$.

2.Canuto, C., Tabacco, A. Mathematical Analysis I,93p.

5-ta'rif. ([2], p. 81, Def. 3.21) Agar $\forall \varepsilon > 0$ son olinganda ham shunday $\delta > 0$ son topilsaki, $\forall x \in X \bigcap (U_\delta(x_0) \setminus \{x_0\})$ uchun $f(x) > \varepsilon$ tengsizlik bajarilsa, $f(x)$ funksiyaning x_0 nuqtadagi limiti $+\infty$ deb ataladi va

$$\lim_{x \to x_0} f(x) = +\infty$$

kabi belgilanadi.

Definition 3.21 *Let f be defined in a neighbourhood of $x_0 \in \mathbb{R}$, except possibly at x_0. The function f* **has limit** $+\infty$ *(or* **tends to** $+\infty$*)* **for** x **approaching** x_0, *in symbols*

$$\lim_{x \to x_0} f(x) = +\infty,$$

if for any $A > 0$ there is a $\delta > 0$ such that

$$\forall x \in \operatorname{dom} f, \qquad 0 < |x - x_0| < \delta \quad \Rightarrow \quad f(x) > A. \qquad (3.14)$$

2.Canuto, C., Tabacco, A. Mathematical Analysis I,81p.

7-misol. Aytaylik, $X=(0,+\infty)$, $x_0=+\infty$, $f(x)=\frac{1}{x}$ bo‘lsin. U holda

$$\lim_{x\to+\infty}\frac{1}{x}=0$$

bo‘ladi.

◀ Haqiqatan ham, $\forall\varepsilon>0$ sonni olaylik. Ravshanki, $\forall x>0$ uchun

$$\left|\frac{1}{x}-0\right|=\frac{1}{x}<\varepsilon \quad\Leftrightarrow\quad x>\frac{1}{\varepsilon}.$$

Demak, $\delta=\frac{1}{\varepsilon}$ deyilsa, unda $\forall x>\delta$ uchun

$$\left|\frac{1}{x}-0\right|=\frac{1}{x}<\frac{1}{\delta}=\varepsilon$$

bo‘ladi. ▶

2.Limitga ega bo'lgan funksiyalarning xossalari.

1^0. Agar $\lim_{x\to a} f(x) = b$ bo'lib, $b > p, (b < q)$ bo'lsa, u holda x ning a ga yetarlicha yaqin $x \neq a$ qiymatlarida $f(x) > p, (f(x) < q)$ bo'ladi.

Xususiy holda, $\lim_{x\to a} f(x) = b$ bo'lib, $b > 0, (b < 0)$ bo'lsa, x ning a ga yetarlicha yaqin $(x \neq a)$ qiymatlarida $f(x) > 0, (f(x) < 0)$ bo'ladi.

Bu xossani ketma-ketlikdagi kabi isbotlash mumkin.

2^0. Agar $\lim\limits_{x \to a} f(x) = b$ limit mavjud bo'lsa, x ning a ga yetarlicha yaqin $(x \neq a)$ qiymatlarida $f(x)$ funksiya chegaralangan bo'ladi.

Isbot. Ta'rifga ko'ra har bir $\varepsilon > 0$ uchun $\delta > 0$ topilib, x ning $0 < |x - a| < \delta$ tengsizlikni qanoatlantiruvchi qiymatlarida $|f(x) - a| < \varepsilon$ tengsizlik o'rinli bo'ladi. $|f(x) - a| < \varepsilon \Leftrightarrow a - \varepsilon < f(x) < a + \varepsilon$, demak, $f(x)$ funksiya x ning $(a - \delta; a + \delta)$ atrofida chegaralangan.

3^0. Agar x ning a nuqtaning biror $(a-\delta; a+\delta)$ atrofidan olingan barcha qiymatlarida $f(x) \le g(x) \le \varphi(x)$ tengsizlik o'rinli va $\lim_{x \to a} f(x)$ $\lim_{x\to a}\varphi(x)$ limitlar mavjud bo'lib, $\lim_{x \to a} f(x) = \lim_{x \to a} \varphi(x) = b$ bo'lsa, u holda $\lim_{x \to a} g(x) = b$ bo'ladi.

Isbot. $\lim_{x \to a} f(x) = b$ bo'lganidan $\varepsilon > 0$ uchun $\delta_1 > 0$ topilib, tengsizlik o'rinli bo'ladigan barcha x larda $b-\varepsilon < f(x) < b+\varepsilon$, $\lim_{x\to a}\varphi(x) = b$ bo'lganidan $\varepsilon > 0$ uchun $\delta_2 > 0$ topilib, $0 < |x-a| < \delta_2$ tengsizlik o'rinli bo'ladigan barcha x larda $b-\varepsilon < \varphi(x) < b+\varepsilon$ tengsizlik o'rinli bo'ladi. $\delta = \min\{\delta_1, \delta_2\}$ deb olsak, $0 < |x-a| < \delta$ tengsizlikni qanoatlantiruvchi barcha x larda $b-\varepsilon < f(x) < b+\varepsilon$ $b-\varepsilon < \varphi(x) < b+\varepsilon$ tengsizliklarning ikkalasi ham o'rinli bo'ladi.

Bulardan va $f(x) \le g(x) \le \varphi(x)$ tengsizlikdan $b-\varepsilon < g(x) < b+\varepsilon$ tengsizlik kelib chiqadi. Bundan $\lim_{x \to a} g(x) = b$ bo'ladi.

Limitning yagonaligi. Agar $f(x)$ funksiya $x \to a$ da limitga ega bo'lsa, bu limit yagona bo'ladi.

Isboti ketma-ketlikdagi kabi ko'rsatiladi.

$$\lim_{x\to\pm\infty} \sin x, \quad \lim_{x\to\pm\infty} \cos x, \quad \lim_{x\to\pm\infty} \tan x \quad \text{do not exist}$$

$$\lim_{x\to\left(\frac{\pi}{2}+k\pi\right)^{\pm}} \tan x = \mp\infty, \quad \forall k \in \mathbb{Z}$$

$$\lim_{x\to\pm 1} \arcsin x = \pm\frac{\pi}{2} = \arcsin(\pm 1)$$

$$\lim_{x\to+1} \arccos x = 0 = \arccos 1, \qquad \lim_{x\to-1} \arccos x = \pi = \arccos(-1)$$

$$\lim_{x\to\pm\infty} \arctan x = \pm\frac{\pi}{2}$$

2.Canuto, C., Tabacco, A. Mathematical Analysis I,102p.

Asosiy elementar funksiyalarning limiti.

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$$

$$\lim_{x\to \pm\infty} \left(1+\frac{a}{x}\right)^x = e^a \qquad (a \in \mathbb{R})$$

$$\lim_{x\to 0} (1+x)^{1/x} = e$$

$$\lim_{x\to 0} \frac{\log_a(1+x)}{x} = \frac{1}{\log a} \ (a>0); \text{ in particular, } \lim_{x\to 0} \frac{\log(1+x)}{x} = 1$$

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \log a \quad (a>0); \qquad \text{in particular, } \lim_{x\to 0} \frac{e^x - 1}{x} = 1$$

2.Canuto, C., Tabacco, A. Mathematical Analysis I,106p.

1-misol. Ushbu

$$\lim_{x \to 1} \frac{x + x^2 + x^3 + ... + x^n - n}{x - 1}$$

limit hisoblansin.

◄ Bu limitni yuqoridagi xossalardan foydalanib hisoblaymiz:

$$\lim_{x \to 1} \frac{x + x^2 + x^3 + ... + x^n - n}{x - 1} = \lim_{x \to 1} \frac{(x-1) + (x^2 - 1) + (x^3 - 1) + ... + (x^n - 1)}{x - 1} =$$

$$= \lim_{x \to 1} \frac{(x-1)[1 + (x+1) + (x^2 + x + 1) + ... + (x^{n-1} + x^{n-2} + x + 1)]}{x - 1} =$$

$$= 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2} \; . ►$$

2-misol. Ushbu

$$\lim_{x\to0}\frac{1-\cos x}{x^2}$$

limit hisoblansin.

◀ Ma'lumki, $1-\cos x = 2\sin^2\frac{x}{2}$. Shuni hisobga olib topamiz:

$$\lim_{x\to0}\frac{1-\cos x}{x^2}=\lim_{x\to0}\frac{2\sin^2\frac{x}{2}}{x^2}=\lim_{x\to0}\frac{1}{2}\cdot\left[\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right]^2=$$

$$=\frac{1}{2}\cdot\left[\lim_{x\to0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\cdot\lim_{x\to0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right]=\frac{1}{2}.$$ ▶

Mashqlar

1. Ushbu

$$\lim_{x\to+\infty} f(x)=+\infty,\ \lim_{x\to-\infty} f(x)=+\infty,$$
$$\lim_{x\to+\infty} f(x)=-\infty,\ \lim_{x\to-\infty} f(x)=-\infty$$

limitlarning ta'riflari keltirilsin.

2. Ushbu $f(x)=\sin\frac{\pi}{x}$ funksiya $x_0=0$ nuqtada limitga ega emasligi isbotlansin.

3. Limit ta'rifidan foydalanib, $\lim_{x\to 1}\frac{1}{x-1}=\infty$ bo'lishi isbotlansin.

4. $f(x)$ funksiya a nuqtada b limitga ega bo'lishi uchun uning shu nuqtadagi o'ng va chap limitlari mavjud bo'lib,

$$f(a+0)=f(a-0)=b$$

tengliklar o'rinli bo'lishi zarur va etarli bo'lishi isbotlansin.

Xulosa

1. To'plamning limit nuqtasi - a nuqtaning ixtiyoriy $(a-\varepsilon; a+\varepsilon)$ atrofida X to'plamning cheksiz ko'p nuqtalari mavjud bo'lsa, a X to'plamning limit nuqtasi deyiladi.

2. Funksiyaning limiti - a nuqta X to'plamning limit nuqtasi bo'lib, a ga yetarlicha yaqin $x \in X, x \neq a$ larda $f(x)$ son b songa yetarlicha yaqin bo'lsa, b son $f(x)$ funksiyaning $x \to a$ da limiti deyiladi.

3. Cheksiz kichik funksiya - $\lim_{x \to a} a(x) = 0$ bo'lsa, $a(x)$ funksiya $x \to a$ da cheksiz kichik funksiya deyiladi.

4. Cheksiz katta funksiya - $\lim_{x \to a} f(x) = \infty$ bo'lsa $f(x)$ funksiya $x \to a$ da cheksiz katta funksiya deyiladi.