9-MAVZU. RATSIONAL FUNKSIYALARNI INTEGRALLASH.

REJA

Ratsional kasrlar

Ratsional kasrlarni integrallash.

Misollar yechish.

1.Ratsional kasrlar. Sodda ratsional kasrlar deb nomlanadigan kasrlar asosan to‘rt xil bo‘ladi. Ratsional funksiyalarni integrallash shu to‘rt xil sodda kasrlarni integrallashga keltiriladi. Shu sababli bu to‘rt xil kasrniIntegrallash masalasi alohida ahamiyat kasb etadi. Ularning ko‘rinishi quyidagicha:

$$\frac{A}{x-a}, \quad \frac{A}{(x-a)^k},$$

$$\frac{Mx+N}{x^2+px+q}$$

$$\frac{Mx+N}{(x^2+px+q)^k},$$

bunda *A, M, N, a, p* va *q* lar haqiqiy sonlar, $k>1$natural son va $p^2-4q<0$ deb hisoblanadi.

1. **Ratsional kasrni integrallash**. Integralni hisoblash uchun umumiy usullar bo‘lmagani uchun ayrim funksiyalar sinflarini integrallash yo‘llari o‘rganilgan. Hozir biz ana shunday funksiyalar sinflaridan biri bilan tanishib chiqamiz.

Ma'lumki, $R_n(x)=a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n$ ko'phad butun ratsional funksiya,

$$\frac{P_n(x)}{Q_m(x)}=\frac{a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n}{b_0x^m+b_1x^{m-1}+\ldots+b_{m-1}x+b_m} \quad (a_0\neq 0,\ b_0\neq 0)$$

esa kasr ratsional funksiyalar deb ataladi. Butun va kasr ratsional funksiyalar umuman ratsional funksiyalar deb aytiladi. Butun ratsional funksiyani integrallash quyidagicha bajariladi:

$$\int R_n(x)dx=\int a_0x^ndx+\int a_1x^{n-1}dx+\ldots+\int a_{n-1}xdx+\int a_ndx=$$

$$=\frac{a_0}{n+1}x^{n+1}+\frac{a_1}{n}x^n+\ldots+a_{n-1}\cdot\frac{x^2}{2}+a_nx+C.$$

Endi kasr ratsional funksiyalarni integrallash masalasiga o'tamiz.

Ushbu $f(x)=\dfrac{P_n(x)}{Q_m(x)}$ kasr ratsional funksiya berilgan bo'lsin.

Agar $n<m$ bo'lsa, u holda $f(x)$ - to'g'ri, $n\geq m$ bo'lsa, $f(x)$ - noto'g'ri kasr ratsional funksiya deyiladi.

Consider maps of the general form

$$f(x) = \frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ denote polynomials of degrees n, m $(m \geq 1)$ respectively. We want to prove they admit primitives in terms of rational functions, logarithms and inverse tangent functions.

Canuto, C., Tabacco, A. Mathematical Analysis I, p.310

Misollar.

$$\frac{3x}{x^2+1}, \quad \frac{1}{x}$$

- to‘g‘ri kasr ratsional funksiyalar;

$$\frac{x^2+3}{x^2+5}, \quad \frac{x^3+x+5}{x^2+4}$$ -

noto‘g‘ri kasr ratsional funksiyalar bo‘ladi.

To‘g‘ri ratsional kasrni integrallashni o‘rganamiz.

$$\frac{P_n(x)}{Q_m(x)}$$

($n<m$) to‘g‘ri ratsional kasr berilgan bo‘lsin. Uni chekli sondagi sodda ratsional kasrlarning yig‘indisi ko‘rinishda ifodalash mumkin. Shu maqsadda kasrning mahrajini chiziqli va kvadrat ko‘paytuvchilarga ajratish lozim, buning uchun $Q_m(x)=0$, ya’ni

$$b_0x^m+b_1x^{m-1}+\ldots+b_m=0 \qquad (1)$$

tenglamani yechish kerak. Algebraning asosiy teoremasiga ko‘ra $Q_m(x)=0$ tenglama karrali ildizlarini hisobga olganda m ta ildizga ega bo‘ladi. Bu ildizlar haqiqiy (sodda yoki karrali) va kompleks (sodda va karrali) bo‘lishi mumkin.

Ma'lumki, agar $x=\alpha$ qaralayotgan $Q_m(x)$ ko'phadning sodda (k karrali) ildizi bo'lsa, u holda $Q_m(x)$ ko'phad $x-\alpha$ $((x-\alpha)^k)$ ga qoldiqsiz bo'linadi va

$Q_m(x)=(x-\alpha)Q_{m-1}(x)$ $(Q_m(x)=(x-\alpha)^k Q_{m-k}(x))$

tenglik o'rinli bo'ladi.

Agar $z=u+iv$ kompleks son $Q_m(x)$ ko'phadning sodda ildizi bo'lsa, u holda unga qo'shma bo'lgan $\overline{z}=u-iv$ kompleks son ham $Q_m(x)$ ko'phadning ildizi bo'ladi. Bu holda ko'phad $(x-z)(x-\overline{z})=x^2+px+q$ ga qoldiqsiz bo'linadi, bu yerda $p=-(z+\overline{z})=-2u$, $q=z\overline{z}=u^2+v^2$, $p^2/4-q<0$ va uni $Q_m(x)=(x^2+px+q)Q_{m-2}(x)$ ko'rinishda ifodalash mumkin. Shunga o'xshash, agar z kompleks son s karrali ildizi bo'lsa, u holda $Q_m(x)=(x^2+px+q)^s Q_{m-2s}(x)$ tenglik o'rinli bo'ladi.

Faraz qilaylik, (1) tenglamaning barcha haqiqiy va kompleks ildizlari topilgan bo'lsin. U holda $Q_m(x)$ ko'phadni chiziqli va kvadrat ko'paytuvchilarga ajratish mumkin:

$Q_m(x)= b_0(x-\alpha)^{k_1}(x-\beta)^{k_2}\dots(x-\gamma)^{k_t}(x^2+p_1x+q_1)^{s_1}(x^2+p_2x+q_2)^{s_2}\dots(x^2+p_rx+q_r)^{s_r}$,

bu yerda $k_1+k_2+\dots+k_t+2s_1+2s_2+\dots 2s_r=m$.

Theorem 9.15 *A polynomial $Q(x)$ of degree m with real coefficients decomposes uniquely as a product*

$$Q(x) = d(x-\alpha_1)^{r_1} \cdots (x-\alpha_h)^{r_h} (x^2+2p_1x+q_1)^{s_1} \cdots (x^2+2p_kx+q_k)^{s_k}, \quad (9.11)$$

where d, α_i, p_j, q_j are real and r_i, s_j integers such that

$$r_1 + \cdots + r_h + 2s_1 + \cdots + 2s_k = m\,.$$

The α_i, all distinct, are the real roots of Q counted with multiplicity r_i. The factors $x^2+2p_jx+q_j$ are pairwise distinct and irreducible over $\mathbb{R}$, i.e., $p_j^2-q_j < 0$, and have two complex(-conjugate) roots $\beta_{j,\pm}$ of multiplicity s_j.

Canuto, C., Tabacco, A. Mathematical Analysis I, p.313

Algebra kursida to‘g‘ri ratsional kasr elementar (sodda) kasrlar yig‘indisi shaklida yozilishi ko‘rsatiladi:

$$\frac{P_n(x)}{Q_m(x)} = \frac{A_1}{x-\alpha} + \frac{A_2}{(x-\alpha)^2} + \ldots + \frac{A_{k_1}}{(x-\alpha)^{k_1}} + \frac{B_1}{x-\beta} + \frac{B_2}{(x-\beta)^2} + \ldots + \frac{B_{k_2}}{(x-\beta)^{k_2}} + \ldots +$$

$$+ \frac{L_1}{x-\gamma} + \frac{L_2}{(x-\gamma)^2} + \ldots + \frac{L_{k_l}}{(x-\gamma)^{k_l}} + \frac{M_1 x + N_1}{x^2 + p_1 x + q_1} + \ldots + \frac{M_{s_1} x + N_{s_1}}{(x^2 + p_1 x + q_1)^{s_1}} + \ldots +$$

$$+ \frac{U_1 x + V_1}{x^2 + p_r x + q_r} + \ldots + \frac{U_{s_r} x + V_{s_r}}{(x^2 + p_r x + q_r)^{s_r}}, \qquad (2)$$

$$\frac{R(x)}{Q(x)} = \frac{1}{d}\left[F_1(x) + \cdots + F_h(x) + \bar{F}_1(x) + \cdots + \bar{F}_k(x)\right], \qquad (9.12)$$

where each $F_i(x)$ takes the form

$$F_i(x) = \frac{A_{i1}}{x-\alpha_i} + \frac{A_{i2}}{(x-\alpha_i)^2} + \cdots + \frac{A_{ir_i}}{(x-\alpha_i)^{r_i}},$$

while $\bar{F}_j(x)$ are like

$$\bar{F}_j(x) = \frac{B_{j1}x + C_{j1}}{x^2 + 2p_j x + q_j} + \frac{B_{j2}x + C_{j2}}{(x^2 + 2p_j x + q_j)^2} + \cdots + \frac{B_{j\bar{r}_j}x + C_{j\bar{r}_j}}{(x^2 + 2p_j x + q_j)^{s_j}},$$

for suitable constants $A_{i\ell}, B_{j\mu}, C_{j\mu}$. Note the total number of constants is $r_1 + \cdots r_h + 2s_1 + \cdots + 2s_k = m$.

Canuto, C., Tabacco, A. Mathematical Analysis I, p.314

Yuqoridagi formulani koeffitsientlarni topmagan holda bir necha misollarda ko‘rsatamiz:

1) $$\frac{x^2+2}{(x^3-1)(x^2+1)}=\frac{x^2+2}{(x-1)(x^2+x+1)(x^2+1)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}+\frac{Dx+E}{x^2+1};$$

2) $$\frac{3x-2}{(x+4)(x-2)^3}=\frac{A}{x+4}+\frac{B}{x-2}+\frac{C}{(x-2)^2}+\frac{D}{(x-2)^3};$$

3) $$\frac{x^2-2x+3}{(x-1)^3(x^2+2)^2(x+5)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}+\frac{Dx+E}{x^2+2}+\frac{Fx+G}{(x^2+2)^2}+\frac{H}{x+5}.$$

(2) yoyilmadagi koeffitsientlarni topish uchun *noma’lum koeffitsientlar metodi* yoki *xususiy qiymatlar metodi*dan foydalaniladi.

Masalan

iii) Let $g(x) = \dfrac{1}{x^2 + 2px + q}$, with $p^2 - q < 0$, so that the denominator has no real roots and is positive. Putting

$$s = \sqrt{q - p^2} > 0,$$

a little algebra shows

$$x^2 + 2px + q = x^2 + 2px + p^2 + (q - p^2) = (x+p)^2 + s^2 = s^2 \left[1 + \left(\frac{x+p}{s}\right)^2\right].$$

Now substitute $y = \varphi(x) = \dfrac{x+p}{s}$

$$\int \frac{1}{x^2 + 2px + q}\,\mathrm{d}x = \frac{1}{s^2} \int \frac{1}{1 + y^2}\, s\,\mathrm{d}y.$$

Recalling (9.1) f) we may conclude

$$\boxed{\int \frac{1}{x^2 + 2px + q}\,\mathrm{d}x = \frac{1}{s} \arctan \frac{x+p}{s} + c.}$$

Canuto, C., Tabacco, A. Mathematical Analysis I, p.311

Noma’lum koeffitsientlar metodining mohiyati quyidagidan iborat. Aytaylik to‘g‘ri ratsional kasrning (2) ko‘rinishdagi noma’lum koeffitsientli sodda kasrlar yig‘indisi shaklidagi yoyilmasi berilgan bo‘lsin. Sodda kasrlarni $Q_m(x)$ umumiy mahrajga keltiramiz va suratda hosil bo‘lgan ko‘phadni $P_n(x)$ ga tenglashtiramiz.

Ma’lumki, ikkita ko‘phad aynan teng bo‘lishi uchun bu ko‘phadlardagi x ning bir xil darajalari oldidagi koeffitsientlarning teng bo‘lishi zarur va yetarli. Shuni hisobga olgan holda hosil bo‘lgan ayniyatning o‘ng va chap tomonidagi x ning bir xil darajalari oldidagi koeffitsientlarni tenglashtiramiz va yuqoridagi noma’lum koeffitsientlarga nisbatan m ta chiziqli tenglamalar sistemasini hosil qilamiz. Shu sistemani yechib, noma’lum koeffitsientlarni topamiz.

3. Misollar yechish.

1-*misol*.

$$\frac{x^2}{x^3-8}$$

ratsional kasrni sodda kasrlarga yoying.

Yechish. $x^3-8=(x-2)(x^2+2x+4)$ bo‘lganligi sababli (8) formulaga ko‘ra

$$\frac{x^2}{x^3-8}=\frac{x^2}{(x-2)(x^2+2x+4)}=\frac{A}{x-2}+\frac{Bx+C}{x^2+2x+4},$$

bu yerda A, B va C lar noma’lum koeffitsientlar. Bu tenglikning o‘ng tomonini umumiy mahrajga keltiramiz, u holda

$$\frac{x^2}{x^3-8}=\frac{A(x^2+2x+4)+(Bx+C)(x-2)}{(x-2)(x^2+2x+4)}$$ bo‘ladi. Bundan

$x^2=(A+B)x^2+(2A+C-2B)x+4A-2C.$

Endi x ning bir xil darajalari oldidagi koeffitsientlarni tenglashtirib, A, B, C larn topish uchun ushbu tenglamalar sistemasiga ega bo'lamiz:

$$\begin{array}{l|c} x^2 & 1 = A + B, \\ x^1 & 0 = 2A + C - 2B, \\ x^0 & 0 = 4A - 2C \end{array}\left.\vphantom{\begin{array}{c}1\\0\\0\end{array}}\right\} \Rightarrow A = \frac{1}{3},\ B = \frac{2}{3},\ C = \frac{2}{3}$$

Shunday qilib,

$$\frac{x^2}{x^3 - 8} = \frac{1}{3(x-2)} + \frac{2(x+1)}{3(x^2+2x+4)}.$$

2-*misol*. Ushbu $\dfrac{7x^2 + 26x - 9}{x^4 + 4x^3 + 4x^2 - 9}$ ratsional kasrni sodda kasrlarga yoying.

Yechish. Kasrning mahrajini ko'paytuvchilarga ajratamiz:

$x^4+4x^3+4x^2-9=(x^2+2x)^2-9=(x^2+2x-3)(x^2+2x+3)=(x-1)(x+3)(x^2+2x+3)$.

(8) formuladan foydalanib yoyilmani yozamiz:

$$\frac{7x^2 + 26x - 9}{(x-1)(x+3)(x^2+2x+3)} = \frac{A}{x-1} + \frac{B}{x+3} + \frac{Cx + D}{x^2 + 2x + 3}.$$

Tenglamaning o‘ng tomonini umumiy mahrajga keltiramiz. U holda

$$\frac{7x^2+26x-9}{(x-1)(x+3)(x^2+2x+3)}=$$

$$=\frac{A(x+3)(x^2+2x+3)+B(x-1)(x^2+2x+3)+(Cx+D)(x+3)(x-1)}{(x-1)(x+3)(x^2+2x+3)}$$ bo‘ladi. Bu kasrlarning suratlarini tenglashtiramiz so‘ngra x oldidagi koeffitsientlarni tenglashtirib quyidagiga ega bo‘lamiz:

$$\left.\begin{array}{l|c} x^3 & 0=A+B+C, \\ x^2 & 7=5A+B+2C+D, \\ x^1 & 26=9A+B-3C+2D, \\ x^0 & -9=9A-3B-3D, \end{array}\right\} \Rightarrow A=1,\ B=1,\ C=-2,\ D=5.$$

Demak,

$$\frac{7x^2+26x-9}{(x-1)(x+3)(x^2+2x+3)}=\frac{1}{x-1}+\frac{1}{x+3}+\frac{-2x+5}{x^2+2x+3}.$$

Noma’lum koeffitsientlarni topishda x ning bir xil darajalari oldidagi koeffitsientlarni solishtirish o‘rniga x o‘zgaruvchiga bir nechta (noma’lum koeffitsientlar soniga teng) qiymatlar berib, noma’lum koeffitsientlarga nisbatan tenglamalar sistemasini hosil qilish mumkin. Bu metod *xususiy qiymatlar metodi* deb yuritiladi. Bu metod ayniqsa $\frac{P_n(x)}{Q_m(x)}$ ratsional kasr mahraji ildizlari sodda va haqiqiy bo‘lganda qo‘l keladi. Bunda x ga shu ildizlarga teng qiymatlar berish qo‘lay bo‘ladi.

3-*misol.*

$$\frac{4x^2+16x-8}{x^3-4x}$$

ni sodda kasrlarga ajrating.

Yechish. (2) formulaga ko‘ra

$$\frac{4x^2+16x-8}{x^3-4x}=\frac{4x^2+16x-8}{x(x+2)(x-2)}=\frac{A}{x}+\frac{B}{x+2}+\frac{C}{x-2}\ .$$

Ushbu tenglikning o‘ng tomonini umumiy mahrajga keltiramiz va suratlarini tenglashtiramiz:

$4x^2+16x-8=A(x+2)(x-2)+Bx(x-2)+Cx(x+2)$.

x ga ketma-ket $x=0$, $x=-2$ va $x=2$ qiymatlar berib quyidagini hosil qilamiz:

$$\left.\begin{array}{l|l} x=0 & -8=-4A \\ x=-2 & -24=8B \\ x=2 & 40=8C \end{array}\right\} \Rightarrow \begin{cases} A=2, \\ B=-3, \\ C=5. \end{cases}$$

Shunday qilib, $\dfrac{4x^2+16x-8}{x(x+2)(x-2)}=\dfrac{2}{x}-\dfrac{3}{x+2}+\dfrac{5}{x-2}$.

Ba'zi hollarda yuqorida ko'rgan ikkala metoddan birgalikda foydalanish ham mumkin, ya'ni noma'lum koeffitsientlar uchun tenglamalar sistemasini hosil qilish uchun x ga bir qator xususiy qiymatlar berish va x ning oldidagi koeffitsientlarni tenglashtirish mumkin.

Endi ratsional kasr funksiyalarni integrallash qoidasini keltiramiz. Ratsional kasrni integrallash uchun quyidagi ishlarni bajarish lozim:

1) agar qaralayotgan ratsional kasr noto'g'ri ($n \geq m$) bo'lsa, u holda uni ko'phad va to'g'ri ratsional kasr yig'indisi ko'rinishda ifodalab olamiz:

$$\frac{P_n(x)}{Q_m(x)} = R(x) + \frac{P_k(x)}{Q_m(x)}, \quad k<m;$$

2) agar qaralayotgan $\frac{P_n(x)}{Q_m(x)}$ ratsional kasr to'g'ri ($n < m$) bo'lsa, u holda uni (8) formula yordamida sodda kasrlarga yoyyamiz;

3) ratsional kasr integralini uning butun qismi va sodda ratsional kasrlar integrallari yig'indisi ko'rinishida yozib olamiz va har bir integralni hisoblaymiz.

4-misol. $\int \frac{x^3+1}{x(x-1)^3}dx$ hisoblang.

Yechish. Integral ostidagi funksiya to‘g‘ri kasrdan iborat. Uni quyidagi ko‘rinishda yozib olamiz:

$$\frac{x^3+1}{x(x-1)^3} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}.$$

Bundan $x^3+1=A(x-1)^3+Bx(x-1)^2+Cx(x-1)+Dx$ kelib chiqadi. Endi x o‘zgaruvchiga 0, 1, 2 va -1 qiymatlar berib, quyidagi tenglamalar sistemasini hosil qilamiz:

$$\begin{cases} -A = 1, \\ D = 2, \\ A + 2B + 2C + 2D = 9, \\ -8A - 4B + 2C - D = 0. \end{cases}$$

Bundan A=-1, B=2, C=1, D=2 ni topamiz.

Demak, $$\int \frac{x^3+1}{x(x-1)^3}dx = -\int \frac{dx}{x} + 2\int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2} + 2\int \frac{dx}{(x-1)^3} =$$

$$= -\ln|x| + 2\ln|x-1| - \frac{1}{x-1} - \frac{1}{(x-1)^2} + C$$

5-*misol*. $I=\int \frac{x^5+x^4-8}{x^3-4x}dx$ integralni hisoblang.

Yechish. Integral ostidagi kasr-noto‘g‘ri kasr. Uning butun va to‘g‘ri qismlarini ajratib olamiz:

$$\frac{x^5+x^4-8}{x^3-4x}=x^2+x+4+\frac{4x^2+16x-8}{x(x-2)(x+2)}.$$

To‘g‘ri qismi $\frac{4x^2+16x-8}{x^3-4x}$ ni sodda kasrlarga ajratamiz (qarang 3-misol), natijada $\frac{x^5+x^4-8}{x^3-4x}=x^2+x+4+\frac{2}{x}-\frac{3}{x+2}+\frac{5}{x-2}$ tenglikka ega bo‘lamiz.

Bundan

$$I=\int\left(x^2+x+4+\frac{2}{x}-\frac{3}{x+2}+\frac{5}{x-2}\right)dx=\frac{x^3}{3}+\frac{x^2}{2}+4x+2\int\frac{dx}{x}-3\int\frac{dx}{x+2}+5\int\frac{dx}{x-2}=$$

$$=\frac{x^3}{3}+\frac{x^2}{2}+4x+2\ln|x|-3\ln|x+2|+5\ln|x-2|+\ln C=\frac{x^3}{3}+\frac{x^2}{2}+4x+$$

$$+\ln\left|\frac{Cx^2(x-2)^5}{(x+2)^3}\right|.$$

6-*misol*. $\int \frac{x^2}{x^3-8}dx$

integralni hisoblang.

Yechish. Integral ostidagi funksiya to‘g‘ri kasrdan iborat. Uni sodda kasrlarga ajratishni 1-misolda ko‘rgan edik. Shu yoyilmadan foydalanib integralni hisoblaymiz:

$$\int \frac{x^2}{x^3-8}dx =$$

$$\int \frac{x^2}{(x-2)(x^2+2x+4)}dx = \int\left(\frac{A}{x-2}+\frac{Bx+C}{x^2+2x+4}\right)dx = \left| \begin{array}{l} A=\frac{1}{3}, \\ B=C=\frac{2}{3} \end{array} \right| =$$

$$= \frac{1}{3}\int\frac{dx}{x-2}+\frac{1}{3}\int\frac{2x+2}{x^2+2x+4}dx = \frac{1}{3}\ln|x-2|+\frac{1}{3}\int\frac{d(x^2+2x+4)}{x^2+2x+4} = \frac{1}{3}\ln|x-2|$$

$$+\frac{1}{3}\ln|x^2+2x+4|+\frac{1}{3}\ln C = \ln|(C(x-2)(x^2+2x+4))^{\frac{1}{3}}| = \ln\sqrt[3]{C(x^3-8)}.$$

Izoh. Integrallarni hisoblashda har doim ham tayyor sxemalardan foydalanishga harakat qilavermaslik kerak. Xususan, yuqoridagi misolda

$$x^2dx = \frac{1}{3}d(x^3-8)$$

ekanligidan foydalanish mumkin edi. U holda

$$\int\frac{x^2}{x^3-2}dx\,\frac{1}{3}\int\frac{d(x^3-8)}{x^3-8}dx = \frac{1}{3}\ln|x^3-8|+\frac{1}{3}\ln C = \ln\sqrt[3]{C(x^3-8)}.$$