10-MAVZU: IRRATSIONAL IFODALARNI INTEGRALLASH

REJA.

1. $\int R(x, \sqrt[n_1]{x^{m_1}}, \sqrt[n_2]{x^{m_2}}, \ldots, \sqrt[n_k]{x^{m_k}})dx \quad (m_1, n_1, m_2, n_2, \ldots, m_k, n_k$ -butun sonlar)

ko'rinishdagi integrallar.

2. I= $\int R\left(x, \left(\frac{ax+b}{cx+d}\right)^{\alpha_1}, \ldots, \left(\frac{ax+b}{cx+d}\right)^{\alpha_n}\right)dx$ ko'rinishdagi integral.

3. $I_1 = \int \frac{dx}{\sqrt{ax^2+bx+c}}, \quad I_2 = \int \frac{(Ax+B)dx}{\sqrt{ax^2+bx+c}}, \quad I_3 = \int \frac{dx}{x\sqrt{ax^2+bx+c}}$

ko'rinishdagi integrallar.

1. $\int R(x,\sqrt[n_1]{x^{m_1}},\sqrt[n_2]{x^{m_2}},...,\sqrt[n_k]{x^{m_k}})dx$ $(m_1,n_1,m_2,n_2,...,m_k,n_k$ **-butun sonlar) ko‘rinishdagi integrallar.**

Har qanday ratsional funksiyaning boshlang‘ich funksiyalari elementar funksiya bo‘lishini va ularni hisoblash usullarini ko‘rib chiqdik. Lekin har qanday irratsional funksiyaning boshlang‘ich funksiyalari elementar funksiya bo‘lavermaydi. Biz hozir boshlang‘ich funksiyalari elementar bo‘ladigan ba’zi bir sodda irratsional funksiyalarni integrallash bilan shug‘ullanamiz. Ular asosan biror almashtirish yordamida ratsional funksiyaga keltiriladigan funksiyalardir.

$\int R(x,\sqrt[n_1]{x^{m_1}},\sqrt[n_2]{x^{m_2}},...,\sqrt[n_k]{x^{m_k}})dx$ $(m_1,n_1,m_2,n_2,...,m_k,n_k$ **-butun sonlar) ko‘rinishdagi integrallar.**

Bu integral $x=t^s$, bu yerda s $\frac{m_1}{n_1},\frac{m_2}{n_2},...,\frac{m_k}{n_k}$ kasrlarning eng kichik umumiy maxraji, almashtirish natijasida ratsional funksiya integraliga keltiriladi.

$$\int R(x,\sqrt[n_1]{x^{m_1}},\sqrt[n_2]{x^{m_2}},...,\sqrt[n_k]{x^{m_k}})dx=\int R(t^s,t^{r_1},t^{r_2},...,t^{r_k})st^{s-1}dt$$

1-*misol*. $\int \frac{\sqrt{x}dx}{\sqrt{x}-\sqrt[3]{x}}$ ni hisoblang.

Yechish: 1/2 va 1/3 kasrlarning eng kichik umumiy maxraji 6 ga teng bo‘lganligi sababli $x=t^6$ almashtirish bajaramiz. U holda $dx=6t^5dt$ bo‘ladi.

$$\int \frac{\sqrt{x}dx}{\sqrt{x}-\sqrt[3]{x}} = \int \frac{t^3 6t^5}{t^3-t^2}dt = 6\int \frac{t^6}{t-1}dt = 6\int (t^5+t^4+t^3+t^2+t+1+\frac{1}{t-1})dt =$$

$$= t^6 + \frac{6}{5}t^5 + \frac{3}{2}t^4 + 2t^3 + 3t^2 + 6t + 6\ln|t-1| + C = x + \frac{6}{5}\sqrt[6]{x^5} +$$

$$+\frac{3}{2}\sqrt[3]{x^2} + 2\sqrt{x} + 3\sqrt[3]{x} + 6\sqrt[6]{x} + 6\ln\left|\sqrt[6]{x}-1\right| + C$$

2. I= $\int R\left(x, \left(\frac{ax+b}{cx+d}\right)^{\alpha_1}, \ldots, \left(\frac{ax+b}{cx+d}\right)^{\alpha_n}\right)dx$ **ko‘rinishdagi integral.**

Bu integralda R-o'z argumentlarining ratsional funksiyasi, a, b, c, d lar haqiqiy sonlar va $\alpha_1, \alpha_2, ..., \alpha_n$ - ratsional sonlar bo'lib, ularning eng kichik umumiy maxraji m va $ad - bc \neq 0$ bo'lsin. (Agar ad-bc=0 bo'lsa, u holda $\frac{ax+b}{cx+d}$=*const* va $R\left(x, \left(\frac{ax+b}{cx+d}\right)^{\alpha_1}, ..., \left(\frac{ax+b}{cx+d}\right)^{\alpha_n}\right)$ ifoda x ga nisbatan ratsional funksiya bo'ladi).

Quyidagi

$$t=\sqrt[m]{\frac{ax+b}{cx+d}} \text{ yoki } t^m=\frac{ax+b}{cx+d}$$

almashtirishni kiritamiz. U holda

$$x=\frac{t^m d-b}{a-ct^m} \text{ va } dx=\frac{m(ad-bc)t^{m-1}dt}{(a-ct^m)^2}$$

bo'ladi. Natijada, berilgan integral t ga nisbatan ratsional funksiyani integrallashga keltiriladi, ya'ni

$$I=\int R\left(\frac{dt^m-b}{a-ct^m}, t^{\alpha_1 \cdot m}, ..., t^{\alpha_n m}\right)\frac{m(ad-bc)t^{m-1}}{(a-ct^m)^2}\,dt.$$

Bundan avval R ning argumentlari irratsional ifodalardan tashkil bo'lsa, endi argumentlar ratsional va butun ratsional funksiyalarga keltirildi.

Qisqacha qilib yozsak, $I=\int R_1(t)dt$, bunda $R_1(t)$ - ratsional funksiya. Avval olingan natijalarga ko'ra bunday integral elementar funksiyalar orqali ifodalanadi.

2-*misol*. $I=\int\frac{dx}{\sqrt{x+1}-\sqrt[3]{x+1}}$ integralni hisoblang.

Yechish. Integral ostidagi funksiya $R(x,\sqrt{x+1},\sqrt[3]{x+1})$ ko'rinishdagi funksiya bo'lib, bu yerda $\alpha_1=\frac{1}{2}$, $\alpha_2=\frac{1}{3}$. Bu kasrlarning eng kichik umumiy mahraji $m=6$. U holda $t^6=x+1$, $x=t^6-1$, $dx=6t^5dt$, $\sqrt{x+1}=t^3$, $\sqrt[3]{x+1}=t^2$ almashtirishlar bajarib, quyidagi $I=\int\frac{6t^5dt}{t^3-t^2}=6\int\frac{t^3dt}{t-1}$ integralga kelamiz.

Natijada $I=6\int(t^2+t+1+\frac{1}{t-1})dt=2t^3+3t^2+6t+6\ln|t-1|+C=$

$=2\sqrt{x+1}+3\sqrt[3]{x+1}+6\sqrt[6]{x+1}+6\ln|\sqrt[6]{x+1}-1|+C$ bo'ladi.

3. $I_1 = \int \frac{dx}{\sqrt{ax^2+bx+c}}$, $I_2 = \int \frac{(Ax+B)dx}{\sqrt{ax^2+bx+c}}$, $I_3 = \int \frac{dx}{x\sqrt{ax^2+bx+c}}$

ko'rinishdagi integrallar. I_1 integralni hisoblash uchun ildiz ostidagi ifodadan to'la kvadrat ajratiladi:

$ax^2+bx+c = a((x+\frac{b}{2a})^2 + (\frac{c}{a} - \frac{b^2}{4a^2})) = a((x+\frac{b}{2a})^2 \pm k^2)$. Keyin esa $x+\frac{b}{2a} = u$, $dx = du$ almashtirish bajariladi. Natijada integral jadvaldagi ushbu $\int \frac{du}{\sqrt{u^2 \pm k^2}}$ ko'rinishdagi integralga keltiriladi.

I_2 integral suratida ildiz ostidagi ifodaning differensiali ajratib olinadi va bu integral ikkita integral yig'indisi ko'rinishida ifodalanadi.

$$I_2 = \int \frac{(Ax+B)dx}{\sqrt{ax^2+bx+c}} = \int \frac{\frac{A}{2a}(2ax+b)+(B-\frac{Ab}{2a})}{\sqrt{ax^2+bx+c}}dx = \frac{A}{2a}\int \frac{d(ax^2+bx+c)}{\sqrt{ax^2+bx+c}} +$$

$$+(B-\frac{AB}{2a})I_1 = \frac{A}{2a}\int (ax^2+bx+c)^{-\frac{1}{2}} d(ax^2+bx+c) + (B-\frac{Ab}{2a})I_1 =$$

$$= \frac{A}{a}\sqrt{ax^2+bx+c} + (B-\frac{Ab}{2a})I_1,$$

bu yerda I_1 yuqorida hisoblangan integral.

I_3 integralni hisoblash $x = \frac{1}{u}$, $dx = -\frac{1}{u^2}du$ almashtirish yordamida I_1 ga keltiriladi.

3-*misol*. $\int \frac{3x-1}{\sqrt{x^2+2x+2}}dx$ ni hisoblang.

Yechish. Berilgan integral I_2 ko'rinishidagi integral.

$$\int \frac{(3x-1)}{\sqrt{x^2+2x+2}}dx = \int \frac{\frac{3}{2}(2x+2)-4}{\sqrt{x^2+2x+2}}dx = \frac{3}{2}\int (x^2+2x+2)^{-\frac{1}{2}}d(x^2+2x+2) -$$

$$-4\int \frac{d(x+1)}{\sqrt{(x+1)^2+1}} = 3\sqrt{x^2+2x+2} - 4\ln\left|x+1+\sqrt{x^2+2x+2}\right| + C$$

4-*misol*. $\int \frac{dx}{x\sqrt{x^2+2x-1}}$ ni hisoblang.

Yechish. Ushbu integral I_3 ko'rinishdagi integral.

$$\int \frac{dx}{x\sqrt{x^2+2x-1}} = \left| \begin{matrix} x = \frac{1}{u} \\ dx = -\frac{1}{u^2}du \end{matrix} \right| = -\int \frac{udu}{u^2\sqrt{\frac{1}{u^2}+\frac{2}{u}-1}} = -\int \frac{du}{\sqrt{1+2u-u^2}} =$$

$$= \int \frac{d(u-1)}{\sqrt{2-(u-1)^2}} = -\arcsin\frac{\frac{1}{x}-1}{\sqrt{2}} + C = \arccos\frac{1-x}{\sqrt{2}x} + C.$$