11-MAVZU: TRIGONOMETRIK IFODALARNI INTEGRALLASH.

REJA.

1. $\int \sin nx \cdot \cos mx dx, \quad \int \cos nx \cdot \cos mx dx, \quad \int \sin nx \cdot \sin mx dx$ **ko'rinishdagi integrallar.**

2. $I = \int \sin^n x \cdot \cos^m x dx$ **ko'rinishdagi integrallar.** (m, n- butun sonlar) .

3. $\int R(\sin x, \cos x) dx$ **integralni hisoblash.**

1. $\int \sin nx \cdot \cos mx dx, \qquad \int \cos nx \cdot \cos mx dx, \qquad \int \sin nx \cdot \sin mx dx$

ko'rinishdagi integrallar.

Bu integrallarni hisoblash uchun ushbu

$$\sin nx \cdot \cos mx = \frac{1}{2}(\sin(n-m)x + \sin(n+m)x),$$

$$\cos nx \cdot \cos mx = \frac{1}{2}(\cos(n-m)x + \cos(n+m)x),$$

$$\sin nx \cdot \sin mx = \frac{1}{2}(\cos(n-m)x - \cos(n+m)x),$$

formulalardan foydalanib, berilgan integrallarni yig'indining integraliga keltirish mumkin.

1-misol. $\int \sin 5x \cdot \cos 3x dx$ ni hisoblang.

Yechish. $\int \sin 5x \cdot \cos 3x dx = \frac{1}{2}\int (\sin(5x-3x)+\sin(5x+3x))dx =$

$= \frac{1}{2}\int \sin 2x + \frac{1}{2}\int \sin 8x dx = -\frac{1}{4}\cos 2x - \frac{1}{16}\cos 8x + C$.

2-misol.

$$\int \sin 4x \cos 2x dx = \frac{1}{2}\int (\sin 6x + \sin 2x)dx = -\frac{1}{12}\cos 6x - \frac{1}{4}\cos 2x + C$$

3-misol.

$$\int \cos 2x \cos 8x dx = \frac{1}{2}\int (\cos 6x + \cos 10x)dx = \frac{1}{12}\sin 6x + \frac{1}{20}\sin 10x + C$$

2. $I=\int \sin^n x\cdot\cos^m x dx$ **ko'rinishdagi integrallar.** (*m*, *n*- butun sonlar) .

b) $I=\int \sin^n x\cdot\cos^m x dx$ integralni qaraylik. Bunda *m*, *n*- butun sonlar. Quyidagi uchta holni ko‘ramiz:

1) *m* va *n* lardan hech bo‘lmaganda biri toq son bo‘lsin. Masalan, *m*- toq son, ya'ni $m=2k+1$, *k*-butun son. U holda $t=\sin x$, $dt=\cos x dx$, $\cos^{2k}x=(1-\sin^2 x)^k=(1-t^2)^k$ almashtirishlar natijasida

$I=\int \sin^n x\cdot\cos^m x dx=\int \sin^n x\cos^{2k}x\cos x dx=\int t^n\cdot(1-t^2)^k dt$ bo‘ladi.

Demak, *t* ga nisbatan ratsional funksiyaning integraliga ega bo‘lamiz.

4-*misol.* $\int \sin^4 2x \cdot \cos^3 2x dx$ integralni hisoblang.

Yechish. $\int \sin^4 2x \cdot \cos^3 2x dx = \int \sin^4 2x(1-\sin^2 2x)\cos 2x dx =$

$= \frac{1}{2}\int t^4(1-t^2)dt = \frac{1}{10}t^5 - \frac{1}{14}t^7 + C = \frac{1}{10}\sin^5 2x - \frac{1}{14}\sin^7 2x + C$ kelib chiqadi.

2) m va n musbat juft sonlar bo‘lsin, ya’ni $m=2s$, $n=2k$, s, k- natural sonlar. Bu holda ushbu

$$\cos^2 x = \frac{1+\cos 2x}{2}, \quad \sin^2 x = \frac{1-\cos 2x}{2}, \quad \sin 2x = 2\sin x \cos x$$

formulalardan foydalanish maqsadga muvofiqdir. Bu formulalar orqali *sinx* va *cosx* larning darajalarini pasaytirish mumkin bo‘ladi.

5-*misol*. $\int \sin^4 x \cdot \cos^2 x dx$ ni hisoblang.

Yechish. $\int \sin^4 x \cdot \cos^2 x dx = \int \sin^2 x (\sin x \cos x)^2 dx =$

$$= \int \frac{1}{2}(1 - \cos 2x)(\frac{1}{2}\sin 2x)^2 dx = \frac{1}{8}\int \sin^2 2x dx - \frac{1}{8}\int \sin^2 2x \cdot \cos 2x dx =$$

$$= \frac{1}{16}\int (1 - \cos 4x) dx - \frac{1}{16}\int \sin^2 2x d(\sin 2x) = \frac{1}{16}x - \frac{1}{64}\sin 4x - \frac{1}{48}\sin^3 2x + C.$$

3) Agar m va n lar juft sonlar bo‘lib, ularning kamida biri manfiy bo‘lsa, yuqorida bayon qilingan usul maqsadga olib kelmaydi. Bunda $tgx=t$ almashtirishni bajarish lozim bo‘ladi.

c) $\int tg^n x dx$, $\int ctg^n x dx$, n – natural son, $n>1$ ko‘rinishdagi integrallar mos ravishda $tgx=t$ va $ctgx=t$ almashtirishlar yordamida hisoblanadi.

Masalan, $tgx=t$, $x=arctgt$, $dx = \frac{dt}{1+t^2}$ almashtirishlarni bajarsak, $\int tg^n x dx = \int \frac{t^n}{1+t^2} dt$ hosil bo‘ladi. Demak, berilgan integral ratsional funksiyani integrallashga keltiriladi.

6-*misol*. $\int tg^5 x dx$ ni hisoblang.

Yechish. Yuqoridagi almashtirishlarni bajarsak,

$$\int tg^5 x dx = \int \frac{t^5}{1+t^2} dt = \int (t^3 - t + \frac{t}{t^2+1}) dt = \frac{t^4}{4} - \frac{t^2}{2} + \frac{1}{2}\int \frac{d(t^2+1)}{t^2+1} =$$

$$= \frac{t^4}{4} - \frac{t^2}{2} + \frac{1}{2}\ln(t^2+1) + C = \frac{tg^4 x}{4} - \frac{tg^2 x}{2} + \frac{1}{2}\ln(tg^2 x + 1) + C$$

hosil bo‘ladi.

To break up $\int \sin^m x \cos^n x\, dx$ into forms to which you can apply the Power Rule, use the following identities.

$$\sin^2 x + \cos^2 x = 1 \qquad \text{Pythagorean identity}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \text{Half-angle identity for } \sin^2 x$$

$$\cos^2 x = \frac{1 + \cos 2x}{2} \qquad \text{Half-angle identity for } \cos^2 x$$

GUIDELINES FOR EVALUATING INTEGRALS INVOLVING POWERS OF SINE AND COSINE

1. If the power of the sine is odd and positive, save one sine factor and convert the remaining factors to cosines. Then, expand and integrate.

$$\int \sin^{\overbrace{2k+1}^{\text{Odd}}} x \cos^n x\, dx = \int \overbrace{(\sin^2 x)^k}^{\text{Convert to cosines}} \cos^n x \overbrace{\sin x\, dx}^{\text{Save for } du} = \int (1 - \cos^2 x)^k \cos^n x \sin x\, dx$$

2. If the power of the cosine is odd and positive, save one cosine factor and convert the remaining factors to sines. Then, expand and integrate.

$$\int \sin^m x \cos^{\overbrace{2k+1}^{\text{Odd}}} x\, dx = \int \sin^m x \overbrace{(\cos^2 x)^k}^{\text{Convert to sines}} \overbrace{\cos x\, dx}^{\text{Save for } du} = \int \sin^m x (1 - \sin^2 x)^k \cos x\, dx$$

3. If the powers of both the sine and cosine are even and nonnegative, make repeated use of the identities

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \text{and} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

to convert the integrand to odd powers of the cosine. Then proceed as in guideline 2.

Larson Edvards. /Calculus/ 2010. P.536.

3. $\int R(\sin x, \cos x)dx$ **integralni hisoblash.**

$I = \int R(\sin x, \cos x)dx$ integralni qaraylik. Ushbu integralni hisoblash uchun umumiy usul mavjud. Haqiqatdan ham, $t = tg\frac{x}{2}$ almashtirishni bajarsak,

$$x = 2arctgt, \quad dx = \frac{2dt}{1+t^2}, \quad \cos x = \frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}, \quad \sin x = \frac{2tg\frac{x}{2}}{1+tg^2\frac{x}{2}} = \frac{2t}{1+t^2}$$

kelib chiqadi. Bu ifodani integralga qo‘ysak,

$$I = \int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right) \cdot \frac{2dt}{1+t^2} = \int R_1(t)dt$$

hosil bo‘ladi. Bunda *R* o‘z argumentlarining ratsional funksiyasi bo‘lgani uchun R_1 ham ratsional funksiya bo‘ladi. Demak, berilgan integral ratsional funksiyalarni integrallashga keltiriladi.

iii) f is rational in $\sin x$ and/or $\cos x$. In this case

$$t = \tan \frac{x}{2},$$

together with the identities

$$\sin x = \frac{2t}{1+t^2}, \qquad \cos x = \frac{1-t^2}{1+t^2},$$

does the job, because then $x = 2 \arctan t$, hence

$$dx = \frac{2}{1+t^2}\, dt.$$

Canuto, C., Tabacco, A. Mathematical Analysis I, p.321

7-*misol*. $\int \frac{dx}{1+\sin x}$ ni hisoblang.

Yechish. Bunda $tg\frac{x}{2}=t$ almashtirishni bajaramiz. U holda

$$\int \frac{dx}{1+\sin x}=\int \frac{1}{1+\frac{2t}{1+t^2}}\cdot\frac{2dt}{1+t^2}=\int \frac{2dt}{1+2t+t^2}=2\int \frac{d(t+1)}{(t+1)^2}=\frac{-2}{t+1}+C=-\frac{2}{1+tg\frac{x}{2}}+C$$

bo‘ladi.

Shuni ta’kidlash kerakki, $t=tg\frac{x}{2}$ universal almashtirish yordamida

$\int \frac{dx}{a\cos x+b\sin x+c}$ ko‘rinishdagi integrallarni hisoblash osonlashadi.

8-*misol.* $\int \frac{dx}{9+8\cos x+\sin x}$ integralni hisoblang.

Yechish. $tg\frac{x}{2}=t$ almashtirishdan foydalanamiz. U holda

$$\int \frac{dx}{9+8\cos x+\sin x}=\int \frac{2dt}{(1+t^2)\left(9+\frac{8(1-t^2)}{1+t^2}+\frac{2t}{1+t^2}\right)}=\int \frac{2dt}{t^2+2t+17}=$$

$$=2\int \frac{d(t+1)}{(t+1)^2+16}=\frac{1}{2}arctg\frac{t+1}{4}+C=\frac{1}{2}arctg\frac{tg\frac{x}{2}+1}{4}+C.$$

Ko‘pgina hollarda bunday universal almashtirish murakkab ratsional funksiyalarni integrallashga olib keladi. Shuning uchun, ba’zi hollarda boshqa almashtirishlardan foydalanish ancha qulay bo‘ladi.

a) *R(sinx, -cosx)=-R(sinx, cosx)* bo‘lsa, u holda *sinx=t* almashtirish bajariladi.

Agar *R(-sinx, cosx)=-R(sinx, cosx)* bo‘lsa, u holda *cosx=t* almashtirish bajariladi. Nihoyat,

R(-sinx, -cosx)=R(sinx, cosx) bo‘lsa, u holda *tgx=t* almashtirishdan foydalaniladi.

9-*misol*. $\int \frac{dx}{\cos^4 x}$ integralni hisoblang.

Yechish. Bu holda integral ostidagi funksiya uchun

$$R(-\sin x, -\cos x)=R(\sin x, \cos x)$$

shart bajariladi, $tgx=t$ almashtirishdan foydalanamiz. Natijada

$$\int \frac{dx}{\cos^4 x} = \int (1+tg^2 x)d(tgx) = \int (1+t^2)dt = t + \frac{t^3}{3} + C = tgx + \frac{tg^3 x}{3} + C$$ bo‘ladi.

10-misol.

Find $\int \cos^4 x \, dx$.

Solution Because m and n are both even and nonnegative $(m = 0)$, you can replace $\cos^4 x$ by $[(1 + \cos 2x)/2]^2$.

$$\int \cos^4 x \, dx = \int \left(\frac{1 + \cos 2x}{2}\right)^2 dx \qquad \text{Half-angle identity}$$

$$= \int \left(\frac{1}{4} + \frac{\cos 2x}{2} + \frac{\cos^2 2x}{4}\right) dx \qquad \text{Expand.}$$

$$= \int \left[\frac{1}{4} + \frac{\cos 2x}{2} + \frac{1}{4}\left(\frac{1 + \cos 4x}{2}\right)\right] dx \qquad \text{Half-angle identity}$$

$$= \frac{3}{8}\int dx + \frac{1}{4}\int 2\cos 2x \, dx + \frac{1}{32}\int 4\cos 4x \, dx \qquad \text{Rewrite.}$$

$$= \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C \qquad \text{Integrate.}$$

Use a symbolic differentiation utility to verify this. Can you simplify the derivative to obtain the original integrand? ■

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