12-MAVZU. ANIQ INTEGRAL VA UNING XOSSALARI.

REJA

1. Aniq integralning ta’rifi.

2. Aniq integralning asosiy xossalari.

1. Aniq integralning ta'rifi.

Aniq integral - matematik analizning eng muhim tushunchalaridan biridir. Egri chiziq bilan chegaralangan yuzalarni, egri chiziqli yoylar uzunliklarini, hajmlarni, bajarilgan ishlarni, yo'llarni, inersiya momentlarini va hokazolarni hisoblash masalasi shu tushuncha bilan bog'liq. $[a, b]$ kesmada $y=f(x)$ uzluksiz funksiya berilgan bo'lsin. Quyidagi amallarni bajaramiz.

1. $[a, b]$ kesmani qo'yidagi nuqtalar bilan ixtiyoriy n ta qismga bo'lamiz, va ularni qismiy intervallar deb ataymiz.

$$a=x_0<x_1<x_2<x_3<\ldots x_{i-1}<x_i\ldots<x_n=b$$

2.Qismiy intervallarning uzunliklarini bunday belgilaymiz:

$$\Delta x_1=x_1-a \qquad \Delta x_2=x_2-x_1 \ \ldots \quad \Delta x_i=x_i-x_{i-1} \ \ldots \quad \Delta x_n=b-x_{n-1}$$

DEFINITION OF RIEMANN SUM

Let f be defined on the closed interval $[a, b]$, and let Δ be a partition of $[a, b]$ given by

$$a = x_0 < x_1 < x_2 < \cdot \cdot \cdot < x_{n-1} < x_n = b$$

where Δx_i is the width of the ith subinterval. If c_i is *any* point in the ith subinterval $[x_{i-1}, x_i]$, then the sum

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i, \quad x_{i-1} \leq c_i \leq x_i$$

is called a **Riemann sum** of f for the partition Δ.

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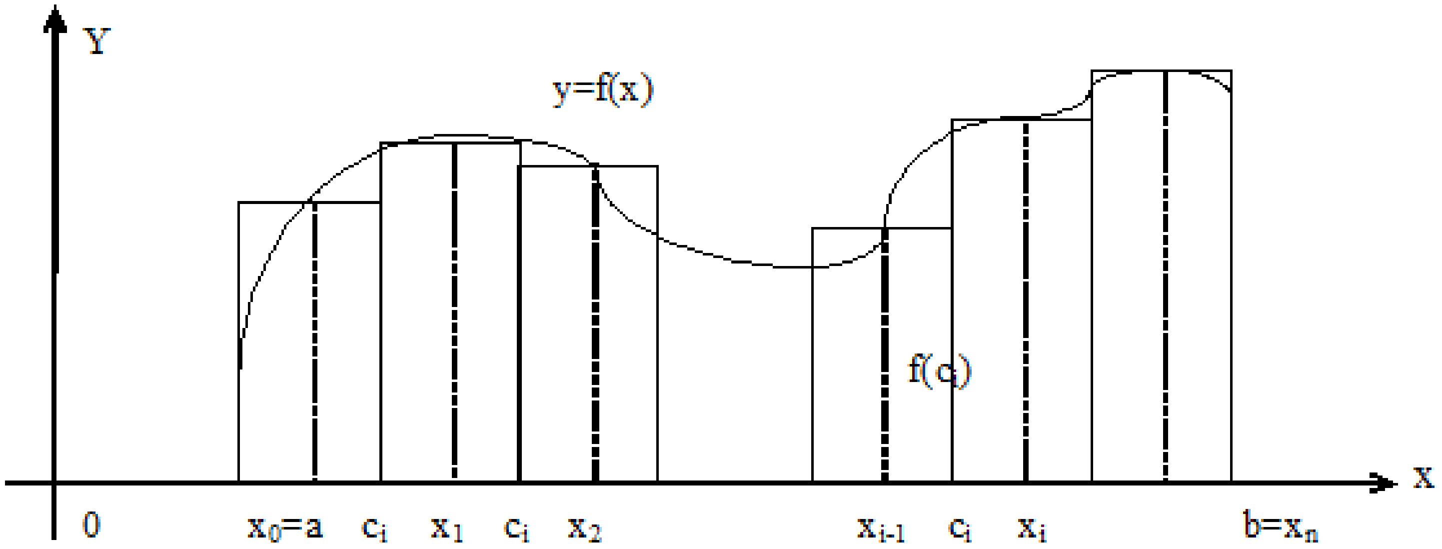
σ_n yig'indi $f(x)$ funksiya uchun $[a, b]$ kesmada tuzilgan integral yig'indi deb ataladi. σ_n integral yig'indi qisqacha bunday yoziladi:

$$\sigma_n = \sum_{i=1}^{n} f(c_i)\Delta x_i .$$



1-rasm.

Integral yig'indining geometrik ma'nosi ravshan: Agar $f(x) \geq 0$ bo'lsa, u holda σ_n - asoslari Δx_1, Δx_2, ..., Δx_i,...,Δx_n va balandliklari mos ravishda $f(c_1)$, $f(c_2)$, ..., $f(c_i)$, ..., $f(c_n)$ bo'lgan to'g'ri to'rtburchak yuzlarining yig'indisidan iborat (1-rasm).

Endi bo'lishlar soni n ni orttira boramiz ($n \to \infty$) va bunda eng katta intervalning uzunligi nolga intilishini, ya'ni $\max \Delta x_i \to 0$ deb faraz qilamiz.

Ushbu ta'rifni beramiz.

Ta'rif. Agar σ_n integral yig'indi $[a, b]$ kesmani qismiy $[x_i, x_{i-1}]$ kesmalarga ajratish usuliga va ularning har biridan c_i nuqtani tanlash usuliga bog'liq bo'lmaydigan chekli songa intilsa, u holda shu son $[a, b]$ kesmada $f(x)$ funksiyadan olingan aniq integral deyiladi va bunday belgilanadi:

$$\int_a^b f(x)dx.$$

Bu yerda $f(x)$ integral ostidagi funksiya. $[a, b]$ kesma integrallash oralig'i. a va b sonlar integrallashning qo'yi va yuqori chegarasi deyiladi.

$$\int_a^b f(x)dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

DEFINITION OF DEFINITE INTEGRAL

If f is defined on the closed interval $[a, b]$ and the limit of Riemann sums over partitions Δ

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\, \Delta x_i$$

exists (as described above), then f is said to be **integrable** on $[a, b]$ and the limit is denoted by

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\, \Delta x_i = \int_a^b f(x)\, dx.$$

The limit is called the **definite integral** of f from a to b. The number a is the **lower limit** of integration, and the number b is the **upper limit** of integration.

THEOREM 4.4 CONTINUITY IMPLIES INTEGRABILITY

If a function f is continuous on the closed interval $[a, b]$, then f is integrable on $[a, b]$. That is, $\int_a^b f(x)\, dx$ exists.

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Aniq integralning ta'rifidan ko'rinadiki, aniq integral hamma vaqt mavjud bo'lavermas ekan. Biz qo'yida aniq integralning mavjudlik teoremasini isbotsiz keltiramiz.

THEOREM 4.5 THE DEFINITE INTEGRAL AS THE AREA OF A REGION

If f is continuous and nonnegative on the closed interval $[a, b]$, then the area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by

$$\text{Area} = \int_a^b f(x)\,dx.$$

(See Figure 4.21.)

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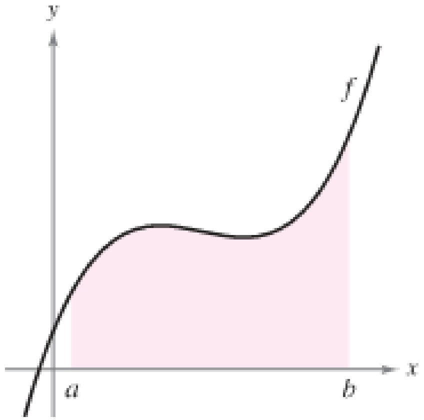

You can use a definite integral to find the area of the region bounded by the graph of f, the x-axis, $x = a$, and $x = b$.

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Teorema. Agar $u=f(x)$ funksiya $[a, b]$ kesmada uzluksiz bo'lsa, u integrallanuvchidir, ya'ni bunday funksiyaning aniq integrali mavjuddir.

Agar yuqoridan $y=f(x)\geq 0$ funksiyaning grafigi, qo'yidan OX o'qi, yon tomonlaridan esa $x=a$, $x=b$ to'g'ri chiziqlar bilan chegaralangan sohani egri chiziqli trapetsiya deb atasak, u holda

$$\int_a^b f(x)dx.$$

Aniq integralning geometrik ma'nosi ravshan bo'lib qoladi: $f(x) \geq 0$ bo'lganda u shu egri chiziqli trapetsiyaning yuziga son jihatdan teng bo'ladi.

1-izoh. Aniq integralning qiymati funksiyaning ko'rinishiga va integrallash chegarasiga bog'liq. Masalan:

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(z)dz.$$

2-izoh. Aniq integralning chegaralari almashtirilsa, integralning ishorasi o'zgaradi.

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

3-izoh. Agar aniq integralning chegaralari teng bo'lsa, har qanday funksiya uchun ushbu tenglik o'rinli bo'ladi:

$$\int_a^b f(x)dx = 0$$

18.2. Aniq integralning asosiy xossalari.

1-xossa. O'zgarmas ko'paytuvchini aniq integral belgisidan tashqariga chiqarish mumkin:

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx \qquad (1)$$

2-xossa. Bir nechta funksiyaning algebraik yig'indisining aniq integrali qo'shiluvchilar integralining yig'indisiga teng (ikki qo'shiluvchi bo'lgan hol bilan chegaralanamiz):

$$\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx \qquad (2)$$

THEOREM 4.7 PROPERTIES OF DEFINITE INTEGRALS

If f and g are integrable on $[a, b]$ and k is a constant, then the functions kf and $f \pm g$ are integrable on $[a, b]$, and

1. $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

2. $\displaystyle\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$

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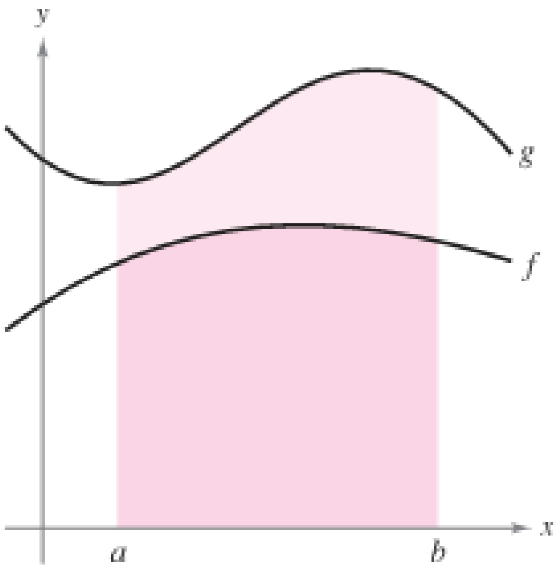
3-xossa. Agar $[a, b]$ kesmada ikki $f(x)$ va $g(x)$ funksiya $(a<b)$ $f(x) \le g(x)$ shartni qanoatlantirsa, ushbu tengsizlik o'rinli.

$$\int_a^b f(x)dx \ge \int_a^b g(x)dx. \qquad (3)$$



$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$

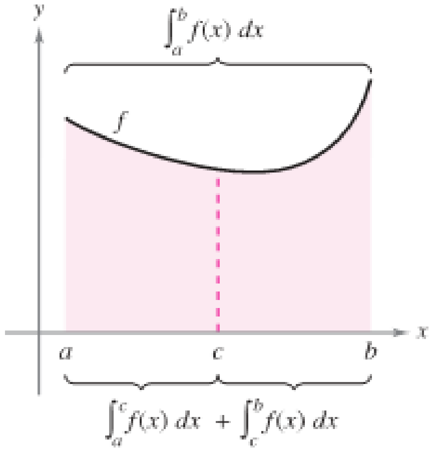
4-xossa. Agar [*a*, *b*] kesma bir necha qismga bo'linsa, u holda [*a*, *b*] kesma bo'yicha aniq integral har bir qism bo'yicha olingan aniq integrallar yig'indisiga teng. [*a*, *b*] kesma ikki qismga bo'lingan hol bilan cheklanamiz, ya'ni $a<c<b$ bo'lsa, u holda

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \qquad (4)$$

THEOREM 4.6 ADDITIVE INTERVAL PROPERTY

If f is integrable on the three closed intervals determined by a, b, and c, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$



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5-xossa. Agar m va M sonlar $f(x)$ funksiyaning $[a, b]$ kesmada eng kichik va eng katta qiymati bo'lsa, ushbu tengsizlik o'rinli.

$$m(b-a) \leq \int_a^b f(x)dx. \leq M(b-a) \qquad (5)$$

Isboti Shartga ko'ra $m \leq f(x) \leq M$ ekani kelib chiqadi. 3-xossaga asosan qo'yidagiga ega bo'lamiz:

$$\int_a^b m dx \leq \int_a^b f(x)dx \leq \int_a^b M dx \qquad (5^*)$$

Biroq

$$\int_a^b m dx = m\int_a^b dx = m \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} \Delta x_i = m(b-a)$$

$$\int_a^b M dx = M\int_a^b dx = M \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} \Delta x_i = M(b-a)$$

bo'lgani uchun (5*) tengsizlik

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

bo'ladi.

6-xossa (o'rta qiymat haqidagi teorema).

Agar $f(x)$ funksiya $[a, b]$ kesmada uzluksiz bo'lsa, bu kesmaning ichida shunday $x=s$ nuqta topiladiki, bu nuqtada funksiyaning qiymati uning shu kesmadagi o'rtacha qiymatiga teng bo'ladi, ya'ni

$$f(s)=\frac{1}{(b-a)}\int_a^b f(x)dx.$$

Isboti. Faraz qilaylik, m va M sonlar $f(x)$ uzluksiz funksiyaning $[a, b]$ kesmadagi eng kichik va eng katta qiymati bo'lsin. Aniq integralni baholash haqidagi xossaga ko'ra qo'yidagi qo'sh tengsizlik to'g'ri:

$$m(b-a)\leq\int_a^b f(x)dx.\leq M(b-a)$$

tengsizlikning hamma qismlarini b-a>0 ga bo'lamiz. Natijada

$$m \le \frac{1}{(b-a)}\int_a^b f(x)dx \le M$$

ni hosil qilamiz. Ushbu $\mu = \frac{1}{(b-a)}\int_a^b f(x)dx.$ belgilashni kiritib, qo'sh tengsizlikni qayta yozamiz. $m \le \mu \le M$

$f(x)$ funksiya $[a, b]$ kesmada uzluksiz bo'lgani uchun u m va M orasidagi hamma oraliq qiymatlarni qabul qiladi.

Demak, biror $x=s$ qiymatda $\mu=f(s)$ bo'ladi, ya'ni

$$f(s) = \frac{1}{(b-a)}\int_a^b f(x)dx.$$

Teorema isbotlandi.

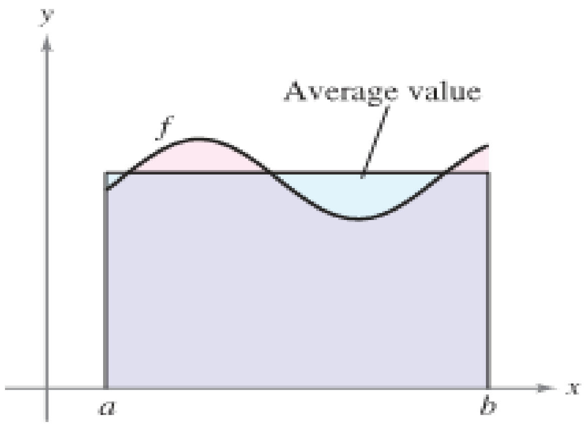
Average Value of a Function

The value of $f(c)$ given in the Mean Value Theorem for Integrals is called the **average value** of f on the interval $[a, b]$.

DEFINITION OF THE AVERAGE VALUE OF A FUNCTION ON AN INTERVAL

If f is integrable on the closed interval $[a, b]$, then the **average value** of f on the interval is

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$



$$\text{Average value} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

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