13-MAVZU. ANIQ INTEGRALDA O'ZGARUVCHINI ALMASHTIRISH VA BO'LAKLAB INTEGRALLASH.

REJA

1. Nyuton-Leybnits formulasi.
2. O’zgaruvchini almashtirish.
3. Aniq integralni bo’laklab integrallash.

1. Nyuton-Leybnits formulasi.

Aniq integrallarni integral yig'indining limiti sifatida bevosita hisoblash ko'p hollarda juda qiyin, uzoq hisoblashlarni talab qiladi va amalda juda kam qo'llaniladi. Integrallarni topish formulasi Nyuton-Leybnits teoremasi bilan beriladi.

Teorema. Agar *F(x)* funksiya *f(x)* funksiyaning [*a*, *b*] kesmadagi boshlang'ich funksiyasi bo'lsa, u holda aniq integral boshlang'ich funksiyaning integrallash oralig'idagi orttirmasiga teng, ya'ni

$$\int_a^b f(x)dx = F(b) - F(a) \qquad (1)$$

(1)tenglik Nyuton-Leybnits formulasi deyiladi.

1. *Provided you can find* an antiderivative of f, you now have a way to evaluate a definite integral without having to use the limit of a sum.
2. When applying the Fundamental Theorem of Calculus, the following notation is convenient.

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b) - F(a)$$

For instance, to evaluate $\int_1^3 x^3\,dx$, you can write

$$\int_1^3 x^3\,dx = \frac{x^4}{4}\Big]_1^3 = \frac{3^4}{4} - \frac{1^4}{4} = \frac{81}{4} - \frac{1}{4} = 20.$$

3. It is not necessary to include a constant of integration C in the antiderivative because

$$\int_a^b f(x)\,dx = \Big[F(x) + C\Big]_a^b = [F(b) + C] - [F(a) + C] = F(b) - F(a).$$

Larson Edvards. /Calculus/ 2010. P.283.

Isboti. $F(x)$ funksiya $f(x)$ funksiyaning biror boshlang'ich funksiyasi bo'lsin, u holda 1-teoremaga ko'ra $\int_a^x f(t)dt$ funksiya ham $f(x)$ funksiyaning boshlang'ich funksiyasi bo'ladi. Berilgan funksiyaning ikkita istalgan boshlang'ich funksiyalari bir-biridan o'zgarmas C qo'shiluvchiga farq qiladi, ya'ni $F(x)=F(x)+C$.

Shuning uchun:

$$\int_a^x f(t)dt = F(x) + C$$

C-o'zgarmas miqdorni aniqlash uchun bu tenglikda x=a deb olamiz:

$$\int_a^a f(t)dt = F(a) + C, \quad \int_a^a f(t)dt = 0$$

bo'lgani uchun $F(a)+C=0$. Bundan, $S=-F(a)$. Demak, $\int_a^x f(t)dt = F(x) - F(a)$

Endi $x=b$ deb Nyuton-Leybnits formulasini hosil qilamiz:

$$\int_a^b f(t)dt = F(b)\text{-}F(a)$$

yoki integrallash o'zgaruvchisini x bilan almashtirsak:

$$\int_a^b f(x)dx = F(b)\text{-}F(a)$$

$F(b)\text{-}F(a)=F(x)\big|_a^b$ belgilash kiritib, oxirgi formulani qo'yidagicha qayta yozish mumkin:

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b)\text{-}F(a)$$

Teorema isbotlandi.

Integral ostidagi funksiyaning boshlang'ich funksiyasi ma'lum bo'lsa, u golda Nyuton-Leybnits formulasi aniq integrallarni hisoblash uchun amalda qulay usulni beradi. Faqat shu formulaning kashf etilishi aniq integralni hozirgi zamonda matematik analizda tutgan o'rnini olishga imkon bergan. Nyuton-Leybnits formulasi aniq integralning tatbiqi sohasini ancha kengaytirdi, chunki matematika bu formula yordamida xususiy kurinishdagi turli masalalarni yechish uchun umumiy usulga ega bo'ldi.

Misollar.

1) $\int_0^1 \frac{dx}{1+x^2} = arctgx|_0^1 = arctg1 - arctg0 = \frac{\pi}{4}$

2) $\int_{\sqrt{3}}^{\sqrt{8}} \frac{xdx}{\sqrt{1+x^2}} = \frac{1}{2}\int_{\sqrt{3}}^{\sqrt{8}} \frac{d(1+x^2)}{\sqrt{1+x^2}} = \frac{1}{2}\int_{\sqrt{3}}^{\sqrt{8}} (1+x^{2)^{-\frac{1}{2}}} d(1+x^2) = (1+x^2)^{\frac{1}{2}}|_{\sqrt{3}}^{\sqrt{8}} =$

$= \sqrt{9} - \sqrt{4} = 3 - 2 = 1.$

3) $\int_0^{\frac{\pi}{2}} \sin^2 xdx = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1-\cos 2x)dx = \frac{1}{2}(x - \frac{1}{2}\sin 2x)|_0^{\frac{\pi}{2}} = \frac{\pi}{4}.$

2. O'zgaruvchini almashtirish.

Bizga $\int_a^b f(x)dx$ aniq integral berilgan bo'lsin, bunda $f(x)$ funksiya $[a, b]$ kesmada uzluksizdir.

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Aniq integral (2) formula bo'yicha hisoblaganda yangi o'zgaruvchidan eski o'zgaruvchiga qaytish kerak emas, balki eski o'zgaruvchining chegaralarini keyingi boshlang'ich funksiyaga qo'yish kerak.

THEOREM 4.15 CHANGE OF VARIABLES FOR DEFINITE INTEGRALS

If the function $u = g(x)$ has a continuous derivative on the closed interval $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Larson Edvards. /Calculus/ 2010. P.303.

Misollar.

1) $\int_3^8 \frac{xdx}{\sqrt{x+1}}$ integralni hisoblang.

Yechish. x+1=u^2 deb almashtirsak, x=u^2-1, dx=2udu bo'ladi. Integrallashning yangi chegaralari: x=3 bo'lganda t=2.

x=8 bo'lganda u=3 u holda:

$$\int_3^8 \frac{xdx}{\sqrt{x+1}} = \int_2^3 \frac{(t^2-1)2udu}{t} = 2\int_2^3 (u^2-1)du = 2(\frac{u^3}{3}-u)|_2^3 = 2(6-\frac{2}{3}) = \frac{32}{3};$$

2) $\int_0^1 \sqrt{1-x^2}dx$ integralni hisoblang.

Yechish. x=sin$u$ deb almashtirsak, dx=cos$u$$du$, 1-$x^2$=cos$^2u$ bo'ladi. Integrallashning yangi chegaralarini aniqlaymiz: x=0 bo'lganda u=0

x=1 bo'lganda u=π/2

U holda:

$$\int_0^1 \sqrt{1-x^2}dx = \int_0^{\pi/2} \cos^2 udu = \frac{1}{2}\int_0^{\pi/2}(1+\cos 2udu = \frac{1}{2}(u+\frac{1}{2}\sin 2u)\Big|_0^{\pi/2} = \frac{\pi}{4}$$

3. Aniq integralni bo'laklab integrallash.

Faraz qilaylik, $u(x)$ va $v(x)$ funksiyalar $[a, b]$ kesmada differensiallanuvchi funksiyalar bo'lsin. U holda: $(uv)'=u'v+uv'$

Bu tenglikni ikkala tomonini a dan b gacha bo'lgan oraliqda integrallaymiz.

$$\int_a^b (uv)'dx = \int_a^b u'vdx + \int_a^b uv'dx \qquad (3)$$

Lekin $\int (uv)'dx = uv + C$ bo'lgani sababli

$$\int (uv)'dx = uv\Big|_a^b$$

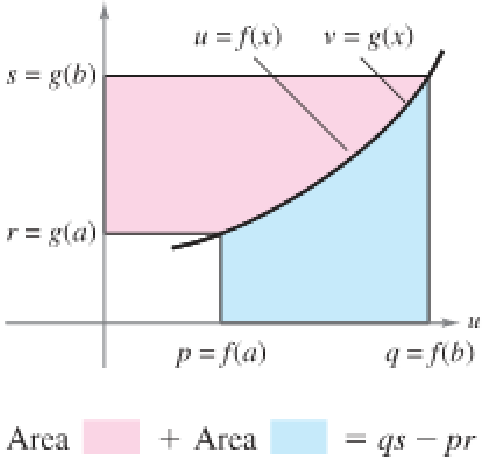
Demak, (3) tenglikni qo'yidagi ko'rinishda yozish mumkin:

$$uv\Big|_a^b = \int_a^b vdu + \int_a^b udv$$

Bundan

$$\int_a^b udv = uv\,\Big|_a^b - \int_a^b vdu \qquad (4)$$

Bu formula aniq integralni bo'laklab integrallash formulasi deyiladi.



$$\int_r^s u\,dv + \int_q^p v\,du = \Big[uv\Big]_{(p,r)}^{(q,s)}$$

$$\int_r^s u\,dv = \Big[uv\Big]_{(p,r)}^{(q,s)} - \int_q^p v\,du$$

Misol.

1) $\int_0^1 arctgx dx$ integral hisoblansin.

$$\int_0^1 arctgx dx = \begin{vmatrix} u = arctgx & du = \dfrac{dx}{1+x^2} \\ dv = dx & v = x \end{vmatrix} = xarctgx \Big|_0^1 - \int_0^1 \frac{xdx}{1+x^2} = arctg1 -$$

$$\frac{1}{2}\ln\left(1+x^2\right)\Big|_0^1 = \frac{\pi}{4} - \frac{1}{2}\ln 2$$

2) $\int_0^1 xe^{-x} dx$ integral hisoblansin.

$$\int_0^1 xe^{-x} dx = \begin{vmatrix} u = x & du = dx \\ dv = e^{-x} dx & v = -e^{-x} \end{vmatrix} = -xe^{-x}\Big|_0^1 + \int_0^1 e^{-x} dx = -e^{-1} - e^{-x}\Big|_0^1 =$$

$$= -e^{-1} - e^{-1} + 1 = 1 - \frac{2}{e};$$

Izoh: Ba'zi integrallarni hisoblashda bo'laklab integrallash formulasini bir necha marta qo'llash mumkin.

3) $\int_0^1 \arcsin x dx$ integral hisoblansin.

Evaluate $\int_0^1 \arcsin x \, dx$.

Solution Let $dv = dx$.

$$dv = dx \quad \Rightarrow \quad v = \int dx = x$$

$$u = \arcsin x \quad \Rightarrow \quad du = \frac{1}{\sqrt{1 - x^2}} dx$$

Integration by parts now produces

$$\int u \, dv = uv - \int v \, du$$ Integration by parts formula

$$\int \arcsin x \, dx = x \arcsin x - \int \frac{x}{\sqrt{1 - x^2}} dx$$ Substitute.

$$= x \arcsin x + \frac{1}{2} \int (1 - x^2)^{-1/2} (-2x) \, dx$$ Rewrite.

$$= x \arcsin x + \sqrt{1 - x^2} + C.$$ Integrate.

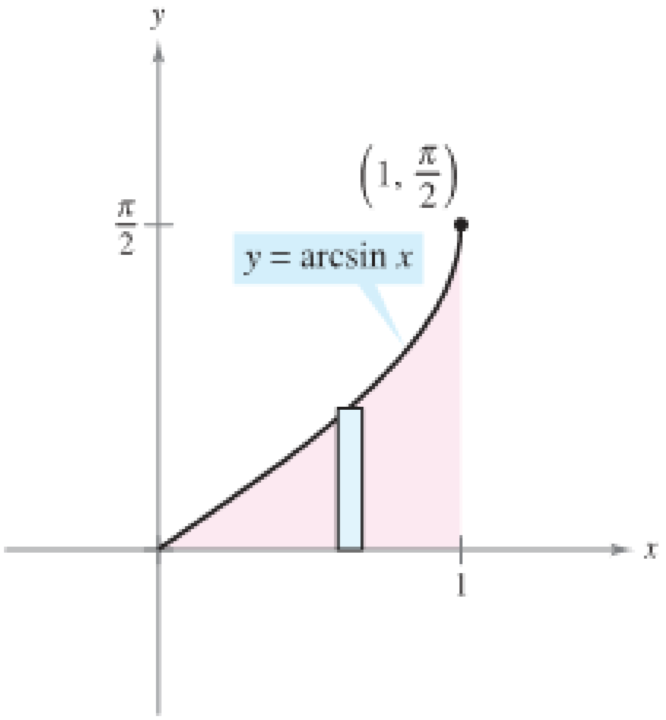
Using this antiderivative, you can evaluate the definite integral as follows.

$$\int_0^1 \arcsin x \, dx = \left[x \arcsin x + \sqrt{1 - x^2} \right]_0^1$$

$$= \frac{\pi}{2} - 1$$

$$\approx 0.571$$

The area represented by this definite integral is shown in Figure 8.2.



The area of the region is approximately 0.571.
Figure 8.2

Larson Edvards. /Calculus/ 2010. P.529