14-MAVZU: XOSMAS INTEGRALLAR.

REJA.

1. Trapesiyalar formulasi.
2. Simpson formulasi.
3. Xosmas integrallar.

Tayanch ibora va tushunchalar

Xosmas integral, cheksiz oraliq bo'yicha integral, xosmas integral yaqinlashuvchi, xosmas integrpl uzoqlashuvchi, chegaralanmagan funksiyaningchekli oraliq bo'yicha xosmas integrali.

Taqribiy hisoblash, trapesiyalar formulasi, Simpson formulasi, hisoblash xatosi, *parabolik trapesiya*, xosmas integral, cheksiz oraliq bo'yicha integral, xosmas integral yaqinlashuvchi, xosmas integrpl uzoqlashuvchi, chegaralanmagan funksiyaning chekli oraliq bo'yicha xosmas integrali.

Hisoblash amaliyotida ko'pincha boshlang'ich funksiyalari elementar bo'lmagan, ya'ni chekli ko'rinishda ifodalab bo'lmaydigan funksiyalardan olingan integrallar bilan, shuningdek, jadval yoki grafik usulda berilgan funksiyalardan olingan integrallar bilan ish ko'rishga to'g'ri keladi. Bunday hollarda Nyuton - Leybnis formulasini qo'llab bo'lmaydi va integral taqribiy usullar yordamida hisoblanadi.

Hisoblash mashinalarining jadal taraqqiy etib borishi natijasida aniq integrallarni hisoblashning taqribiy usullari keng tatbiq qilinmoqda.

Integral ostidagi funksiya elementar boshlang'ich funksiyaga ega bo'lsada, biroq, uni Nyuton - Leybnis formulasi bo'yicha hisoblash murakkab va katta hajmdagi hisoblash ishlarini talab etadigan hollarda ham taqribiy hisoblash usullari afzal bo'ladi.

Aniq integralni taqribiy hisoblashning bir necha usullari mavjud bo'lib ulardan ko'proq ishlatiladiganlari trapesiyalar va Simpson usullaridir.

1.Trapesiyalar formulasi

Trapesiyalar formulasi

$$\int_a^b f(x)dx$$

aniq integralni hisoblash talab etilsin $y = f(x)$ funksiya $[a,b]$ kesmada uzluksiz $[a,b]$ kesmani $a = x_0 < x_1 < x_2 < .... < x_n = b$ nuqtalar orqali n ta teng qismiy kesmalarga ajratamiz. Funksiyaning x_i nuqtalaridagi $y_i = f(x_i)$ qiymatlarini hisoblaymiz $(i = 1, n)$. $[x_{i-1}, x_i]$ qismiy kesmalarning uzunligi $\frac{(b-a)}{n}$ kattalik integrallash qadami deyiladi. Bo'linish nuqtalaridan $y_0, y_1, y_2, ..., y_n$ ordinatlarni o'tkazamiz. Ordinatlar oxirlarini to'g'ri chiziqlar bilan tutashtirib trapesiyalar hosil qilamiz.

Aniq integralning taqribiy qiymati uchun, hosil bo'lgan trapesiyalar yuzlarining yig'indisini olamiz. Bu holda

Bu holda

$$S = \int_a^b f(x)dx \approx \frac{y_0 + y_1}{2} \cdot \frac{b-a}{n} + \frac{y_1 + y_2}{2} \cdot \frac{b-a}{n} + \frac{y_2 + y_3}{2} \cdot \frac{b-a}{n} + \ldots + \frac{y_{n-1} + y_n}{2} \cdot \frac{b-a}{n}$$

Shunday qilib, natijada

$$S = \int_a^b f(x)dx \approx \frac{b-a}{n}\left[\frac{y_0 + y_n}{2} + y_1 + y_2 + y_3 + \ldots y_{n-1}\right] \qquad (1)$$

formulani olamiz. (1) formulaga *trapesiyalar formulasi* deb ataladi. Bu formulada egri chiziqli trapesiyalarning yuzlarini to'g'ri chiziqli trapesiyalar yuzlari bilan taqriban almashtirdik. n o'sib borishi bilan to'g'ri chiziqli trapesiyalarning yuzi egri chiziqli trapesiyalar yuzlariga cheksiz yaqinlashib boradi.

Bu taqribiy hisoblashda yo'l qo'yilgan *absolyut xato*

$$M_2 \frac{(b-a)^3}{12n^2}$$

ifodadan katta emasligini ko'rsatish mumkin, bunda M_2, $|f''(x)|$ ning $[a,b]$ kesmadagi eng katta qiymati.

2. Simpson formulasi.

2. Simpson formulasi. $[a,b]$ kesmani $n=2m$ ta juft miqdordagi teng qismlarga bo'lamiz. Uchta $(x_0,y_0),(x_1,y_1),(x_2,y_2)$ nuqtalar olib ulardan parabola o'tkazamiz. Bu parabola bilan $y=f(x)$ funksiyaning $[x_0,x_2]$ kesmadagi grafigini almashtiramiz. Xuddi shunga o'xshash $y=f(x)$ funksiyaning grafigini $[x_2,x_4]$, $[x_4,x_6]$ va boshqa kesmalarda ham almashtiramiz.

Shunday qilib, bu usulda berilgan $y=f(x)$ egri chiziq bilan chegaralangan trapesiyaning yuzini $[x_0,x_2]$, $[x_2,x_4]$, $[x_4,x_6]$,..... kesmalarda parabolalar bilan chegaralangan egri chiziqli trapesiyalar yuzlarining yig'indisi bilan almashtiriladi. Bunday egri chiziqli trapesiya *parabolik trapesiya* deyiladi.
Parabolik trapesiyalar yuzlarini qo'shib,

$$S=\int_a^b f(x)dx \approx \frac{b-a}{6m}[y_0+y_{2m}+4(y_1+y_3+....+y_{2m-1})+2(y_2+y_4+...+y_{2m-2})] \quad (2)$$

Bu formula *Simpson (parabolalar) formulasi* deyiladi. Simpson formulasining *absolyut xatosi* $M_4\dfrac{(b-a)^5}{2880n^4}$ dan katta bo'lmaydi, bunda M_4, $|f^5(x)|$ funksiyaning $[a,b]$ kesmadagi eng katta qiymati. Xatolarni baholash ifodalaridan ma'lumki n^4 kattalik n^2 kattalikka nisbatan tezroq o'sgani uchun Simpson formulasining xatoligi trapesiyalar formulasi xatosiga nisbatan ancha tez kamayadi.

1-misol. $\int_0^1 e^{-x^2}dx$ aniq integral trapesiyalar va Simpson formulalaridan foydalanib taqribiy hisoblansin.

Yechish. $[0,1]$ kesmani $x_0 = 0,\ x_1 = 0.2,\ x_2 = 0.4,\ x_3 = 0.6,\ x_4 = 0.8,\ x_5 = 1$ nuqtalar yordamida 5 ta teng bo'lakka bo'lamiz. Keyin $f(x) = e^{-x^2}$ funksiyaning shu nuqtalardagi qiymatlarini hisoblaymiz.

$$f(x_0) = f(0) = e^0 = 1, \qquad f(x_1) = f(0.2) \approx 0.96079,$$
$$f(x_2) = f(0.4) \approx 0.85214, \qquad f(x_3) = f(0.6) \approx 0.69768,$$
$$f(x_4) = f(0.8) \approx 0.52729, \qquad f(x_5) = f(2) \approx 0.36788.$$

Trapesiyalar formulasi bo'yicha

$$\int_0^1 e^{-x^2} dx \approx \frac{1}{5}\left[\frac{1+0.36788}{2} + 0.96079 + 0.85214 + 0.69769 + 0.52729\right] = 0.74805.$$

Simpson formulasi bo'yicha, hisoblash uchun $[0,1]$ kesmani $x_0 = 0,\ x_1 = \frac{1}{4},\ x_2 = \frac{1}{2},\ x_3 = \frac{3}{4},\ x_4 = 1$ nuqtalar orqali 4 ta teng bo'laklarga ajratamiz va bu nuqtalarda funksiyaning qiymatlari

$$y_0 = 1, \quad y_1 = (0.25) \approx 0.9394, \quad y_2 = (0.5) \approx 0.7788,$$
$$y_3 = (0.75) \approx 0.5698, \quad y_4 = (1) \approx 0.3679$$

bo'ladi.

Simpson formulasiga asosan:

$$\int_0^1 e^{-x^2}\ dx \approx \frac{1}{12}[1 + 0.3679 + 4(0.9394 + 0.5698) + 2 \cdot 0.7788] \approx 0.7469$$

bo'ladi.

3. Xosmas integrallar.

Aniq integralning ta'rifida integrallash chegaralari chekli va integral ostidagi funksiya $[a,b]$ oraliqda chegaralangan deb olingan edi. Bu shartlardan hyech bo'lmaganda birortasi bajarilmasa, integralning yuqoridagi ta'rifi ma'nosini yo'qotadi.

Biroq nazariy va amaliy mulohazalarga muvofiq aniq integralning ta'rifi bu cheklanishlar bajarilmaydigan hollar uchun ham umumlashtirilishi mumkin.

Bunday integrallar bizga tanish bo'lgan aniq integrallarga xos bo'lmagan qisqacha *xosmas integrallar* deb aytiladi.

Xosmas integrallarning ikki asosiy turini qaraymiz:

1) Uzluksiz funksiyalarning cheksiz oraliq bo'yicha integrallari. $f(x)$ funksiya $[a,+\infty)$ oraliqda berilgan va uning istalgan qismi $[a,+A]$ da integrallanuvchi, ya'ni istalgan $A > a$ da aniq integral mavjud bo'lsin. Bu holda

$$\lim_{A\to\infty}\int_a^A f(x)dx = J$$

limitga $f(x)$ funksiyaning $[a,\infty)$ oraliqdagi xosmas integrali deyiladi va quyidagicha belgilanadi:

$$J = \int_a^\infty f(x)dx. \qquad (1)$$

J limit chekli bo'lsa, xosmas integral *yaqinlashuvchi* deyiladi. Limit mavjud bo'lmasa, xosmas integral *uzoqlashuvchi* deyiladi.

$f(x)$ funksiyadan $(-\infty, a]$ oraliq bo'yicha olingan xosmas integral ham xuddi yuqoridagiga o'xshash aniqlanadi:

$$\int_{-\infty}^{a} f(x)dx = \lim_{A \to -\infty} \int_{A}^{a} f(x)dx. \qquad (2)$$

$f(x)$ funksiyadan $(-\infty, +\infty)$ oraliq bo'yicha olingan xosmas integral qo'yidagicha aniqlanadi.

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx \qquad (3)$$

bu yerda a istalgan son. (3) integrallarda o'ng tomondagi ikkala integral ham yaqinlashsa chap tomondagi integral ham yaqinlashuvchi deyiladi. O'ng tomondagi integrallardan aqalli bittasi uzoqlashsa, chap tomondagi integral ham uzoqlashuvchi bo'ladi.

DEFINITION OF IMPROPER INTEGRALS WITH INFINITE INTEGRATION LIMITS

1. If f is continuous on the interval $[a, \infty)$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx.$$

2. If f is continuous on the interval $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a \to -\infty} \int_a^b f(x)\,dx.$$

3. If f is continuous on the interval $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx$$

where c is any real number (see Exercise 120).

In the first two cases, the improper integral **converges** if the limit exists—otherwise, the improper integral **diverges.** In the third case, the improper integral on the left diverges if either of the improper integrals on the right diverges.

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Xosmas itegrallarni hisoblash uchun Nyuton-Leybnis formulasidan foydalaniladi. $F(x)$ funksiya $[a,+\infty)$ oraliqda $f(x)$ uchun boshlang'ich funksiya bo'lsa,

$$\int_a^{\infty} f(x)dx = \lim_{A\to\infty}\int_a^{A} f(x)dx = \lim_{A\to\infty}\left[F(A)-F(a)\right] = F(+\infty)-F(a) = F(x)$$

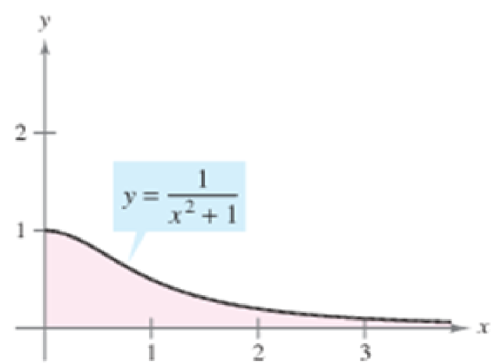
bo'lib, bu yerda: $F(+\infty)=\lim_{A\to\infty}F(A)$ integralning yaqinlashishini yoki uzoqlashishini aniqlaydi.

1-misol. $\int_1^{\infty}\frac{dx}{(1+x^2)}$ integralning yaqinlashishini tekshiring.

Yechish: $\int_0^{\infty}\frac{dx}{1+x^2} = arctgx\Big|_1^{\infty} = \lim_{x\to\infty} arctgx - arctg0 = \frac{\pi}{2}-0=\frac{\pi}{2}$.

Demak, integral yaqinlashuvchi va $\frac{\pi}{2}$ ga teng.

b. $$\int_0^{\infty}\frac{1}{x^2+1}\,dx = \lim_{b\to\infty}\int_0^{b}\frac{1}{x^2+1}\,dx = \lim_{b\to\infty}\Big[\arctan x\Big]_0^b = \lim_{b\to\infty}\arctan b = \frac{\pi}{2}$$



The area of the unbounded region is $\pi/2$.
Figure 8.20

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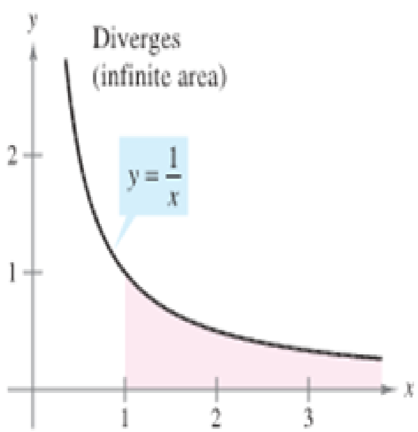
2-misol. $\int_1^{\infty}\frac{dx}{x}=\lim_{A\to\infty}\int_1^{A}\frac{dx}{x}=\lim_{A\to\infty}\ln x\Big|_1^A=\lim_{A\to\infty}(\ln A-\ln 1)=\lim_{A\to\infty}\ln A=\infty$

bo'lib, bu integral uzoqlashuvchi.



This unbounded region has an infinite area.

Figure 8.18

Evaluate $\int_1^{\infty}\frac{dx}{x}$.

Solution

$$\int_1^{\infty}\frac{dx}{x}=\lim_{b\to\infty}\int_1^{b}\frac{dx}{x}$$ Take limit as $b\to\infty$.

$$=\lim_{b\to\infty}\Big[\ln x\Big]_1^b$$ Apply Log Rule.

$$=\lim_{b\to\infty}(\ln b-0)$$ Apply Fundamental Theorem of Calculus.

$$=\infty$$ Evaluate limit.

See Figure 8.18.

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2) Chegaralanmagan funksiyalarning chekli oraliq bo'yicha xosmas integrallari. $(a,b]$ intervalda uzluksiz va $x=a$ da aniqlanmagan yoki uzilishga ega bo'lgan $f(x)$ funksiyaning xosmas integrali quyidagicha belgilanib aniqlanadi:

$$\int_a^b f(x)dx = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^b f(x)dx \qquad (4)$$

Oxiri limit mavjud bo'lsa, xosmas integral yaqinlashuvchi aks holda uzoqlashuvchi deyiladi. Bunday integrallarga 2-tur xosmas integral deyiladi.

Integral ostidagi $f(x)$ funksiya uchun $F(x)$ boshlang'ich funksiya ma'lum bo'lsa, Nyuton - Leybnis formulasini qo'llash mumkin:

$$\int_a^b f(x)dx = \lim_{\varepsilon \to 0} F(x)\Big|_{a+\varepsilon}^{b} = \lim_{\varepsilon \to 0}\left[F(b) - F(a+\varepsilon)\right] = F(b) - F(a).$$

Shunday qilib, $x \to a$ da $F(x)$ boshlang'ich funksiyaning limiti mavjud bo'lsa, xosmas integral yaqinlashuvchi, mavjud bo'lmasa, xosmas integral uzoqlashuvchi bo'ladi.

$[a,b)$ intervalda $x=b$ nuqtada uzilishga ega bo'lgan $f(x)$ funksiya xosmas integrali ham shunga o'xshash bo'ladi, ya'ni

$$\int_a^b f(x)dx = \lim_{\varepsilon\to 0}\int_a^{b-\varepsilon} f(x)dx = \lim_{\varepsilon\to 0} F(x)\Big|_a^{b-\varepsilon} = \lim_{\varepsilon\to 0}[F(b-\varepsilon)-F(a)] = F(b)-F(a).$$

bunda $F(b)$, $F(x)$ boshlang'ich funksiyaning $x \to b$ dagi limiti.

$f(x)$ funksiya $[a,b]$ kesmaning biror $x=c$ nuqtasida uzilishga ega bo'lsa xosmas integral quyidagicha aniqlanadi:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \qquad (5)$$

O'ng tomondagi integrallardan aqalli bittasi uzoqlashuvchi bo'lsa, xosmas integral uzoqlashuvchidir. O'ng tomondagi ikkala integral ham yaqinlashuvchi bo'lsa, chap tomondagi xosmas integral yaqinlashuvchi bo'ladi.

DEFINITION OF IMPROPER INTEGRALS WITH INFINITE DISCONTINUITIES

1. If f is continuous on the interval $[a, b)$ and has an infinite discontinuity at b, then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.$$

2. If f is continuous on the interval $(a, b]$ and has an infinite discontinuity at a, then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx.$$

3. If f is continuous on the interval $[a, b]$, except for some c in (a, b) at which f has an infinite discontinuity, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In the first two cases, the improper integral **converges** if the limit exists—otherwise, the improper integral **diverges.** In the third case, the improper integral on the left diverges if either of the improper integrals on the right diverges.

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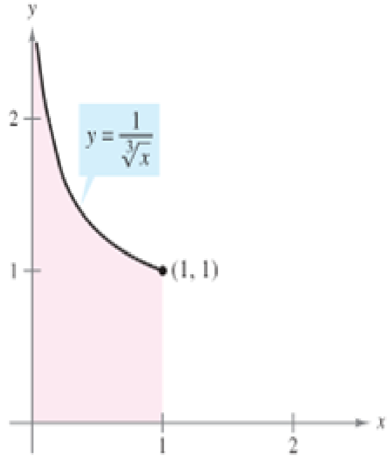
3-misol. $\int_0^1 \frac{dx}{\sqrt[3]{x}}$ integralning yaqinlashuvchiligini tekshiring.

Yechish: $x \to 0$ da $f(x) = \frac{1}{\sqrt[3]{x}} \to \infty$ demak,

$$\lim_{b\to 0} \int_0^1 \frac{1}{\sqrt[3]{x}} dx = \lim_{b\to 0} \frac{3}{2}\left(1 - b^{2/3}\right) = \frac{3}{2}.$$



Infinite discontinuity at $x = 0$

Figure 8.24

EXAMPLE 6 An Improper Integral with an Infinite Discontinuity

Evaluate $\int_0^1 \frac{dx}{\sqrt[3]{x}}$.

Solution The integrand has an infinite discontinuity at $x = 0$, as shown in Figure 8.24. You can evaluate this integral as shown below.

$$\int_0^1 x^{-1/3}\,dx = \lim_{b\to 0^+} \left[\frac{x^{2/3}}{2/3}\right]_b^1$$

$$= \lim_{b\to 0^+} \frac{3}{2}(1 - b^{2/3})$$

$$= \frac{3}{2}$$

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$\int_0^1 \frac{dx}{\sqrt[3]{x}}$ integral yaqinlashuvchi.

4-misol. $\int_{-1}^{8} \frac{dx}{\sqrt[3]{x^2}}$ integralning yaqinlashuvchiligini tekshiring.

Yechish: $x \to 0$ da $f(x) = \frac{1}{\sqrt{x^2}} \to +\infty$, $x = 0$ nuqta $[-1, 8]$ kesmaning ichki nuqtasi. (5) formuladan foydalansak,

$$\int_{-1}^{8} \frac{dx}{\sqrt[3]{x^2}} = \int_{-1}^{8} x^{-\frac{2}{3}} 3dx = 3\sqrt[3]{x}\Big|_{-1}^{0} + 3\sqrt[3]{x}\Big|_{0}^{8} = 0 + 3 + 6 = 9$$

bo’ladi.

Demak, berilgan xosmas integral yaqinlashuvchi.

5-misol. $\int_0^2 \frac{dx}{x^3}$ integralning yaqinlashuvchiligini tekshiring.

Evaluate $\int_0^2 \frac{dx}{x^3}$.

Solution Because the integrand has an infinite discontinuity at $x = 0$, you can write

$$\int_0^2 \frac{dx}{x^3} = \lim_{b \to 0^+} \left[-\frac{1}{2x^2} \right]_b^2$$

$$= \lim_{b \to 0^+} \left(-\frac{1}{8} + \frac{1}{2b^2} \right)$$

$$= \infty.$$

So, you can conclude that the improper integral diverges.

Demak integral uzoqlashuvchi.

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