19-MAVZU: FUNKSIONAL QATORLARNI DIFFERENSIALLASH VA INTEGRALLASH.TEYLOR VA MAKLOREN QATORLARI.

REJA.

1. Funksional qatorlarni differensiallash va integrallash.
2. Teylor va Makloren qatorlari.

1. Funksional qatorlarni integrallash va differensiallash

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + u_3(x) + \ldots + u_n(x) + \ldots.. \quad (1)$$

ko'rinishdagi ifoda funksional qator deb ataladi. Uning har bir hadi x ga bog'liq funksiyadir. x ga har xil sonli qiymatlarni berib turli tuman sonli qatorlarni hosil qilish mumkin. Ularning ayrimlari yaqinlashuvchi, ayrimlari esa uzoqlashuvchi bo'ladi.

Ta'rif. Funksional qatorni yaqinlashuvchi qatorga aylantiradigan x larning sonli qiymatlar to'plami uning yaqinlashish sohasi deyiladi.

Tabiiyki, yaqinlashish sohasida funksional qatorning yig'indisi x ga bog'liq bo'lgan birorta funksiyadan iborat bo'ladi. Shuning uchun funksional qatorning yig'indisi $S(x)$ orqali belgilanadi.

Masalan,

$$\sum_{n=1}^{\infty} \ln^n x$$

qatorning yaqinlashish sohasi topilsin.

Yechish. Berilgan qator maxraji $q=\ln x$ ga teng bo'lgan cheksiz geometrik progressiyani ifodalaydi. Geometrik progressiya $|q|<1$ shart bajarilgandagina yaqinlashgani uchun, berilgan qator $|\ln x|<1$ ya'ni $-1<\ln x<1$ tengsizlik bajarilganda absolyut yaqinlashadi, demak, $e^{-1}<x<e$ tengsizlik berilgan qatorning yaqinlashish sohasini ifodalaydi. Shunday qilib (e^{-1},e) oraliqda berilgan qatorning yig'indisini

$$S(x)=\frac{\ln x}{1-\ln x}$$

formula yordamida hisoblaymiz.

(1) qatorning birinchi n ta hadining yig'indisini $S_n(x)$ deb belgilaylik. Agar bu qator yaqinlashsa va uning yig'indisi $S(x)$ ga teng bo'lsa, u holda

$$S(x)=S_n(x)+r_n(x)$$

tenglikni yozishimiz mumkin, bu yerda

$$r_n(x)=u_{n+1}(x)+u_{n+2}(x)+u_{n+3}(x)+\ldots$$

kattalik (1) qatorning qoldig'i deyiladi. Qatorning yaqinlashish sohasida $\lim\limits_{n\to\infty} s_n(x)=S(x)$ munosabat o'rinli, shuning uchun

$$\lim_{n\to\infty} r_n(x)=\lim_{n\to\infty}\left[S(x)-S_n(x)\right]=0,$$

ya'ni yaqinlashuvchi qatorning qoldig'i $r_n(x)$, $n\to\infty$ da nolga intiladi.

1-teorema. $[a,b]$ kesmada kuchaytirilgan bo'lgan uzluksiz funksiyalarning quyidagi

$$u_1(x)+u_2(x)+u_3(x)+...+u_n(x)+.....$$

qatori berilgan va $s(x)$ bu qatorning yig'indisi bo'lsin. Bu holda $[a,b]$ kesmaga tegishli bo'lgan α dan x gacha chegarada $s(x)$ dan olingan integral berilgan qator hadlaridan shunday chegarada olingan integrallar yig'indisiga teng, ya'ni

$$\int_\alpha^x s(t)dt=\int_\alpha^x u_1(t)dt+\int_\alpha^x u_2(t)dt+\int_\alpha^x u_3(t)dt+...+\int_\alpha^x u_n(t)dt+.....$$

2-teorema. Agar (a, b) kesmada hosilalari uzluksiz bo'lgan funksiyalardan tuzilgan

$$u_1(x)+u_2(x)+u_3(x)+...+u_n(x)+.....$$

qator shu kesmada $s(x)$ yig'indiga yaqinlashsa, va uning hadlarining hosilalaridan tuzilgan

$$u_1'(x)+u_2'(x)+u_3'(x)+...+u_n'(x)+.....$$

qator o'sha kesmada kuchaytirilgan bo'lsa, hosilalar qatorining yig'nidisi boshlang'ich qator yig'indisining hosilasiga teng, ya'ni

$$s'(x)=u_1'(x)+u_2'(x)+u_3'(x)+...+u_n'(x)+.....$$

bo'ladi.

2.Teylor va Makloren qatorlari.

BROOK TAYLOR (1685–1731)

Although Taylor was not the first to seek polynomial approximations of transcendental functions, his account published in 1715 was one of the first comprehensive works on the subject.

Larson Edvards. /Calculus/ 2010. P.652.

Agar $y=f(x)$ funksiya $x=a$ nuqtaning atrofida $(n+1)$-nchi tartibgacha hosilaga ega bo'lsa Teylor formulasi deb ataluvchi

$$f(x)=f(a)+\frac{x-a}{1!}f^{|}(a)+\frac{(x-a)^2}{2!}f^{||}(a)+\ldots+ \\ +\frac{(x-a)^n}{n!}f^{(n)}(a)+R_n(x). \qquad (1)$$

formula bizga ma'lum, bu yerda

$$R_n(x)=\frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}[a+\theta(x-a)]$$

qoldiq had edi, $0<\theta<1$.

Agar $f(x)$ funksiya $x=a$ nuqtaning atrofida istalgan tartibgacha hosilaga ega bo'lsa, Teylor formulasidagi n istalgancha katta qilib olinishi mumkin. Faraz qilaylik $\lim\limits_{n\to\infty}R_n(x)=0$ bajarilsin, u holda (1) formulada $n\to\infty$ da limitga o'tib, o'ng tomonda qator hosil qilinadi va u Teylor qatori deb ataladi:

$$f(x)=f(a)+\frac{x-a}{1!}f^{|}(a)+\frac{(x-a)^2}{2!}f^{||}(a)+\ldots+\frac{(x-a)^n}{n!}f^{(n)}(a)+\ldots \qquad (2)$$

(2) tenglik $\lim\limits_{n\to\infty}R_n(x)\to 0$ bajarilgandagina o'rinlidir.

Agar Teylor qatorida a=0 desak uning xususiy ko'rinishi bo'lgan Makloren qatori hosil bo'ladi:

$$f(x) = f(0) + \frac{x}{1!}f^{|}(0) + \frac{x^2}{2!}f^{||}(0)+...+\frac{x^n}{n!}f^{(n)}(0)+... \quad (3)$$

Berilgan $f(x)$ funksiyani Teylor qatoriga yoyish uchun:

a) $f(x)$ funksiyaning barcha tartibdagi hosilalarining $x=a$ nuqtadagi qiymatlari hisoblanadi va Teylor qatorining yoyilmasiga olib borib qo'yiladi;

b) hosil bo'lgan qatorning yaqinlashish sohasi topiladi.

Misol. $f(x)=2^x$ funksiya x ning darajalari bo'yicha Teylor qatoriga yoyilsin.

Yechish. a) 2^x funksiyaning barcha tartibdagi hosilalarini x=0 nuqtadagi qiymatlarini topamiz:

$f(x)=2^x$, $\quad f(0)=1$;

$f^{|}(x)=2^x\ln 2$, $\quad f^{|}(0)=\ln 2$;

$f^{||}(x)=2^x\ln^2 2$, $\quad f^{||}(0)=\ln^2 2$;

....

$f^{(n)}(x)=2^x\ln^n 2$, $\quad f^{(n)}(0)=\ln^n 2$;

....

Endi topilgan qiymatlarni (3) ifodaga qo'yib, 2^x funksiya uchun x ning darajalari bo'yicha Teylor qatorini hosil qilamiz:

$$2^x = 1 + \frac{\ln 2}{1!}x + \frac{\ln^2 2}{2!}x^2 + ... + \frac{\ln^n 2}{n!}x^n + ...$$

b) hosil bo'lgan qatorning yaqinlashish sohasini topamiz:

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{\ln^n 2(n+1)!}{n! \ln^{n+1} 2} = \infty$$ dan ko'rinadiki topilgan qator x ning har qanday qiymatlarida yaqinlashadi.

Taylor and Maclaurin Polynomials

The polynomial approximation of $f(x) = e^x$ given in Example 2 is expanded about $c = 0$. For expansions about an arbitrary value of c, it is convenient to write the polynomial in the form

$$P_n(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots + a_n(x-c)^n.$$

In this form, repeated differentiation produces

$$P_n'(x) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots + na_n(x-c)^{n-1}$$
$$P_n''(x) = 2a_2 + 2(3a_3)(x-c) + \cdots + n(n-1)a_n(x-c)^{n-2}$$
$$P_n'''(x) = 2(3a_3) + \cdots + n(n-1)(n-2)a_n(x-c)^{n-3}$$
$$\vdots$$
$$P_n^{(n)}(x) = n(n-1)(n-2)\cdots(2)(1)a_n.$$

Letting $x = c$, you then obtain

$$P_n(c) = a_0, \quad P_n'(c) = a_1, \quad P_n''(c) = 2a_2, \quad \ldots, \quad P_n^{(n)}(c) = n!a_n$$

and because the values of f and its first n derivatives must agree with the values of P_n and its first n derivatives at $x = c$, it follows that

$$f(c) = a_0, \quad f'(c) = a_1, \quad \frac{f''(c)}{2!} = a_2, \quad \ldots, \quad \frac{f^{(n)}(c)}{n!} = a_n.$$

With these coefficients, you can obtain the following definition of **Taylor polynomials,** named after the English mathematician Brook Taylor, and **Maclaurin polynomials,** named after the English mathematician Colin Maclaurin (1698–1746).

DEFINITIONS OF nTH TAYLOR POLYNOMIAL AND nTH MACLAURIN POLYNOMIAL

If f has n derivatives at c, then the polynomial

$$P_n(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdot \cdot \cdot + \frac{f^{(n)}(c)}{n!}(x - c)^n$$

is called the **nth Taylor polynomial for f at c.** If $c = 0$, then

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdot \cdot \cdot + \frac{f^{(n)}(0)}{n!}x^n$$

is also called the **nth Maclaurin polynomial for f.**

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2. Asosiy funksiyalar yoyilmasining jadvali:

1. $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \qquad (|x| < \infty);$$

Find the nth Maclaurin polynomial for $f(x) = e^x$.

Solution From the discussion on page 651, the nth Maclaurin polynomial for

$$f(x) = e^x$$

is given by

$$P_n(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdot \cdot \cdot + \frac{1}{n!}x^n.$$

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$$2.\ \sin x=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots\quad(|x|<\infty);$$

$$3.\ \cos x=\sum_{n=0}^{\infty}\frac{(-1)^{n}x^{2n}}{(2n)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots\quad(|x|<\infty);$$

$$4.\ \frac{1}{1-x}=\sum_{n=0}^{\infty}x^n=1+x+x^2+x^3+x^4+\ldots\qquad(|x|<1);$$

$$5.\ (1+x)^m=1+\sum_{n=1}^{\infty}\frac{m(m-1)(m-2)\ldots(m-n+1)}{n!}x^n=$$

$$=1+mx+\frac{m(m-1)}{2!}x^2+\frac{m(m-1)(m-2)}{3!}x^3+\ldots\ (|x|<1)$$

(binomial m-istalgan haqiqiy son);

$$6.\ln(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\quad(-1<x\le 1);$$

$$7.\ arctgx=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n-1}}{2n-1}=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots\quad(|x|\le 1);$$

Function	*Interval of Convergence*
$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - \cdots + (-1)^n (x-1)^n + \cdots$	$0 < x < 2$
$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots + (-1)^n x^n + \cdots$	$-1 < x < 1$
$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots + \frac{(-1)^{n-1}(x-1)^n}{n} + \cdots$	$0 < x \le 2$
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!} + \cdots$	$-\infty < x < \infty$
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$	$-\infty < x < \infty$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$	$-\infty < x < \infty$
$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$	$-1 \le x \le 1$
$\arcsin x = x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots + \frac{(2n)!x^{2n+1}}{(2^n n!)^2(2n+1)} + \cdots$	$-1 \le x \le 1$

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