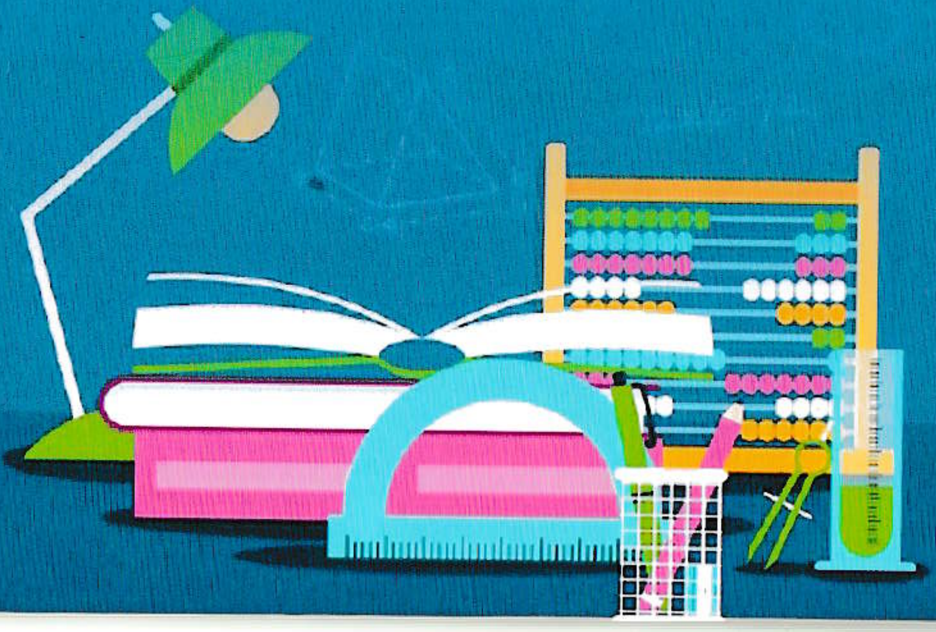
F.S.Aktamov, E.M.Mahkamov,
G.B.Quzmanova

MATEMATIK ANALIZ



O'ZBEKISTON RESPUBLIKASI OLIY TA'LIM, FAN VA
INNOVATSIYALAR VAZIRLIGI

CHIRCHIQ DAVLAT PEDAGOGIKA UNIVERSITETI

F.S.Aktamov, E.M.Mahkamov, G.B.Quzmanova

MATEMATIK ANALIZ

O'QUV QO'LLANMA
I QISM

Chirchiq-2023
Yangi chirchiq prints

UO'K 378;51;004

KBK 22.1;32.97

A-99

Aktamov F.S., Mahkamov E.M., Quzmanova G.B. / Matematik analiz / O'quv qo'llanma –Chirchiq.: «Yangi chirchiq prints», 2023 y. 368 b.

Ushbu o'quv qo'llanma pedagogika oily ta'lim muassasalarining matematika va informatika yo'nalishlari bakalavrlari uchun «Matematik analiz»fani dasturi asosida yozilgan bo'lib, to'plamlar haqida tushuncha, ketma-ketliklar, funksiya limiti, hosila va differensial, aniq va aniqmas integrallar, sonli hamda funksional qatorlarga oid materiallarni o'z ichiga oladi. Qo'llanmada zarur nazariy tushunchalar qisqachacha bayon etilgan va namunaviy misol va masalalar ishlab ko'rsatilgan. Har bir mavzu yuzasidan talabalar mustaqil ishlashi uchun topshiriqlar berilgan. Topshiriqlarni bajarish uchun kerakli ko'rsatmalar keltirilgan, ularning mohiyati misol va masalalar yechimlarida tushuntirilgan. Har bir mustaqil ish topshirig'iga oid misol va masala namuna sifatida yechib ko'rsatilgan.

Taqrizchilar:

A.J. Seytov t.f.d.(DSc),Chirchiq davlat pedagogika universiteti "Algebra va matematik analiz" kafedra dotsenti.

J.K. Adashev f-m.f.d.(DSc), V.I.Romonovskiy nomidagi matematika instituti katta ilmiy xodimi.

O'zbekiston Respublikasi Oliy ta'lim, fan va innovatsiyalar vazirligining 2023 yil 29 sentyabrdagi 438-sonli qaroriga asosan o'quv qo'llanma sifatida nashr qilishga tavsiya etilgan.

ISBN 978-9910-9950-1-9

SO'Z BOSHI

Matematik analiz fanini o'qitishda mutaxassislar oldida turgan muhim muammolardan biri ma'lumotni taqdim yetishning darajasini tanlash, uzviylikni ta'minlash asosida talabalarning bilim ko'nikmalarida paydo bo'lishi mumkin bo'lgan bo'shliqni to'ldirish, mavzularni o'qitishdagi takomillashgan uslublarni ishlab chiqish, talabalarning mantiqiy tafakkurini rivojlantirish va zarur bilim, ko'nikmalarini hamda amaliy ko'nikmalarini shakllantirish o'rtasida to'g'ri muvozanatni o'rnatishga alohida e'tibor qaratishdan iboratdir.

Bugungi kunda oliy ta'lim muassasalari talabalarini matematik analiz fanini o'zlashtirishlari uchun bir qator innovatsion metodlar yaratish, matematik analiz fanini o'qitishda samaraga erishish usullari ishlab chiqish, fanning bo'limlari va mavzularini o'qitish metodlari ustida bir nechtalab ishlar amalga oshirilmoqda. Ushbu o'quv qo'llanma pedagogika oliy ta'lim muassasalari matematika va informatika yo'nalishlari talabalari uchun mo'ljallangan bo'lib, quyidagi vazifalarning hal qilishga qaratilgan:

Bulardan tashqari ushbu o'quv qo'llanma o'quv dasturining kredik moodle dasturidagi asosiy vazifalardan, talabalarning mustaqil ta'limini tashkillashtirish vazifalari asosida shakllantirilgan.

Ushbu o'quv qo'llanma Davlat ta'lim standartlariga mos keladi va fanning o'quv dasturlariga to'la javob beradigan tarzda bayon qilingan.

Qo'llanmaning har bir bo'limi zarur nazariy tushunchalar, ta'riflar, teoremalar va formulalar bilan boshlangan, ularning mohiyati misol va masalalarning yechimlarida tushuntirilgan, shu bo'limga oid amaliy mashg'ulot darslarida va mustaqil uy ishlarida bajarishga mo'ljallangan ko'p sondagi mustahkamlash uchun mashqlar berilgan. Har bir bo'limning oxirida talabalarning mustaqil ishlari uchun topshiriqlar variantlari keltirilgan. Qo'llanmani yozishda pedagogika oliy o'quv yurtlarining bakalavrlari uchun matematik analiz fanining amaldagi dasturida tavsiya qilingan adabiyotlardan hamda o'zbek tilida chop etilgan zamonaviy darslik va o'quv qo'llanmalardan keng foydalanilgan.

Mualliflar o`quv qo`llanmani takomillashtirishda (yaxshilashda) bergan foydali maslahatlari uchun mas'ul muharrir va taqrizchilarga, hamda matnni tahrir qilgani uchun F.U. Jo'rayevga o`z minnatdorchiliklarini bildiradilar.

O'quv qo'llanmani tayyorlashda xato va kamchiliklar bo`lishi mumkin. Xato va kamchiliklar haqidagi fikrlaringizni feruzaktamov28@gmail.com elektron manziliga jo`natishlaringizni so`raymiz Qo'llanma haqida bildirilgan fikr va mulohazalar mamnuniyat bilan qabul qilinadi.

Mualliflar

MUNDARIJA

X BOB
FUNKSIONAL KETMA-KETLIK VA QATORLAR

I BOB
TO'PLAMLAR. TO'PLAMLAR USTIDA AMALLAR.

1.1. To'plamlar

1⁰.To'plam tushunchasi. To'plam matematikaning boshlang'ich, ayni paytda muhim tushunchalaridan biri. Uni ixtiyoriy tabiatli narsalarning (predmetlarni) ma'lum belgilar bo'yicha birlashmasi (majmuasi) sifatida tushuniladi. Masala, javondagi kitoblar to'plami, bir nuqtadan o'tuvchi to'g'ri chiziqlar to'plami, $x^2-5x+6=0$ tenglamaning ildizlari to'plami deyilishi mumkin.

To'plamni tashkil etgan narsalar uning elementlari deyiladi. Matematikada to'plamlar bosh xarflar bilan, ularning elementlari esa kichik xarflar bilan belgilanadi. Masalan, A,B,C - to'plamlar, a,b,c -to'plamning elementlari. Ba'zan to'plamlar ularning elementlarini ko'rsatish bilan yoziladi:

$$A=\{2,4,6,8\,10,12\},$$
$$N=\{1,2,3,...,n,...\}$$
$$Z=\{...,-2,-1,0,1,2,...\}.$$

Agar a biror A to'plamning elementi bo'lsa, $a\in A$ kabi yoziladi va «a element A to'plamga tegishli» deb o'qiladi. Agar a shu to'plamga tegishli bo'lmasa,u $a\notin A$ kabi yoziladi va «a element A to'plamga tegishli emas» deb o'qiladi. Masalan, yuqoridagi to'plamda $10\in A, 15\notin A$.

Agar A chekli sondagi elementlardan tashkil topgan bo'lsa, u chekli to'plam, aks holda cheksiz to'plam deyiladi. Masalan, $A=\{2,4,6,8,10,12\}$ chekli to'plambir nuqtadan o'tuvchi barcha to'g'ri chiziqlar to'plami esa cheksiz to'plam bo'ladi.

1-ta'rif. A va B to'plamlari berilgan bo'lib, A to'plamning barcha elementlari B to'plamga tegishli bo`lsa, $A$ to`plam B ning qismi (qismiy to'plam) deyiladi va

$$A\subset B \text{ (yoki } B\supset A)$$

kabi yoziladi.

A to'planing elementlari orasida biror xususiyatga (bu xususiyatni P bilan belgilaymiz) ega bo'ladiganlari bo'lishi mumkin. Bunday xususiyatli elementlardan tuzilgan to'plam quyidagicha

$$\{x\in A\mid P\}$$

belgilanadi. Ravshanki,

$$\{x\in A\mid P\}\subset A$$

bo'ladi.

Agar A to'plam elementlari orasida P xususiyatli elementlar bo'lmasa, u holda

$$\{x\in A\mid P\}$$

bitta ham elementga ega bo'lmagan to'plam bo'lib, uni **bo'sh to'plam** deyiladi.

Bo'sh to'plam $\varnothing$ kabi belgilanadi. Masalan, $x^2+x+1=0$ tenglamaning haqiqiy ildizlaridan iborat A bo'sh to'plam bo'ladi:

$$\varnothing=\{x\in A\mid x^2+x+2=0\}.$$

Har qanday A to'plam uchun

$$A\subset A,\ \varnothing\subset A$$

deb qaraladi.

Odatda, A to'plamning barcha qismiy to'plamlaridan iborat to'plam $F(A)$ kabi belgilanadi. Masalan, $A=\{a,b,c\}$ to'plam uchun

$$F(A)=\{\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},\{a,b,c\},\varnothing\}$$

bo'ladi.

2-ta`rif. A va B to'plamlar berilgan bo'lib,

$$A\subset B,\ B\subset A$$

bo'lsa, A va B bir biriga teng to'plamlar deyiladi va

$$A=B$$

kabi yoziladi.

Demak, $A=B$ tenglik A va B to'plamlarning bir xil elementlardan tashkil topganligini bildiradi.

2^0. To'plamlar ustida amallar. Ikki A va B to'plamlar berilgan bo'lsin.

**3-ta`rif.** $A$ va $B$ to'plamlarning barcha elementlaridan tashkil topgan $E$ to'plam $A$ va $B$ to`plamlar yig`indisi (birlashmasi) deyiladi va $A\bigcup B$ kabi belgilanadi:

$$E=A\bigcup B.$$

Demak,bu holda $a\in A\bigcup B$ dan $a\in A$, yoki $a\in B$, yoki bir vaqtda $a\in A$, $a\in B$ bo'lishi kelib chiqadi.

4-ta`rif. A va B to'plamlarning barcha elementlaridan tashkil topgan F to'plam A va B to'plamlar ko'paytmasi (kesishmasi) deyiladi va $A\bigcap B$ kabi belgilanadi:

$$F=A\bigcap B.$$

Demak,bu holda $a\in A\bigcap B$ dan bir vaqtda $a\in A$, $a\in B$ bo'lishi kelib chiqadi.

5-ta`rif. A to'plamning B to'plamga tegishli bo'lmagan barcha elementlaridan tashkil topgan G to'plam A to'plamdan B to'plamning ayirmasi deyiladi va $A\setminus B$ kabi belgilanadi:

$$G=A\setminus B.$$

Demak, $a\in A\setminus B$ dan $a\in A$, $a\notin B$ bo'lishi kelib chiqadi.

6-ta`rif. A to'plamning B ga tegishli bo'lmagan barcha elementlaridan va B to'plamning ga tegishli bo'lmagan barcha elementlaridan tuzilgan to'plam A va B **to'plamlarning simmetrik ayirmasi** deyiladi va $A\Delta B$ kabi belgilanadi:

$$A\Delta B=(A\setminus B)\cup(B\setminus A).$$

Demak, $a\in A\Delta B$ bo'lishidan $a\in A, a\notin B$ yoki $a\in B$, $a\notin A$ bo'lishi kelib chiqadi.

7-ta`rif. Aytaylik, $a\in A$, $a\in B$ bo'lsin. Barcha tartiblangan (a,b) ko'rinishdagi juftliklardan tuzilgan to'plam A va B **to'plamlarning dekart ko'paytmasi** deyiladi va $A\times B$ kabi belgilanadi. Demak,

$$A\times B=\{(a,b)\mid a\in A, b\in B\}.$$

8-ta`rif. Aytaylik, S va A to'plamlar berilgan bo'lib, $A\subset S$ bo'lsin. Ushbu

$$S\setminus A$$

to'plam A to'plamni S ga **to'ldiruvchi to'plam** deyiladi va CA yoki C_SA kabi belgilanadi:

$$CA=S\setminus A.$$

To'plamlar ustida bajariladigan amallarning ba`zi xossalarini keltiramiz.

A, B va D to'plamlari berilgan bo'lsin.

1) $A\subset B,\ B\subset D$ bo'lsa, $A\subset D$ bo'ladi;

2) $A\cup A=A,\ A\cap A=A$ bo'ladi;

3) $A\subset B$ bo'lsa, $A\cup B=B,\ A\cap B=A$ bo'ladi;

4) $A\cup B=B\cup A,\ A\cap B=B\cap A$ bo'ladi;

5) $(A\cup B)\cup D=A\cup(B\cup D),\ (A\cap B)\cap D=A\cap(B\cap D)$ bo'ladi;

6) $A\subset S$ bo'lsa, $A\cap CA=\varnothing$;

7) $C(A\cup B)=CA\cap CB$ bunda $A\subset S, B\subset S$ $A\cap CA=\varnothing$;

8) $C(A\cap B)=CA\cup CB$, bunda $A\subset S, B\subset S$.

Bu xossalarning isboti yuqorida keltirilga ta`riflardan kelib chiqadi.

1-misol. Ushbu

$$(A\setminus B)\bigcup(B\setminus A)=(A\bigcup B)\setminus(A\bigcap B) \tag{1}$$

tenglik isbotlansin.

$a\in(A\setminus B)\bigcup(B\setminus A)$ bo'lsin. U holda

$$a\in(A\setminus B):\ a\in A,\ a\notin B$$

yoki

$$a\in(B\setminus A):\ a\in B,\ a\notin A$$

bo'ladi. Bundan esa

$$a\in(A\bigcup B),\ a\notin(A\bigcap B)$$

bo'lib,

$$a\in(A\bigcup B)\setminus(A\bigcap B)$$

bo'lishi kelib chiqadi. Demak,

$$(A\setminus B)\bigcup(B\setminus A)\subset(A\bigcup B)\setminus(A\bigcap B). \tag{2}$$

Aytaylik, $a\in(A\bigcup B)\setminus(A\bigcap B)$ bo'lsin.

U holda,

$$a\notin(A\cap B):\ a\notin A, a\notin B \text{ yoki } a\in A, a\notin B \text{ yoki } a\notin A, a\in B$$

bo'ladi.

Bundan esa

$$a \in A \setminus B \text{ yoki } a \in B \setminus A$$

bo'lib,

$$a \in (A \setminus B) \cup (B \setminus A)$$

bo'lishi kelib chiqadi. Demak,

$$(A \cup B) \setminus (A \cap B) \subset (A \setminus B) \cup (B \setminus A) \qquad (3)$$

(2) va (3) munosabatlardan (1) tenglikning o'rinli bo'lishi topiladi.

To'plamlar ustida bajariladigan amallarni bayon etishda to'plamlarning qanday tabiatli elementlardan tuzilganligiga e`tabor qilinmaydi.

Aslida, keltirilgan amallar biror universal to'plam deb ataluvchi to'plamningr qismiy to'plamlari ustida bajariladi deb qaraladi. Masalan,natural sonlar to'plamlari ustida amallar bajariladigan bo'lsa, universal to'plam sifatida barcha natural sonlardan iborat N to'plamni olish mumkin.

3^0. Matematik belgilar. Matematikada tez-tez uchraydigan so'z va so'z birikmalari o'rnida maxsus belgilar ishlatiladi. Ulardan muhimlarini keltiramiz:

1) «agar ... bo'lsa, u holda ... bo'ladi» iborasi «$\Rightarrow$ » belgi orqali yoziladi;

2) ikki iboraning ekvivalenti ushbu «$\Leftrightarrow$» belgi orqali yoziladi;

3) «har qanday», «ixtiyoriy», «barchasi uchun» so'zlari o'rniga «$\forall$ » belgi ishlatiladi;

4) «mavjudki», «topiladiki» so'zlari o'rniga «$\exists$» mavjudlik belgisi ishlatiladi.

1.2. Haqiqiy sonlar

Son tushunchasi uzoq o'tmishdan ma'lum. Odamlar sanash taqozosi bilan dastlab $1,2,3,...$ - natural sonlarni qo'lla-ganlar. So'ngra manfiy son, ratsional son va nihoyat, haqiqiy son tushunchasi kiritilgan va o'rganilgan.

Biz o'quvchiga o'rta maktab, kollej va litseylarda matematika kursidan natural, butun, ratsional sonlarni, ular ustida bajariladigan amallarni, amallarning xossalarini, shuningdek to'g'ri chiziqda (sonlar o'qida) geometrik ifodalanishini ma'lum deb hisoblaymiz.

Haqiqiy sonlarning matematik analiz kursida muhimligini e'tiborga olib, ular haqidagi ma'lumotlarni talab darajasida bayon etamiz.

1^0. Ratsional sonlar va cheksiz davriy o'nli kasrlar.

Faraz qilaylik, $\frac{p}{q}$ biror musbat ratsional son bo'lsin. Bo'lish qoidasidan foydalanib p butun sonni q ga bo'lamiz. Agar p ni q ga bo'lish jarayonida biror qadamdan keyin qoldiq nolga teng bo'lsa, u holda bo'lish jarayon to'xtab, $\frac{p}{q}$ kasr

o'nli kasrga aylanadi. Odatda, bunday o'nli kasr chekli o'nli kasr deyiladi. Masalan, $\frac{59}{40}$ kasrda 59 ni 40 ga bo'lib,unu $1,475$ bo'lishini topamiz:

$$\frac{59}{40} = 1{,}475.$$

Agar p ni q ga bo'lish jarayoni cheksiz davom etsa, ma'lum qadamdan keyin yuqorida aytilgan qoldiqlardan biri yana bir marta uchraydi, so'ng undan oldingi raqamlar mos tartibda takrorlanadi.

Odatda bunday kasr cheksiz davriy o'nli kasr deyiladi. Takrorlanadigan raqamlar (raqamlar birlashmasi) o'nli kasrning davri bo'ladi.

Masalan, $\frac{1}{3}$ kasrda 1 ni 3 ga bo'lib, $0{,}333...$ bo'lishini topamiz;

$$\frac{1}{3} = 0{,}333...$$

Ushbu

$$0{,}333..., \quad 1{,}4777..., \quad 2{,}131313...$$

kasrlar cheksiz davriy o'nli kasrlardir. Ularning davri mos ravishda 3, 7, 13 bo'ladi va bu cheksiz davriy o'nli kasrlar quyidagicha

$$0,(3), \quad 1,4(7), \quad 2,(13)$$

yoziladi;

$$0,(3) = 0{,}333...$$

$$1{,}4(7) = 1{,}4777...$$

$$2,(13) = 2{,}131313...\, .$$

Shuni ta'kidlaymizki, davri 9 ga teng bo'lgan cheksiz davriy o'nli kasrni chekli o'nli kasr qilib yoziladi.

Masalan,

$$0{,}4999... = 0{,}4(9) = 0{,}5\,,$$

$$2{,}71999... = 2{,}71(9) = 2{,}72\,.$$

Har qanday chekli o'nli kasrni nollar bilan davom ettirib cheksiz davriy o'nli kasr ko'rinishida yozish mumkin.

Masalan,

$$1{,}4 = 1{,}4000... = 1{,}4(0)$$

$$0{,}75 = 0{,}75000... = 0{,}75(0)\,.$$

Demak,har qanday $\frac{p}{q}$ ratsional son cheksiz davriy o'nli kasr ko'rinishida ifodalanadi. Aksincha, har qanday cheksiz davriy o'nli kasrni $\frac{p}{q}$ ko'rinishida yozish mumkin.

Masalan, ushbu

$$0,(3)=0,333... \ , \ 7,31(06)=7,31060606 ...$$

cheksiz davriy o'nli kasrlarni qaraylik. Avvalo ularni

$$0,(3)=0+\frac{3}{10}+\frac{3}{10^2}+\frac{3}{10^3}+... ,$$

$$7,31(06)=7+\frac{3}{10}+\frac{1}{10^2}+\frac{6}{10^4}+\frac{6}{10^6}+...$$

ko'rinishda yozib, so'ng cheksiz kamayuvchi geometrik progres-siya yig'indisi formulasidan foydalanib topamiz:

$$0,(3)=0,333...=\frac{\frac{3}{10}}{1-\frac{1}{10}}=\frac{3}{10}\cdot\frac{10}{9}=\frac{1}{3},$$

$$7,31(06)=7,31060606...=\frac{731}{100}+\frac{\frac{1}{10^4}}{1-\frac{1}{10^2}}=\frac{731}{100}+\frac{1}{100}\cdot\frac{6}{99}=$$

$$=\frac{1}{100}\left(731+\frac{2}{33}\right)=\frac{965}{132}.$$

Demak, ixtiyoriy ratsional son cheksiz davriy o'nli kasr orqali va aksincha, ixtiyoriy cheksiz davriy o'nli kasr ratsional son orqali ifodalanadi.

2^0. **Haqiqiy son tushunchasi.** Cheksiz davriy bo'lmagan o'nli kasrlar ham bo'ladi. Bu kesmalarni o'lchash jarayonida yuzaga kelishini ko'rsatamiz.

Faraz qilaylik, biror J kesma hamda o'lchov birligi, masalan metr berilgan bo'lsin. J kesmaning uzunligini hisoblash talab etilsin.

Aytaylik, 1 metr J kesmada 5 marta butun joylashib, kesmaning J_1 qismi ortib qolsin. Ravshanki J_1 ning uzunligi 1 metrdan kam bo'ladi. Bu holda J kesmaning uzunligini taxminan 5 m. ga teng deb olish mumkin:

$$J \text{ uzunligi} \approx 5 \text{ m}.$$

Agar bu aniqlik yetarli bo'lmasa, o'lchov birligining $\frac{1}{10}$ qismini, ya'ni 1 dm. ni olib, uni J_1 kesmaga joylashtiramiz. Aytaylik, 1 dm. J_1 kesmada 7 marta butunlay joylashib, J_1 kesmaning J_2 qismi ortib qolsin. Bunda J_2 ning uzunligi 1 dm. dan kichik bo'ladi. Bu holda J kesmaning uzunligi taxminan 5,7 m ga teng deb olinishi mumkin:

$$J \text{ uzunligi} \approx 5,7 \text{m}.$$

Bu jarayonni davom ettira borish natijasida ikki holga duch kelamiz:

1) biror qadamdan keyin, masalan $n+1$ qadamdan keyin o'lchov birligining $\frac{1}{10^n}$ qismi J_n kesmaga α_n marta butunlay joylashadi. Bu holda o'lchov jarayoni to'xtatilib,

$$J \text{ uzunligi} = 5,\underbrace{7...\alpha_n}_{n \text{ ta raqam}}$$

bo'lishi topiladi.

2) o'lcham jarayoni to'xtovsiz davom (cheksiz davom) etadi. Bu holda J kesmaning uzunligining aniq qiymati deb ushbu

$$5,7...\alpha_n...$$

cheksiz o'nli kasr olinadi:

$$J \text{ uzunligi } = 5,7...\alpha_n...$$

Aytaylik, to'g'ri chiziqda biror O nuqta (koordinata boshi) hamda o'lchov birligi tayinlangan bo'lsin. U holda O nuqtadan o'ngda joylashgan har bir P nuqtaga, OP kesmani o'lchash natijasida hosil bo'lgan ushbu $\alpha_0,\alpha_1\alpha_2...\alpha_n...$ ushbu cheksiz o'nli kasrni mos qo'yish mumkin. Bunda

$$\alpha_0 \in N\cup\{0\},\ \alpha_n \in \{0,1,2,3,4,5,6,7,8,9\},\ n\geq 1.$$

Bu moslik o'zaro bir qiymatli moslik bo'ladi. Ravshanki, yuqoridagi cheksiz o'nli kasrlar orasida cheksiz davriy o'nli kasrlar bo'lib, ular manfiy bo'lmagan ratsional sonlar bo'ladi. Qolgan kasrlar esa ratsional sonlar bo'lmaydi.

1-ta'rif. Ushbu $a=\alpha_0,\alpha_1\alpha_2...\alpha_n....$, ko'rinishidagi cheksiz o'nli kasr **manfiy bo'lmagan haqiqiy son** deyiladi, bunda

$$\alpha_0 \in N\cup\{0\},\ \alpha_n \in \{0,1,2,3,4,5,6,7,8,9\},\ n\geq 1 .$$

Agar $\exists\, n\geq 0;\alpha_n>0$ bo'lsa, u **musbat haqiqiy son** deyiladi.

Manfiy haqiqiy sonning «–» ishora bilan olingani musbat haqiqiy son sifatida ta'riflanadi.

Barcha haqiqiy sonlardan iborat to'plam R harfi bilan belgilanadi.

Barcha natural sonlar to'plami N, ratsional sonlar to'plami Q, haqiqiy sonlar to'lami R uchun $N \subset Q \subset R$ bo'ladi.

2-ta`rif. Ushbu $R \setminus Q$ to'plam elementi (son) **irratsional son** deyiladi.

Biz yuqorida, davri «9» ga teng bo'lgan cheksiz davriy o'nli kasrni chekli o'nli kasr qilib olinishini aytgan edik. Buning oqibatida bitta son ikki ko'rinishga, masalan, $\frac{1}{2}$ soni

$$\frac{1}{2} = 0{,}5000...$$

$$\frac{1}{2} = 0{,}4999...$$

ko'rinishlarga ega bo'lib qoladi.

Umuman, $\alpha_0, \alpha_1, \alpha_2, ... \alpha_n$ $(\alpha_n \neq 0)$ ratsional son ushbu,

1) $\alpha_0, \alpha_1, \alpha_2, ... \alpha_{n-1}$ $(\alpha_n - 1)999...$,

2) $\alpha_0, \alpha_1, \alpha_2, ... \alpha_n 000...$, ikki ko'rinishda yozilishi mumkin. Haqiqiy sonlarni solishtirishda ratsional sonning 1)- ko'rinishidan foydalanamiz.

Ikkita manfiy bo'lmagan

$$a = \alpha_0, \alpha_1\alpha_2 ... \alpha_n ...,$$

$$b = \beta_0, \beta_1\beta_2 ... \beta_n ....$$

haqiqiy sonlar berilgan bo'lsin.

3-ta'rif. Agar $\forall n \geq 0$ da $\alpha_n = \beta_n$,ya'ni

$$\alpha_0 = \beta_0,\ \alpha_1 = \beta_1,\ \alpha_2 = \beta_2, ..., \alpha_n = \beta_n, ...$$

bo'lsa, a **va** b **sonlar teng** deyiladi va $a = b$ kabi yoziladi.

4-ta`rif. Agar

$$\alpha_0 = \beta_0,\ \alpha_1 = \beta_1,\ \alpha_2 = \beta_2, ..., \alpha_n = \beta_n, ...$$

tengliklarning hech bo'lmaganda bittasi bajarilmasa va birinchi bajarilmagan tenglik $n = k$ da sodir bo'lsa, u holda:

$\alpha_k > \beta_k$ bo'lganda a **soni** b **sonidan katta** deyiladi va $a > b$ kabi belgilanadi.

$\alpha_k < \beta_k$ bo'lganda a **soni** b **sonidan kichik** deyiladi va $a < b$ kabi belgilanadi.

Aytaylik, to'g'ri chiziq, unda tayin olingan O nuqta (koordinata boshi) va o'lchov birligi berilgan bo'lsin.

Haqiqiy sonlar to'plami R bilan to'g'ri chiziq nuqtalari orasidagi bir qiymatli moslik o'rnatish mumkin:

O nuqtadan o'ngda joylashgan P nuqtaga OP kesmaning uzunligiga teng x soni mos qo'yiladi (x son P **nuqtaning koordinatasi** deyiladi);

O nuqtadan chapda joylashgan Q nuqtaga QO kesmaning uzunligiga teng x sonining minus ishorasi bilan olingan $-x$ soni mos qo'yiladi;

O nuqtaga nol soni mos qo'yiladi.

Arximed aksiomasi. Ixtiyoriy chekli haqiqiy a soni uchun shunday natural m soni topiladiki,

$$m > a$$

bo'ladi.

Aytaylik,

$$a = \alpha_0, \alpha_1\alpha_2 ... \alpha_n ... > 0,$$

bo'lsin. $m = \alpha_0 + 1$, $m \in N$ deb olinsa, unda **3-ta'rifga** binoan $a < m$ bo'ladi.

Kurs davomida tez-tez uchrab turadigan haqiqiy sonlar to'plamlarini keltiramiz.

Aytaylik, $a \in R, b \in R, a < b$ bo'lsin:

$[a,b] = \{x \in R \mid a \leq x \leq b\}$ – segment deyiladi,

$(a,b) = \{x \in R \mid a < x < b\}$ – interval deyiladi,

$[a,b) = \{x \in R \mid a \leq x < b\}$ – yarim interval deyiladi,

$(a,b] = \{x \in R \mid a < x \leq b\}$ – yarim interval deyiladi.

Bunda a va b sonlar $[a,b]$, (a,b), $[a,b)$, $(a,b]$ larning chegaralari deyiladi.

Shuningdek,

$$[a,+\infty) = \{x \in R \mid x \geq a\},$$

$$(-\infty, a) = \{x \in R \mid x < a\},$$

$$(-\infty, \infty) = R$$

deb qaraymiz.

Faraz qilaylik, a va b ixtiyoriy haqiqiy sonlar bo'lib, $a < b$ bo'lsin. U holda

$$(a,b) \neq \varnothing$$

bo'ladi.

Haqiqatdan ham,

$$a = \alpha_0, \alpha_1\alpha_2 ... \alpha_n ... \geq 0$$

$$b = \beta_0, \beta_1\beta_2 ... \beta_n ....$$

bo'lib, $m \geq 0$ uchun

$$\alpha_0 = \beta_0, \alpha_1 = \beta_1, ... \alpha_{m-1} = \beta_{m-1} \text{ va } \alpha_m < \beta_m$$

bo'lsin. Agar k natural son m dan katta sonlar ichida eng kichigi bo'lsa, $(\alpha_k < 9)$ unda

$$r=\alpha_0,\alpha_1\alpha_2...\alpha_m\alpha_{m+1}...(\alpha_k+1)$$

ratsional son uchun $a<r<b$ bo'ladi.Demak, $(a,b)\neq\varnothing$

1.3. Haqiqiy sonlar to'plamining chegaralari

Haqiqiy sonlar to'plamining chegaralanganligi, to'plamning aniq chegaralari tushunchalari matematik analiz kursida muhim rol o'ynaydi.

1^0. Sonlar to'plamining aniq chegaralari. Biror $E\subset R$ to'plam berilgan bo'lsin.

1-ta'rif. Agar E to'planing shunday x_0 elementi $(x_0\in E)$ topilsaki, E to'plamning ixtiyoriy x elementlari uchun

$$x\leq x_0$$

tengsizlik bajarilsa, ya'ni

$$\exists x_0\in E,\quad \forall x\in E:\ x\leq x_0$$

bo'lsa, x_0 soni E to'plamning **eng katta elementi** deyiladi va

kabi belgilanadi.

2-ta'rif. Agar E to'plamning shunday x_0 elamenti $(x_0\in E)$ topilsaki, E to'plamning ixtiyoriy x elementlari uchun

$$x\geq x_0$$

tengsizlik bajarilsa, ya'ni

$$\exists x_0\in E,\ \forall x\in E:\ x\geq x_0$$

bo'lsa, x_0 soni E to'plamning **eng kichik elementi** deyiladi va

$$x_0=\min E$$

kabi belgilanadi.

Masalan,

$$\max\left\{1,\frac{1}{2},\frac{1}{3},...,\frac{1}{n},...\right\}=1$$

$$\min\ \{1,2,3,...,n,...\}=1$$

bo'ladi.

3-ta'rif. Agar shunday M soni $(M\in R)$ topilsaki, E to'plamning ixtiyoriy x elementlari uchun

$$x\leq M$$

tengsizliklar bajarilsa,ya'ni

$$\exists M\in R,\ \forall x\in E:\ x\leq M$$

bo'lsa, E to'plam **yuqoridan chegaralangan** deyiladi, M soni to'plamning **yuqori chegarasi** deyiladi.

4-ta'rif. Agar shunday m soni $(m\in R)$ topilsaki, E to'plamning ixtiyoriy x elementlari uchun

$$x\geq m$$

tengsizliklar bajarilsa,ya'ni

$$\exists m\in R,\ \forall x\in E:\ x\geq m$$

bo'lsa, E to'plam **quyidan chegaralangan deyiladi,** m soni to'plamning **quyi chegarasi** deyiladi.

Ravshanki, to'plam yuqoridan chegaralangan bo'lsa, uning yuqori chegaralari cheksiz ko'p, shuningdek quyidan chegaralan-gan bo'lsa, uning quyi chegaralari cheksiz ko'p bo'ladi.

5-ta'rif. Agar $E\subset R$ to'plam ham quyidan, ham yuqoridan chegaralangan bo'lsa, E **chegaralangan to'plam** deyiladi.

6-ta'rif. Agar ixtiyoriy M soni $(M\in R)$ olinganda ham shunday x_0 elementi $(x_0\in E)$ topilsaki,

$$x_0>M$$

tengsizlik bajarilsa, ya'ni

$$\forall M\in R,\ \exists x_0\in E:\ x_0>M$$

bo'lsa, E to'plam **yuqoridan chegaralanmagan** deyiladi.

7-ta'rif. Agar ixtiyoriy m soni $(m\in R)$ olinganda ham shunday x_0 elementi $(x_0\in E)$ topilsaki,

$$x_0<m$$

tengsizlik bajarilsa, ya'ni

$$\forall m\in R,\ \exists x_0\in E:\ x_0<m$$

bo'lsa, E to'plam **quyidan chegaralanmagan** deyiladi.

Masalan,

1) $E_1=\{...,-2,-1,0\}$ to'plam yuqoridan chegaralangan;

2) $E_2=\{1,2,3,...\}$ to'plam quyidan chegaralangan;

3) $E_3=\left\{1,\frac{1}{2},\frac{1}{3},...\right\}$ to'plam chegaralangan;

4) $E_4=\{x\in R\mid x>0\}$ to'plam yuqoridan chegaralanmagan;

5) $E_5=\{x\in R\mid x<0\}$ to'plam quyidan chegaralanmagan bo'ladi.

Endi sonlar to‘plamining aniq yuqori hamda aniq quyi chegaralari tushunchalarini keltiramiz.

Aytaylik, $E \subset R$ to’plam va $a \in R$ soni berilgan bo‘lsin.

8-ta'rif. Agar

1) a soni E to’plamning yuqori chegarasi bo‘lsa,

2) E to‘plamning ixtiyoriy yuqori chegarasi M uchun $a \le M$ tengsizlik bajarilsa, a soni E to‘plamning aniq yuqori chegarasi deyiladi va $\sup E$ kabi belgilanadi:

$$a = \sup E .$$

Demak, E to‘plamning aniq yuqori chegarasi, uning yuqori chegaralari orasida eng kichigi bo‘ladi.

9-ta'rif. Faraz qilaylik, $E \subset R$ to’plam va $b \in R$ soni berilgan bo‘lsin.

Agar

1) b son E to’plamning quyi chegarasi bo‘lsa,

2) E to‘plamning ixtiyoriy quyi chegarasi m uchun $b \ge m$ tengsizlik bajarilsa, b soni E to‘plamning **aniq quyi chegarasi** deyiladi va $\inf E$ kabi belgilanadi:

$$b = \inf E .$$

Demak, E to‘plamning aniq quyi chegarasi, uning quyi chegaralari orasida eng kattasi bo‘ladi.

“sup” va “inf” lar lotincha “supremum” va “infimum” so‘zlaridan olingan bo‘lib, ular mos ravishda eng yuqori, eng quyi degan ma'nolarni anglatadi.

1-teorema. Faraz qilaylik, $E \subset R$ to’plam va $a \in R$ soni berilgan bo‘lsin. a soni E to‘plamning aniq yuqori chegarasi bo‘lishi uchun

1) a soni E to‘plamning yuqori chegarasi,

2) a sonidan kichik bo‘lgan ixtiyoriy α $(\alpha < a)$ uchun E to’plamda $x > \alpha$ tengsizlikni qanoatlantiruvchi x sonining topilishi zarur va yetarli.

Zarurligi. Aytaylik,

$$a = \sup E$$

bo‘lsin. 8-ta'rifga binoan:

1) $\forall x \in E$ uchun $x \le a$, ya’ni a soni E to‘plamning yuqori chegarasi;

2) a soni yuqori chegaralar orasida eng kichigi. Binobarin, a dan kichik α soni uchun $x > \alpha$ bo’lgan $x \in E$ soni topiladi.

Yetarliligi. Teoremaning ikkala sharti bajarilsin. Bu holda, ravshanki, $\alpha < a$ shartni qanoatlantiruvchi har qanday α soni E to‘plamning yuqori chegarasi bo‘lolmaydi. Demak, a- to‘plamning yuqori chegaralari orasida eng kichigi. Unda ta'rifga ko‘ra

$$a = \sup E$$

bo‘ladi.

Xuddi shunga o‘xshash quyidagi teorema isbotlanadi.

2- teorema. Faraz qilaylik, $E \subset R$ to’plam va $b \in R$ soni berilgan bo‘lsin. b soni E to‘plamning aniq quyi chegarasi bo‘lishi uchun

1) b soni E to‘plamning quyi chegarasi,

2) b sonidan katta bo‘lgan ixtiyoriy β $(\beta > b)$ uchun E to’plamda $x < \beta$ tengsizlikni qanoatlantiruvchi x sonining topilishi zarur va yetarli.

Eslatma. Agar $E \subset R$ to‘plam yuqoridan chegaralanmagan bo‘lsa, u holda

$$\sup E = +\infty ,$$

quyidan chegaralanmagan bo‘lsa, u holda

$$\inf E = -\infty$$

deb olinadi.

2°. Aniq chegaralarning mavjudligi.

Aytaylik,

$$\alpha = \alpha_0, \alpha_1 \alpha_2 \dots \alpha_n \dots$$

musbat haqiqiy son bo‘lsin, bunda

$$\alpha_0 \in N \cup \{0\}, \quad \alpha_n \in N_0 = \{0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9\}, \quad n \ge 1.$$

Ushbu

$$a_n = \alpha_0, \alpha_1 \alpha_2 \dots \alpha_n = \alpha_0 + \frac{\alpha_1}{10} + \frac{\alpha_2}{10^2} + \dots + \frac{\alpha_n}{10^n},$$

$$b_n = \alpha_0, \alpha_1 \alpha_2 \dots (\alpha_n + 1) = \alpha_0 + \frac{\alpha_1}{10} + \frac{\alpha_2}{10^2} + \dots + \frac{\alpha_n + 1}{10^n}$$

ratsional sonlar uchun

$$a_n \le \alpha < b_n$$

bo‘ladi.

Demak, ixtiyoriy haqiqiy son olinganda shunday ikkita ratsional son topiladiki, ulardan biri shu haqiqiy sondan kichik yoki teng, ikkinchisi esa katta bo‘ladi.

Endi sonlar to‘plamining aniq chegaralarining mavjudligi haqidagi teoremalarni keltiramiz.

3-teorema. Agar bo‘sh bo‘lmagan to‘plam yuqoridan chegaralangan bo‘lsa, uning aniq yuqori chegarasi mavjud bo‘ladi.

Bu teoremani

$$E \subset [0, +\infty), \quad E \ne \varnothing$$

to‘plam uchun isbotlaymiz.

E to‘plam yuqoridan chegaralangan bo‘lsin:

$$\exists M \in R,\ \forall x \in E: \quad x \le M.$$

Arximed aksiomasini e'tiborga olib, $M \in N$ deyish mumkin.

Endi E to'plam

$$\alpha = \alpha_0, \alpha_1\alpha_2 \dots \qquad (\alpha \in E)$$

elementlarining butun qismlaridan, ya'ni α_0 laridan iborat to'plamni F_0 deylik:

$$F_0 = \{\alpha_0 \in N \cup \{0\} \mid, \ \alpha = \alpha_0, \alpha_1\alpha_2 \dots \in E\}.$$

Bu to'plam ham yuqoridan M soni bilan chegaralangan va $F_0 \neq \varnothing$. Ravshanki, $F_0 \subset N \cup \{0\}$. Bundan F_0 to'plamning chekli ekanligini topamiz. Demak, F_0 to'plamning eng katta elementi mavjud. Uni c_0 deylik:

$$\max F_0 = c_0 \qquad (1)$$

E to'plamning

$$c_0, \alpha_1\alpha_2 \dots$$

ko'rinishdagi barcha elementlaridan iborat to'plamni E_0 deb olamiz:

$$E_0 = \{c_0, \alpha_1\alpha_2 \dots \in E\}.$$

Ravshanki, $E_0 \subset E$, $E_0 \neq \varnothing$.

Endi E_0 to'plam

$$c_0, \alpha_1\alpha_2 \dots$$

elementlarining α_1 laridan iborat to'plamni olib, uni F_1 deylik:

$$F_1 = \{\alpha_1 \in \{0,1,2,\dots,9\} \mid c_0, \alpha_1\alpha_2 \dots \in E_0\}.$$

Bu chekli to'plam bo'lib, $F_1 \neq \varnothing$ bo'ladi. Shuning uchun uning eng katta elementi mavjud. Uni c_1 deb olamiz:

$$\max F_1 = c_1 \qquad (2)$$

E_0 to'plamning

$$c_0, c_1\alpha_2\alpha_3 \dots$$

ko'rinishdagi barcha elementlaridan iborat to'plamni E_1 deb olamiz:

$$E_1 = \{c_0, c_1\alpha_2\alpha_3 \dots \in E_0\}.$$

Ravshanki, $E_1 \subset E_0$, $E_1 \neq \varnothing$.

Endi E_1 to'plam

$$c_0, c_1\alpha_2\alpha_3 \dots$$

elementlarning α_2 laridan iborat to'plamni olib, uni F_2 deylik:

$$F_2 = \{\alpha_1 \in \{0,1,2,\dots,9\} \mid c_0, c_1, \alpha_2 \dots \in E_1\}.$$

Bu to'plam ham chekli va $F_2 \neq \varnothing$ bo'lib, uning eng katta elementi mavjud:

$$\max F_2 = c_2 \qquad (3)$$

E_1 to'plamning

$$c_0, c_1c_2\alpha_3 \dots$$

ko'rinishdagi barcha elementlaridan iborat to'plamni E_2 deb olamiz:

$$E_2 = \{c_0, c_1c_2\alpha_3 \dots \in E_1\}$$

Bu jarayonni davom ettira borish natijasida

$$a = c_0, c_1c_2 \dots c_n \dots$$

haqiqiy son hosil bo'ladi.

Endi E to'plam va bu a son uchun 1-teoremaning ikkala shartlarini bajarilishini ko'rsatamiz:

1) Yuqoridagi (1) munosabatga ko'ra $\forall\, \alpha_0, \alpha_1\alpha_2\alpha_3 \dots \in E$ uchun $\alpha_0 \leq c_0$ bo'ladi.

Agar $\alpha_0 < c_0$ bo'lsa, u holda $\alpha_0, \alpha_1\alpha_2 \dots < a$ bo'ladi.

Agar $\alpha_0 = c_0$ bo'lsa, u holda $c_0, \alpha_1\alpha_2 \dots \in E_0$ bo'lib, (2) munosabatga ko'ra $\alpha_1 \leq c_1$ bo'ladi.

Agar $\alpha_1 < c_1$ bo'lsa, u holda $\alpha_0, \alpha_1\alpha_2 \dots < a$ bo'ladi.

Agar $\alpha_1 = c_1$ bo'lsa, u holda $c_0, c_1\alpha_2 \dots \in E_1$ bo'lib, (3) munosabatga ko'ra $\alpha_2 \leq c_2$ bo'ladi.

Bu jarayonni davom ettirish natijasida ikki holga duch kelamiz:

a) shunday topiladiki $n \geq 0$ topiladiki,

$$\alpha_0 = c_0, \quad \alpha_1 = c_1, \dots \ \alpha_{n-1} = c_{n-1}, \quad \alpha_n < c_n$$

bo'lib, $\alpha_0, \alpha_1\alpha_2 \dots < a$ bo'ladi.

b) ixtiyoriy $n \geq 0$ da $\alpha_n = c_n$ bo'lib, $\alpha_0, \alpha_1\alpha_2 \dots = a$ bo'ladi.

Demak, har doim $\alpha_0, \alpha_1\alpha_2 \dots \leq a$ munosabat o'rinli bo'ladi;

2) a sondan kichik bo'lgan ixtiyoriy

$$\beta = \beta_0, \beta_1\beta_2 \dots \beta_n \dots$$

haqiqiy sonni olaylik:

$$\beta_0, \beta_1\beta_2 \dots \beta_n \dots < c_0, c_1c_2 \dots c_n \dots$$

Unda shunday $n \geq 0$ topiladiki,

$$\beta_0 = c_0, \quad \beta_1 = c_1, \dots \ \beta_{n-1} = c_{n-1}, \quad \beta_n < c_n$$

bo'ladi. Shuni e'tiborga olib, $\forall x \in E_n \subset E$ uchun

$$x > \beta_0, \beta_1\beta_2 \dots \beta_n \dots$$

bo'lishini topamiz.

Shunday qilib teoremada keltirilgan E to'plam va a soni uchun 1-teoremaning ikkala shartining bajarilishi ko'rsatildi. Unda 1-teoremaga muvofiq E to'plamning aniq yuqori chegarasi mavjud va

$$a = \sup E$$

bo'lishi kelib chiqadi.

Xuddi shunga o'xshash quyidagi teorema isbotlanadi.

4-teorema. Agar bo'sh bo'lmagan to'plam quyidan chegaralangan bo'lsa, uning aniq quyi chegarasi mavjud bo'ladi.

Eslatma. To'plamning aniq quyi hamda aniq yuqori chegaralari shu to'plamga tegishli bo'lishi ham mumkin, tegishli bo'lmasligi ham mumkin.

1.4. Haqiqiy sonlar ustida amallar

1^0. Ikki haqiqiy sonlar yig'indisi, ayirmasi, ko'paytmasi va nisbati. Avval aytganimizdek, ratsional sonlar ustida, xususan chekli o'nli kasrlar ustida bajariladigan amallar va ularning xossalari ma'lum deb hisoblaymiz.

Aytaylik, ikkita musbat

$$a = \alpha_0, \alpha_1\alpha_2 ... \alpha_n ...$$
$$b = \beta_0, \beta_1\beta_2 ... \beta_n ...$$

haqiqiy sonlar berilgan bo'lsin. Unda $n \geq 0$ bo'lganda ushbu

$$a_n^{'} = a_0, a_1a_2 ... a_n = a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + ... + \frac{a_n}{10^n},$$

$$a_n^{''} = a_0, a_1a_2 ...(a_n + 1) = a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + ... + \frac{a_n + 1}{10^n}$$

ratsional sonlar uchun

$$a_n^{'} \leq a \leq a_n^{''}, \tag{1}$$

shuningdek,

$$b_n^{'} = \beta_0, \beta_1\beta_2 ... \beta_n = \beta_0 + \frac{\beta_1}{10} + \frac{\beta_2}{10^2} + ... + \frac{\beta_n}{10^n},$$

$$b_n^{''} = \beta_0, \beta_1\beta_2 ...(\beta_n + 1) = \beta_0 + \frac{\beta_1}{10} + \frac{\beta_2}{10^2} + ... + \frac{\beta_n + 1}{10^n}$$

ratsional sonlar uchun

$$b_n^{'} \leq b \leq b_n^{''} \tag{2}$$

bo'ladi.

Endi (1) va (2) tengsizliklarni qanoatlantiruvchi ratsional sonlarning yig'indisi $a_n^{'} + b_n^{'}$ lardan iborat $\{a_n^{'} + b_n^{'}\}$ 'plamni qaraymiz. Ravshanki, bu to'plam yuqoridan chegaralangan. Unda 4-ma'ruzadagi 3-teoremaga ko'ra $\{a_n^{'} + b_n^{'}\}$ to'plamning aniq yuqori chegarasi mavjud bo'ladi.

1-ta'rif. $\{a_n^{'} + b_n^{'}\}$ to'plamning aniq yuqori chegarasi a va b haqiqiy sonlar yig'indisi deyiladi va $a + b$ kabi belgilanadi:

$$a + b = \sup_{n \geq 0}\{a_n^{'} + b_n^{'}\}.$$

(1) va (2) tengsizliklarni qanoatlantiruvchi ratsional sonlarning ko'paytmasi $a_n^{'} \cdot b_n^{'}$ lardan iborat $\{a_n^{'} \cdot b_n^{'}\}$ to'plamni qaraymiz. Bu to'plam yuqoridan chegaralangan bo'ladi. Shuning uchun uning aniq yuqori chegarasi mavjud bo'ladi.

2-ta'rif. $\{a_n^{'} \cdot b_n^{'}\}$ to'plamning aniq yuqori chegarasi a va b haqiqiy sonlar ko'paytmasi deyiladi va $a \cdot b$ kabi belgilanadi.

$$a \cdot b = \sup_{n \geq 0}\{a_n^{'} \cdot b_n^{'}\}.$$

(1) va (2) tengsizliklarni qanoatlantiruvchi ratsional sonlarning nisbati $\frac{a_n^{'}}{b_n^{''}}$ lardan iborat $\left\{\frac{a_n^{'}}{b_n^{''}}\right\}$ to'plam yuqoridan chegaralangan bo'ladi.

3-ta'rif. $\left\{\frac{a_n^{'}}{b_n^{''}}\right\}$ to'plamning aniq yuqori chegarasi a **sonning** b **songa nisbati** deyiladi va $\frac{a}{b}$ kabi belgilanadi.

$$\frac{a}{b} = \sup_{n \geq 0}\left\{\frac{a_n^{'}}{b_n^{''}}\right\}.$$

Aytaylik a va b musbat haqiqiy sonlar bo'lib, $a > b$ bo'lsin.

4-ta'rif. $\{a_n^{'} - b_n^{''}\}$ to'plamning aniq yuqori chegarasi a **sonidan** b **sonining ayirmasi** deyiladi va $a - b$ kabi belgilanadi.

$$a - b = \sup_{n \geq 0}\{a_n^{'} - b_n^{''}\}.$$

Eslatma. 1) Haqiqiy sonlar ustida bajariladigan qo'shish, ko'paytirish, ayirish va bo'lish amallarini to'plamning aniq quyi chegarasi orqali ham ta'riflash mumkin.

Masalan, a va b haqiqiy sonlar yig'indisi quyidagicha ta'riflanadi:

$$a+b=\inf_{n\ge0}\{a_n''+b_n''\}.$$

Haqiqiy sonlarda, yuqorida kiritilgan amallar o'rta maktab matematika kursida o'rganilgan amallarning barcha xossalarga ega.

2^0. Haqiqiy sonning darajasi. Avval haqiqiy sonning 0-hamda n- darajalari ($n\in N$) quydagicha

$$a^0=1,$$
$$a^n=\underbrace{a\cdot a\cdot\ldots\cdot a}_{n\ ta},\ (n\in N)$$

aniqlanishini ta'kidlaymiz.

Teorema (isbotsiz). Faraz qilaylik, $a>0$ va $n\in N$ bo'lsin. U holda shunday yagona musbat soni topiladiki,

$$x^n=a$$

bo'ladi.

5-ta'rif. Musbat haqiqiy a soninning n darajali ildizi deb ushbu

$$x^n=a$$

tenglikni qanoatlantiruvchi yagona x soniga aytiladi va

$$x=\sqrt[n]{a}=a^{\frac{1}{n}}$$

kabi belgilanadi.

Aytaylik, a musbat haqiqiy son, r esa musbat ratsional son bo'lsin:

$$a>0,\quad r=\frac{m}{n},\quad m,n\in N.$$

Bu holda a soning r- darajasi quydagicha

$$a^r=\left(a^m\right)^{\frac{1}{n}}$$

aniqlanadi.

6-ta'rif. Faraz qilaylik, $a>1,\ b>0$ haqiqiy sonlari berilgan bo'lsin, a soning $b-$ darajasi deb ushbu $\left\{a^{b_n'}\right\}$ to'plamning aniq yuqori chegarasiga aytiladi:

$$a^b=\sup_{n\ge0}\left\{a^{b_n'}\right\}\ \text{bunda}\ b_n'=\beta_0,\beta_1,\beta_2,\ldots\beta_n,\quad b=\beta_0,\beta_1,\beta_2,\ldots\beta_n,\ldots$$

3^0.Haqiqiy sonning absolyut qiymati. Aytaylik $x\in R$ son berilgan bo'lsin. Ushbu

$$|x|=\begin{cases}x, & \text{agar } x\ge0 \text{ bo'lsa},\\ -x, & \text{agar } x<0 \text{ bo'lsa},\end{cases}$$

miqdor x **sonining absolyut qiymati** deyiladi.

Haqiqiy sonning absolyut qiymati quyidagi xossalarga ega:

1) $x\in R$ son uchun

$$|x|\ge0,\quad |x|=|-x|,\quad x\le|x|,\quad -x\le|x|$$

munosabatlar o'rinli,

2) $$|x|<a \Leftrightarrow -a<x<a,$$
$$|x|\le a \Leftrightarrow -a<x<a,\quad (a>0)$$

3) $x\in R,\ \ y\in R$ sonlar uchun

$$|x+y|\le|x|+|y|,$$
$$|x-y|\ge|x|-|y|,$$
$$|x\cdot y|=|x|\cdot|y|,$$
$$\left|\frac{x}{y}\right|=\frac{|x|}{|y|},\quad (y\ne0)$$

bo'ladi.

Bu xossalarning isboti bevosita sonning absolyut qiymati ta'rifidan kelib chiqadi. Ulardan birini, masalan $|x+y|\le|x|+|y|$ bo'lishini isbotlaymiz.

Aytaylik, $x+y>0$ bo'lsin. Unda $|x+y|=x+y$ bo'ladi. $x\le|x|,\ \ y\le|y|$ bo'lishini e'tiborga olib topamiz:

$$|x+y|=x+y\le|x|+|y|.$$

Endi $x+y<0$ bo'lsin.

Unda $|x+y|=-(x+y)=(-x)+(-y)$ bo'ladi. $-x\le|x|,\ -y\le|y|$ bo'lishini e'tiborga olib topamiz:

$$|x+y|=(-x)+(-y)\le|x|+|y|.$$

1-misol. Ushbu

$$|3x-1|\le|2x-1|+|x| \tag{3}$$

tengsizlik x ning qanday qiymatlarida o'rinli bo'ladi?

Sonning absolyut qiymati xossasidan foydalanib topamiz:

$$|3x-1|=|(2x-1)+x|\le|2x-1|+|x|.$$

Demak, (3) tengsizlik ixtiyoriy $x\in R$ uchun o'rinli bo'ladi.

Barcha manfiy bo'lmagan haqiqiy sonlar to'plamini R_+ bilan belgilaylik. Ravshanki, $R_+\subset R$.

Har bir $x \in R$ haqiqiy songa uning absolyut qiymati $|x|$ ni mos qo'yish bilan ushbu

$$f: x \to |x| \qquad (f: R \to R_+)$$

akslantirishga ega bo'lamiz.

Demak haqiqiy sonning absolyut qiymati R to'pplamni R_+ to'plamga akslantirish deb qaralishi mumkin.

Ixtiyoriy $x \in R,\ \ y \in R$ sonlarni olaylik. Ushbu

$$|x-y|$$

miqdor x va y **nuqtalar orasidagi masofa** deyiladi va $d(x,y)$ kabi belgilanadi:

$$d(x,y)=|x-y|.$$

Masofa quyidagi xossalarga ega:

1) $d(x,y) \ge 0$ va $d(x,y)=0 \Leftrightarrow x=y,$

2) $d(x,y)=d(y,x),$

3) $d(x,z) \le d(x,y)+d(y,z), \quad (z \in R).$

4^0. Bernulli tengsizligi. Nyuton binomi formulasi. Ixtiyoriy $x \ge -1 (x \in R)$ hamda ixtiyoriy $n \in N$ uchun ushbu

$$(1+n)^n \ge 1+nx \tag{4}$$

tengsizlik o'rinli.

Bu tengsizlikni matematik induktsiya usuli yordamida isbotlaymiz.

Ravshanki, $n=1$ da (4) tengsizlik (tasdiq) o'rinli bo'ladi

$$1+x=1+x$$

Endi $n \in N$ da (4) munosabat o'rinli deb, uni $n+1$ uchun ham o'rinli bo'lishini ko'rsatamiz. (4) tengsizlikning har ikki tomonini $1+x$ ga ko'paytirib topamiz:

$$(1+x)^{n+1} \ge (1+nx)\cdot(1+x)=1+(n+1)x+nx^2 \ge 1+(n+1)x$$

Matematik induktsiya usuliga binoan (4) munosabat ixtiyoriy $n \in N$ uchun o'rinli bo'ladi.(4) tengsizlik Bernulli tengsizligi deyiladi.

Endi Nyuton binomi formulasini keltiramiz.

Ma'lumki, $a \in R,\ \ b \in R$ da

$$(a+b)^2=a^2+2ab+b^2$$

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3$$

bo'ladi. Umuman, ixtiyoriy $n \in N$ da

$$(a+b)^n=C_n^0\cdot a^n+C_n^1\cdot a^{n-1}\cdot b+\ldots+C_n^k\cdot a^{n-k}\cdot b^k+\ldots+ +C_n^{n-1}ab^{n-1}+C_n^n\cdot b^n=\sum_{k=0}^{n} C_n^k a^{n-k} b^k \tag{5}$$

bo'ladi,bunda

$$C_n^0=1$$

$$C_n^k=\frac{n(n-1)\ldots(n-(k-1))}{k!},\ k!=1\cdot 2\cdot 3\ldots\cdot k,\ \ k=1,2\ldots,n$$

(5) tenglik ham matematik induktsiya usuli yordamida isbotlanadi.

Ravshanki, $n=1$ da $C_1^0\cdot a+C_1^1\cdot b=a+b$. Demak, bu holda (5) tenglik o'rinli. Endi (5) tenglik n uchun o'rinli bo'lsin deb, uni $n+1$ uchun ham o'rinli bo'lishini ko'rsatamiz. (5) tenglikning har ikki tomonini $a+b$ ga ko'paytirib topamiz:

$$(a+b)^{n+1}=a^{n+1}+\sum_{k=1}^{n}(C_n^k+C_n^{k+1})\cdot a^{n+1-k}\cdot b^k+b^{k+1}.$$

Ravshanki,

$$C_n^k+C_n^{k-1}=\frac{n(n-1)\ldots(n-(k-2))}{k!}(n-(k-1)+k)=$$

$$\frac{n(n+1)((n+1)-1)\ldots((n+1)-(k-1))}{k!}=C_{n+1}^k$$

Demak,

$$(a+b)^{n+1}=a^{n+1}+\sum_{k=1}^{n}C_{n+1}^k\cdot a^{n+1-k}\cdot b^k+b^{n+1}=\sum_{k=0}^{n+1}C_{n+1}^k\cdot a^{n+1-k}\cdot b^k$$

bo'ladi. Bu esa (5) tenglik $n+1$ bo'lganda ham bajarilishini ko'rsatadi.

Odatda (5) tenglik Nyuton binomi formulasi deyiladi.

5^0. Ichma-ich joylashgan segmentlar printsipi. Ma'lumki, ushbu

$$\{x \in R:\ a \le x \le b\}=[a,b]$$

to'plam segment deb ataladi.

Aytaylik, $[a_1,b_1]$ va $[a_2,\ b_2]$ segmentlar berilgan bo'lsin. Agar

$$[a_1,b_1]\subset[a_2,b_2]$$

bo'lsa, $[a_1,b_1]$ **segment** $[a_2,b_2]$ **segmentning ichiga joylashgan** deyiladi. Bu holda $a_1 \le a_2 < b_2 \le b_1$ bo'ladi.

7-ta'rif. Agar

$$[a_1,b_1],[a_2,b_2],....,[a_n,b_n],... \qquad (6)$$

segmentlar ketma-ketligi quyidagi

$$[a_1,b_1]\supset[a_2,b_2]\supset...\supset[a_n,b_n]\supset...$$

munosabatda,ya'ni $\forall n\in N$ da

$$[a_n,b_n]\supset[a_{n+1},b_{n+1}]$$

bo'lsa, (6) **ichma-ich joylashgan segmentlar ketma-ketligi** deyiladi.

Teorema. Aytaylik,

$$[a_1,b_1],[a_2,b_2],.....,[a_n,b_n],....$$

segmentlar ketma-ketligi quyidagi shartlarni bajarsin:

1) $\forall n\in N:\ [a_n,b_n]\supset[a_{n+1},b_{n+1}]$,

2) $\forall\varepsilon>0,\ \exists n_0\in N,\ n>n_0:\ b_n-a_n<\varepsilon$ bo'lsin.

U holda shunday $c\in R$ mavjud bo'ladiki, $c\in[a_n,b_n]$, $(n=1,2,3...)$ bo'lib,bunday c yagona bo'ladi.

Teoremada qaralayotgan segmentlar ketma-ketligi ichma-ich joylashgan segmentlar ketma-ketligi bo'ladi. Ravshanki, bu holda ushbu

$$a_1\le a_2\le a_3\le...\le a_n<b_n\le b_{n-1}\le...\le b_2\le b_1$$

munosabat bajariladi.

Endi $a_1,a_2,...,a_n$ sonlaridan tashkil topgan

$$E=\{a_1,a_2,...,a_n\}$$

to'plamni qaraymiz. Bu to'plamning yuqoridan chegaralanganligini ko'rsatamiz.

Ixtiyoriy natural m sonini olib, uni tayinlaymiz.

Agar $n\le m$ bo'lsa, $[a_m,b_m]\subset[a_n,b_n]$ bo'lib, $a_n\le a_m<b_m\le b_n$, ya'ni $a_n<b_m$ bo'ladi.

Agar $n>m$ bo'lsa, $[a_n,b_n]\subset[a_m,b_m]$ bo'lib, $a_m\le a_n<b_n\le b_m$, ya'ni $a_n\le b_m$ bo'ladi.

Aniq yuqori chegara haqidagi teoremaga ko'ra

$$\sup E=c \qquad (c\in R)$$

mavjud bo'ladi.

To'plamning aniq yuqori chegarasi ta'rifiga binoan

$\forall n\in N$ da $a_n\le c$ va $\forall m\in N$ da $c\le b_n$ bo'ladi.

Demak,

$$\forall n\in N \text{ da } c\in[a_n,b_n].$$

Agar shu nuqtadan farqli va barcha segmentlarga tegishli c' $(c'\in[a_n,b_n],\ \forall n\in N)$ mavjud deb qaraladigan bo'lsa, unda

$$b_n-a_n\ge|c-c'|>0$$

bo'lib, bu teoremaning 2-shartiga zid bo'ladi.

Demak, $c=c'$.

Odatda bu teorema ichma-ich joylashgan segmentlar printsipi deyilib, u haqiqiy sonlar to'plamining uzluksizlik (to'liqlik) xossasini ifodalaydi.

Quydagi tengliklarni isbotlang:

1. $A\setminus(A\setminus B)=A\cap B$
2. $(A\setminus B)\cup(B\setminus A)=(A\cup B)\setminus(A\cap B)$
3. $(A\setminus B)\setminus C=A\setminus(B\cup C)$
4. $(A\setminus B)\cap C=(A\cap C)\setminus(B\cap C)$
5. $\overline{A\setminus B}\cap(\overline{A}\cup\overline{B})=\overline{A}$
6. $A\setminus B=A\cap\overline{B}$
7. $\overline{A\setminus B}=\overline{A}\cup B$
8. $A\cap(\overline{A\setminus B})=A\cap B$
9. $(A\setminus B)\setminus C=(A\setminus C)\setminus(B\setminus C)$
10. $A\setminus(B\setminus C)\subset A\cup C$
11. $A\setminus C\subset(A\setminus B)\cup(B\setminus C)$
12. $((A\cup B)'\cap(A'\cup B'))'=A\cup B$
13. $A\subset B\subset C=A\cup B=B\cap C$
14. $A\subset B\Rightarrow A\setminus C\subset B\setminus C$
15. $A\subset B\Rightarrow A\cap C\subset B\cap C$
16. $A\subset B\Rightarrow A\cup C\subset B\cup C$
17. $B\subset A\wedge C=A\setminus B\Rightarrow A=B\cup C$
18. $A\subset C\Rightarrow A\cup(B\cap C)=(A\cup B)\cap C$
19. $A\setminus(B\cap C)=(A\setminus B)\cup(A\setminus C)$
20. $(A\setminus B)\times C=(A\times C)\setminus(B\times C)$
21. $A\times(B\setminus C)=(A\times B)\setminus(A\times C)$
22. $(A\cup B)\times C=(A\times C)\cup(B\times C)$
23. $A\times(B\cap C)=(A\times B)\cap(A\times C)$
24. $(A\cap B)\times(C\cap D)=(A\times C)\cap(B\times D)$
25. $A\subset B\Rightarrow A\times C\subset B\times C$
26. $A\cup B\subset C\Rightarrow A\times B=(A\times B)\cap(C\times B)$
27. $(A\times B)\cup(B\times A)=C\times C\Rightarrow A=B=C$
28. $\overline{A\cup B}=\overline{A}\cap\overline{B}$
29. $\overline{A\cap B}=\overline{A}\cup\overline{B}$
30. $(A\cap B)'=A'\cap B'$

II BOB
SONLAR KETMA-KETLIGI

2.1. Sonlar ketma-ketligi va ularning limiti

1^0. Sonlar ketma-ketligi tushunchasi. Biz birinchi bobda ixtiyoriy E to'plamni F to'plamga akslantirish:

$$f: E \to F$$

tushunchasi bilan tanishgan edik.

Endi $E = N$, $F = R$ deb, har bir natural n songa biror haqiqiy x_n sonni mos qo'yuvchi

$$f: n \to x_n, \quad (n = 1, 2, 3, ...) \tag{1}$$

akslantirishni qaraymiz.

1-ta'rif. 1- akslantirishning akslaridan iborat ushbu

$$x_1,\ x_2,\ x_3,\ ...,\ x_n,\ ... \tag{2}$$

to'plam **sonlar ketma-ketligi** deyiladi. Uni $\{x_n\}$ yoki x_n kabi belgilanadi.

x_n $(n = 1, 2, 3, ...)$ sonlar (2) **ketma-ketlikning hadlari** deyiladi.

Masalan,

1) $x_n = \dfrac{1}{n}$: $1, \dfrac{1}{2}, \dfrac{1}{3}, ..., \dfrac{1}{n}, ...,$

2) $x_n = (-1)^n : -1,\ 1,\ -1,\ ...,\ (-1)^n, ...$

3) $x_n = \sqrt[n]{n} :\ 1,\ \sqrt{2},\ \sqrt[3]{3},\ ...,\ \sqrt[n]{n},\ ...$

4) $x_n = 1 :\ 1, 1, 1, ..., 1, ...$

5) $0,3;\ 0,33;\ 0,333;...;\ \underbrace{0,333...3}_{n\ ta};...$

lar sonlar ketma-ketliklaridir.

Biror $\{x_n\}$ ketma-ketlik berilgan bo'lsin.

2-ta'rif. Agar shunday o'zgarmas M soni mavjud bo'lsaki, ixtiyoriy x_n $(n = 1, 2, 3, ...)$ uchun $x_n \le M$ tengsizlik bajarilsa (ya'ni $\exists M,\ \forall n \in N: x_n \le M$ bo'lsa), $\{x_n\}$ ketma-ketlik yuqoridan chegaralangan deyiladi.

3-ta'rif. Agar shunday o'zgarmas m soni mavjud bo'lsaki,ixtiyoriy x_n $(n = 1, 2, 3, ...)$ uchun $x_n \ge m$ tengsizlik bajarilsa (ya'ni, $\exists m,\ \forall n \in N: x_n \ge m$ bo'lsa), $\{x_n\}$ ketma-ketlik quyidan chegaralangan deyiladi.

4-ta'rif. Agar $\{x_n\}$ ketma-ketlik ham yuqoridan, ham quyidan chegaralangan bo'lsa (ya'ni $\exists m, M,\ \forall n \in N: m \le x_n \le M$ bo'lsa), $\{x_n\}$ ketma-ketlik **chegaralangan** deyiladi.

1-misol. Ushbu $x_n = \dfrac{n}{4 + n^2}$ $\quad (n = 1, 2, 3, ...)$

ketma-ketlikning chegaralanganligi isbotlansin.

Ravshanki, $\forall n \in N$ uchun

$$x_n = \frac{n}{4 + n^2} > 0$$

bo'ladi. Demak, qaralayotgan ketma-ketlik quyidan chegaralangan.

Ma'lumki,

$$0 \le (n-2)^2 = n^2 - 4n + 4$$

bo'lib,unda $4n \le 4 + n^2$ ya'ni,

$$\frac{n}{4 + n^2} \le \frac{1}{4}$$

bo'lishi kelib chiqadi.Bu esa berilgan ketma-ketlikning yuqoridan chegaralanganligini bildiradi.Demak, ketma-ketlik chegaralangan.

5-ta'rif. Agar $\{x_n\}$ ketma-ketlik uchun

$$\forall M \in R,\ \exists n_0 \in N:\ x_{n_0} > M$$

bo'lsa, **ketma-ketlik yuqoridan chegaralanmagan** deyiladi.

2^0. Sonlar ketma-ketligining limiti. Aytaylik, $a \in R$ son hamda ixtiyoriy musbat ε berilgan bo'lsin.

6-ta'rif. Ushbu

$$U_\varepsilon(a) = \{x \in R |\ a - \varepsilon < x < a + \varepsilon\} = (a - \varepsilon,\ a + \varepsilon)$$

to'plam a **nuqtaning** ε **- atrofi** deyiladi.

Faraz qilaylik $\{x_n\}$ ketma-ketlik va $a \in R$ soni beril-gan bo'lsin.

7-ta'rif. Agar ixtiyoriy $\varepsilon > 0$ son olinganda ham shunday n_0 natural soni mavjud bo'lsaki, $n > n_0$ tengsizlikni qanoatlantiruvchi barcha natural sonlar uchun

$$|x_n - a| < \varepsilon \tag{3}$$

tengsizlik bajarilsa, ya'ni

$$\forall \varepsilon > 0,\ \exists n_0 \in N,\ \forall n > n_0 : |x_n - a| < \varepsilon$$

bo'lsa a son $\{x_n\}$ **ketma-ketlikning limiti** deyiladi va

$$a = \lim_{n \to \infty} x_n \text{ yoki } n \to \infty \text{ da } x_n \to a$$

kabi belgilanadi.

Ravshanki, yuqoridagi (3) tengsizlik uchun

$$|x_n - a| < \varepsilon \Leftrightarrow a - \varepsilon < x_n < a + \varepsilon$$

ya'ni, $x_n \in U_\varepsilon(a)$, $(n > n_0)$ bo'ladi. Shuni e'tiborga olib, ketma-ketlikning limitini quyidagicha ta'riflasa bo'ladi.

8-ta'rif. Agar a nuqtaning ixtiyoriy $U_\varepsilon(a)$ atrofi olinganda ham $\{x_n\}$ ketma-ketlikning biror hadidan keyingi barcha hadlari shu atrofga tegishli bo'lsa, a son $\{x_n\}$ ketma-ketlikning limiti deyiladi.

Yuqorida keltirilgan ta'riflardan ko'rinadiki ε ixtiyoriy musbat son bo'lib, natural n_0 soni esa ε ga va qaralayotgan ketma-ketlikka bog'liq ravishda topiladi.

2-misol. Ushbu

$$x_n = c \quad (c \in R,\ n = 1, 2, 3, \ldots)$$

ketma-ketlikning limiti c ga teng bo'ladi.

Haqiqatan ham, bu holda $\forall\ \varepsilon > 0$ ga ko'ra $n_0 = 1$ deyilsa, unda $\forall n > n_0$ uchun $|x_n - c| = 0 < \varepsilon$ bo'ladi.Demak, $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} c = c$

3-misol. Ushbu $x_n = \dfrac{1}{n} \quad (n = 1,2,3,....)$

ketma-ketlikning limiti 0 ga teng bo'lishi isbotlansin:

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Ravshanki,

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n}$$

bo'lib, $\dfrac{1}{n} < \varepsilon\ (\varepsilon > 0)$ tengsizlik barcha $n > \dfrac{1}{\varepsilon}$ bo'lganda o'rinli.Bu holda

$$n_0 = \left[\frac{1}{\varepsilon}\right] + 1$$

deyilsa, $([a] - a$ sonidan katta bo'lmagan uning butun qismi), unda $\forall n > n_0$ uchun

$$\left|\frac{1}{n} - 0\right| < \varepsilon$$

bo'ladi. Ta'rfga binoan

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

4-misol. Aytaylik, $a \in R$, $|a| > 1$ bo'lsin.U holda

$$\lim_{n\to\infty} \frac{1}{a^n} = 0$$

bo'lishi isbotlansin.

$|a| = 1 + \delta$ deylik.Unda $\delta = |a| - 1 > 0$ va Bernulli tengsizligiga ko'ra

$$(1+\delta)^n \geq 1 + n\delta > n\delta$$

bo'lib, $\forall n \in N$ da

$$\frac{1}{|a|^n} < \frac{1}{n\delta}$$

bo'ladi.Demak,

$$\left|\frac{1}{a^n} - 0\right| = \frac{1}{a^n} < \varepsilon \qquad (\varepsilon > 0)$$

tengsizlik barcha

$$n > \frac{1}{\varepsilon\delta}$$

bo'lganda o'rinli.Agar

$$n_0 = \left[\frac{1}{\varepsilon\delta}\right] + 1$$

deyilsa,Ravshanki, $\forall n > n_0$ uchun

$$\left|\frac{1}{a^n} - 0\right| < \varepsilon$$

bo'ladi.Demak,

$$\lim_{n\to\infty} \frac{1}{a^n} = 0$$

5-misol. Ushbu $x_n = \dfrac{n}{n+1}$ $(n = 1, 2, 3, \ldots)$

ketma-ketlikning limiti 1 ga teng bo'lishi isbotlansin.

Ixtiyoriy $\varepsilon > 0$ son olamiz. So'ng ushbu

$$|x_n - 1| < \varepsilon$$

tengsizlikni qaraymiz. Ravshanki,

$$|x_n - 1| = \left|\frac{n}{n+1} - 1\right| = \frac{n}{n+1}$$

Unda yuqoridagi tengsizlik

$$\frac{n}{n+1} < \varepsilon$$

ko'rinishga keladi. Keyingi tengsizlikdan

$$n > \frac{1}{\varepsilon} - 1$$

bo'lishi kelib chiqadi. Demak, limit ta'rifidagi $n_0 \in N$ sifatida $n_0 = \left[\frac{1}{\varepsilon} - 1\right] + 1$ olinsa ($\varepsilon > 0$ ga ko'ra $n_0 \in N$ topilib), $\forall n > n_0$ uchun $|x_n - 1| < \varepsilon$ bo'ladi.Bu esa

$$\lim_{n\to\infty} \frac{n}{n+1} = 1$$

bo'lishini bildiradi.

6-misol. Faraz qilaylik, $a \in R, |a| > 1$ va $\alpha \in R$ bo'lsin.U holda

$$\lim_{n\to\infty} \frac{n^\alpha}{a^n} = 0$$

bo'lishi isbotlansin.

Shunday natural k sonni olamizki $k \geq \alpha + 1$ bo'lsin. Endi $|a|^{\frac{1}{k}} > 1$ bo'lishini e'tiborga olib, $|a|^{\frac{1}{k}} = 1 + \delta$, ya'ni $\delta = |a|^{\frac{1}{k}} - 1 > 0$ ya'ni deymiz. Unda Bernulli tengsizligiga ko'ra

$$|a|^{\frac{n}{k}} = (1+\delta)^n \geq 1 + n\delta > n\delta$$

bo'lib, $\forall n \in N$ da

$$\frac{n^{k-1}}{a^n} < \frac{1}{n\delta^k}$$

bo'ladi.Bu holda

$$n_0 = \left[\frac{1}{\delta^k \cdot \varepsilon}\right] + 1 \quad (\varepsilon > 0)$$

deyilsa, $\forall n > n_0$ uchun

$$\left|\frac{n^\alpha}{a^n} - 0\right| = \frac{n^\alpha}{|a|^n} \leq \frac{n^{k-1}}{|n|^n} < \varepsilon$$

bo'ladi.Demak, $\lim\limits_{n\to\infty} \frac{n^\alpha}{a^n} = 0$.

7-misol. Ushbu $\lim\limits_{n\to\infty} \frac{\lg n}{n} = 0$

tenglik isbotlansin.

Ravshanki, $\forall \varepsilon > 0$ va $\forall n \in N$ uchun

$$0 \leq \frac{\lg n}{n} < \varepsilon \Leftrightarrow \lg n < n\varepsilon \Leftrightarrow n < 10^{n\varepsilon} \Leftrightarrow \frac{n}{(10^\varepsilon)^n} < 1$$

bo'ladi.Agar $10^\varepsilon > 1$ bo'lishini e'tiborga olsak, 6-misolga ko'ra

$$n \to \infty \text{ da } \frac{n}{(10^\varepsilon)^n} \to 0$$

ekanini topamiz. Unda ta'rifga ko'ra 1 soni uchun

$$\exists n_0 \in N, \ \forall n > n_0 : \ \frac{n}{(10^\varepsilon)^n} < 1$$

bo'ladi. Shunday qilib, $\forall n > n_0$ uchun $\frac{\lg n}{n} < \varepsilon$ bo'ladi. Demak, $\lim\limits_{n\to\infty} \frac{\lg n}{n} = 0$.

8-misol. Ushbu $x_n = (-1)^n \quad (n = 1, 2, 3, ...)$ ketma-ketlikning limiti mavjud emasligi isbotlansin.

Teskarisini faraz qilaylik a limitga ega bo'lsin. Unda ta'rifga binoan,

$$\forall \varepsilon > 0, \ \exists n_0 \in N, \ \forall n > n_0 : \ |(-1)^n - a| < \varepsilon$$

bo'ladi.

Ravshanki, n juft bo'lganda $|1 - a| < \varepsilon$, n toq bo'lganda $|(-1) - a| < \varepsilon$, ya'ni $|1 + a| < \varepsilon$ ya'ni bo'ladi. Bu tengsizliklardan foydalanib topamiz:

$$2 = |(1-a) + (1+a)| \leq |1-a| + |1-a| < 2\varepsilon .$$

Bu tengsizlik $\varepsilon > 1$ bo'lgandagina o'rinli. Bunday vaziyat $\varepsilon > 0$ sonining ixtiyoriy bo'lishiga zid. Demak, ketma-ketlik limitga ega emas.

Teorema. Agar $\{x_n\}$ ketma-ketlikning limiti mavjud emasligi isbotlansin.

Teskarisini faraz qilaylik, $\{x_n\}$ketma-ketlik ikkita a va b $(a \neq b)$ limitga ega bo'lsin:

$$\lim_{n\to\infty} x_n = a, \quad \lim_{n\to\infty} x_n = b \quad (a \neq b)$$

Limitning ta'rifiga ko'ra

$$\forall \varepsilon > 0, \ \exists n_0 \in N, \ \forall n > n_0 : \ |x_n - a| < \varepsilon,$$

$$\forall \varepsilon > 0, \ \exists n_0^{'} \in N, \ \forall n > n_0^{'} : \ |x_n - b| < \varepsilon$$

bo'ladi.

Agar n_0 va $n_0^{'}$ sonlarning kattasi $\overline{n}$ desak,unda $\forall n > \overline{n}$ da

$$|x_n - a| < \varepsilon, \ |x_n - b| < \varepsilon$$

bo'lib

$$|x_n - a| + |x_n - b| < 2\varepsilon$$

bo'ladi.

Ravshanki, $|a-b|=|a-x_n+x_n-b|\le|x_n-a|+|x_n-b|$.

Demak, $\forall\ \varepsilon>0$ da $|a-b|<2\varepsilon$ bo'lib,unda $a=b$ bo'lishi kelib chiqadi.

2.2. Yaqinlashuvchi ketma-ketliklarning xossalari

$\{x_n\}$ sonlar ketma-ketligi berilgan bo'lsin.

1-ta'rif. Agar $\{x_n\}$ ketma-ketlik chekli limitga ega bo'lsa, u **yaqinlashuvchi ketma-ketlik** deyiladi.

1^0. Yaqinlashuvchi ketma-ketlikning chegaralanganligi. Tengsizliklarda limitga o'tish.

1-teorema. $\{x_n\}$ ketma-ketlik yaqinlashuvchi bo'lsa, u chegaralangan bo'ladi.

Isbot. Aytaylik,

$$\lim_{n\to\infty}x_n=a \qquad (a\in R)$$

bo'lsin. Limit ta'rifiga ko'ra

$$\forall\varepsilon>0,\ \exists n_0\in N,\ \ \forall n>n_0;\ |x_n-a|<\varepsilon$$

bo'ladi.Demak, $n>n_0$ uchun

$$a-\varepsilon<x_n<a+\varepsilon$$

bo'ladi.Agar

$$\max\{|a-\varepsilon|,\ |a+\varepsilon|,\ |x_1|,\ |x_2|,\dots,|x_{n_0}|\}=M$$

deyilsa,u holda, $\forall n\in N$ uchun

$$|x_n|\le M$$

tengsizlik bajariladi. Bu esa $\{x_n\}$ ketma-ketlikning chegaralanganligini bildiradi.

2-teorema. Agar $\{x_n\}$ ketma-ketlik yaqinlashuvchi va

$$\lim_{n\to\infty}x_n=a$$

bo'lib, $a>p$ $(a<q)$ bo'lsa, u holda shunday $n_0\in N$ topiladiki, $\forall n>n_0$ bo'lganda

$$x_n>p\ \ (x_n<q)$$

bo'ladi.

Isbot. Aytaylik,

$$\lim_{n\to\infty}x_n=a,\quad a>p \quad (p\in R)$$

bo'lsin. $\varepsilon>0$ sonining ixtiyoriyligidan foydalanib, $\varepsilon<a-p$ deb qaraymiz.

Ketma-ketlik limiti ta'rifiga binoan, $\forall\ \varepsilon>0$ uchun, jumladan, $0<\varepsilon<a-p$ uchun,shunday $n_0\in N$ topiladiki, $\forall n>n_0$ bo'lganda

$$|x_n-a|<\varepsilon \quad\Leftrightarrow\quad -\varepsilon<x_n-a<\varepsilon$$

bo'ladi.Ravshanki,

$$0<\varepsilon<a-p\Rightarrow p<a-\varepsilon,$$

$$-\varepsilon<x_n-a<\varepsilon\Rightarrow a-\varepsilon<x_n.$$

Bu tengsizliklardan $\forall n>n_0$ bo'lganda

$$x_n>p$$

bo'lishi kelib chiqadi.

($a<q$ hol uchun ham teorema yuqoridagidek isbot etiladi).

3-teorema. Agar $\{x_n\}$ va $\{y_n\}$ ketma-ketlik yaqinlashuvchi bo'lib,

1) $\lim\limits_{n\to\infty}x_n=a,\quad \lim\limits_{n\to\infty}y_n=b;$

2) $\forall n\in N$ uchun $x_n\le y_n\ (x_n\ge y_n)$

bo'lsa u holda $a\le b$ $(a\ge b)$ bo'ladi.

Isbot. Shartga ko'ra

$$\lim_{n\to\infty}x_n=a,\qquad \lim_{n\to\infty}y_n=b.$$

Ketma-ketlik limiti ta'rifiga binoan:

$$\forall\varepsilon>0,\ \exists n_0'\in N,\ \ \forall n>n_0':\ |x_n-a|<\varepsilon,$$

$$\forall\varepsilon>0,\ \exists n_0''\in N,\ \ \forall n>n_0''\ \ |y_n-b|<\varepsilon$$

bo'ladi.

Agar $n_0=\max\{n_0',\ n_0''\}$ deyilsa, unda $\forall n>n_0$ uchun bir yo'la

$$|x_n-a|<\varepsilon,\quad |y_n-b|<\varepsilon$$

tengsizliklar bajariladi.

Ravshanki,

$$|x_n-a|<\varepsilon \quad\Leftrightarrow\quad a-\varepsilon<x_n<a+\varepsilon,$$

$$|y_n-b|<\varepsilon \quad\Leftrightarrow\quad b-\varepsilon<y_n<b+\varepsilon.$$

Bu tengsizliklardan hamda teoremaning 2-shartidan foydalanib topamiz:

$$a-\varepsilon<x_n\le y_n<b+\varepsilon.$$

Keyingi tengsizliklardan

$$a-\varepsilon<b+\varepsilon,\qquad a-b<2\varepsilon$$

va $\forall\varepsilon>0$ bo'lgani uchun $a-b\le0$, ya'ni $a\le b$ bo'lishi kelib chiqadi.

Xuddi shunga o'xshash, $\lim\limits_{n\to\infty}x_n=a,$ $\lim\limits_{n\to\infty}y_n=b$ hamda $\forall n\in N$ uchun $x_n\ge y_n$ bo'lishidan $a\ge b$ tengsizlik kelib chiqishi ko'rsatiladi.

4-teorema. Agar $\{x_n\}$ va $\{z_n\}$ ketma-ketlik yaqinlashuvchi bo'lib,

1) $\lim\limits_{n\to\infty} x_n = a, \quad \lim\limits_{n\to\infty} z_n = a$

2) $\forall n \in N$ uchun $x_n \le y_n \le z_n$

bo'lsa,u holda $\{y_n\}$ ketma-ketlik yaqinlashuvchi va

$$\lim_{n\to\infty} y_n = a$$

bo'ladi.

Isbot. Shartga ko'ra

$$\lim_{n\to\infty} x_n = a, \quad \lim_{n\to\infty} z_n = a.$$

Limit ta'rifiga binoan:

$$\forall \varepsilon > 0,\ \exists n_0' \in N,\ \forall n > n_0':\ |x_n - a| < \varepsilon,$$

$$\forall \varepsilon > 0,\ \exists n_0'' \in N,\ \forall n > n_0''\ \ |z_n - a| < \varepsilon$$

bo'ladi. Agar $n_0 = \max\{n_0', n_0''\}$ deyilsa,unda $\forall n > n_0$ uchun

$$a - \varepsilon < x_n, \quad z_n < a + \varepsilon$$

tengsizliklar bajariladi. Teoremaning 1-shartidan foydalanib topamiz:

$$a - \varepsilon < x_n \le y_n \le z_n < a + \varepsilon.$$

Keyingi tengsizliklardan

$$a - \varepsilon < y_n < a + \varepsilon, \quad \text{ya'ni} \quad |y_n - a| < \varepsilon$$

bo'lishi kelib chiqadi. Demak,

$$\lim_{n\to\infty} y_n = a.$$

Shuni isbotlash talab qilingan edi.

1-misol. Ushbu $\lim\limits_{n\to\infty} \sqrt[n]{n}$ limit topilsin.

Ravshanki, barcha $n \ge 2$ bo'lganda

$$\sqrt[2n]{n} > 1$$

bo'ladi. Aytaylik,

$$\sqrt[2n]{n} = 1 + \alpha_n$$

bo'lsin. Unda

$$\sqrt[n]{n} = (1 + \alpha_n)^2 \tag{1}$$

va $\sqrt{n} = (1 + \alpha_n)^2$ bo'ladi.

Bernulli tengsizligidan foydalanib topamiz:

$$\sqrt{n} = (1 + \alpha_n)^n \ge 1 + n \cdot \alpha_n > n \cdot \alpha_n. \tag{2}$$

(1) va (2) munosabatlardan

$$a_n < \frac{1}{\sqrt{n}},$$

va

$$1 < \sqrt[n]{n} < \left(1 + \frac{1}{\sqrt{n}}\right)^2$$

tengsizliklar kelib chiqadi. Agar

$$\lim_{n\to\infty}\left(1 + \frac{1}{\sqrt{n}}\right)^2 = 1$$

ekanini e'tiborga olsak, unda 4-teoremaga ko'ra

$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

bo'lishini topamiz.

2-misol. Ushbu $\lim\limits_{n\to\infty} \sqrt[n]{1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}}$

limit topilsin.

Ravshanki,

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} > \frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \ldots + \frac{1}{n} = n \cdot \frac{1}{n} = 1,$$

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} < 1 + 1 + 1 + \ldots + 1 = n.$$

Demak,

$$1 < \sqrt[n]{1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}} < \sqrt[n]{n}.$$

4-teoremadan foydalanib topamiz: $\lim\limits_{n\to\infty} \sqrt[n]{1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}} = 1.$

2⁰. Yaqinlashuvchi ketma-ketliklar ustida amallar. Faraz qilaylik, $\{x_n\}$ hamda $\{y_n\}$ ketma-ketliklar berilgan bo'lsin:

$$\{x_n\}:\ x_1, x_2, x_3, \ldots, x_n, \ldots$$

$$\{y_n\}:\ y_1, y_2, y_3, \ldots, y_n, \ldots$$

Quyidagi

$$x_1 + y_1,\ x_2 + y_2,\ x_3 + y_3,\ \ldots,\ x_n + y_n, \ldots$$

$$x_1 - y_1,\ x_2 - y_2,\ x_3 - y_3,\ \ldots,\ x_n - y_n, \ldots$$

$$x_1 \cdot y_1,\ x_2 \cdot y_2,\ x_3 \cdot y_3, \ldots, x_n \cdot y_n, \ldots$$

$$\frac{x_1}{y_1},\ \frac{x_2}{y_2},\ \frac{x_3}{y_3},\ \ldots,\ \frac{x_n}{y_n}, \ldots \quad (y_n \ne 0,\ \ n = 1, 2, 3, \ldots)$$

ketma-ketliklar mos ravishda $\{x_n\}$ va $\{y_n\}$ **ketma-ketlik-larning yig'indisi, ayirmasi, ko'paytmasi hamda nisbati** deyiladi va ular

$$\{x_n+y_n\},\quad \{x_n-y_n\},\quad \{x_n\cdot y_n\},\quad \left\{\frac{x_n}{y_n}\right\}$$

kabi belgilanadi.

5-teorema. Aytaylik $\{x_n\}$ va $\{y_n\}$ ketma-ketliklari berilgan bo'lib,

$$\lim_{n\to\infty} x_n = a,\qquad \lim_{n\to\infty} y_n = b,\qquad (a\in R,\ \ b\in R)$$

bo'lsin.U holda $n\to\infty$ da $(c\cdot x_n)\to c\cdot a$;

$$x_n+y_n\to a+b;\qquad x_n\cdot y_n\to ab;\qquad \frac{x_n}{y_n}\to\frac{a}{b}\ (b=0),$$ ya'ni

a) $\forall c\in R$ da $\lim\limits_{n\to\infty}(c\cdot x_n)=c\cdot\lim\limits_{n\to\infty} x_n$

b) $\lim\limits_{n\to\infty}(x_n+y_n)=\lim\limits_{n\to\infty} x_n+\lim\limits_{n\to\infty} y_n$;

c) $\lim\limits_{n\to\infty}(x_n\cdot y_n)=\lim\limits_{n\to\infty} x_n\cdot\lim\limits_{n\to\infty} y_n$;

d) $\lim\limits_{n\to\infty}\dfrac{x_n}{y_n}=\dfrac{\lim\limits_{n\to\infty} x_n}{\lim\limits_{n\to\infty} y_n},\quad (b\neq 0)$

bo'ladi.

Teoremaning tasdiqlaridan birini, masalan c)-ning isbotini keltiramiz.

Teoremaning shartiga ko'ra,

$$\lim_{n\to\infty} x_n = a,\qquad \lim_{n\to\infty} y_n = b.$$

Ravshanki,

$$|x_n\cdot y_n-ab|=|x_n\cdot y_n-a\cdot y_n+a\cdot y_n-b|\le$$
$$\le|x_n-a|\cdot|y_n|+|a|\cdot|y_n-b|. \tag{3}$$

$\{y_n\}$ ketma-ketlik yaqinlashuvchi bo'lganligi sababli u 1-teoremaga ko'ra chegaralangan bo'ladi:

$$\exists M>0,\quad \forall n\in N:\quad |y_n|\le M.$$

Ketma-ketlik limiti ta'rifidan foydalanib topamiz:

$\forall\varepsilon>0$ berilgan hamda $\dfrac{\varepsilon}{2M}$ ga ko'ra shunday $n_0'\in N$ topiladiki, $\forall n>n_0'$ uchun

$$|x_n-a|<\frac{\varepsilon}{2M}$$

bo'ladi.

Shuningdek, $\dfrac{\varepsilon}{2(1+|a|)}$ ga ko'ra shunday $n_0''\in N$ topiladiki, $\forall n>n_0''$ uchun

$$|y_n-b|<\frac{\varepsilon}{2(1+|a|)}$$

bo'ladi.

Agar $n_0=\max\{n_0',\ n_0''\}$ deyilsa,unda $\forall n>n_0$ uchun bir yo'la

$$|x_n-a|<\frac{\varepsilon}{2M},\qquad |y_n-b|<\frac{\varepsilon}{2(1+|a|)} \tag{4}$$

bo'ladi.

(3) va (4) munosabatlardan

$$|x_n\cdot y_n-ab|<\frac{\varepsilon}{2M}\cdot M+|a|\cdot\frac{\varepsilon}{2(1+|a|)}<\varepsilon$$

bo'lishi kelib chiqadi. Bu esa

$$\lim_{n\to\infty} x_n\cdot y_n=ab$$

bo'lishini bildiradi.

3^0. Cheksiz kichik hamda cheksiz katta miqdorlar. Faraz qilaylik, $\{\alpha_n\}$ ketma-ketlik berilgan bo'lsin.

2-ta'rif. Agar $\{\alpha_n\}$ ketma-ketlikning limiti nolga teng, ya'ni

$$\lim_{n\to\infty}\alpha_n=0$$

bo'lsa, $\{\alpha_n\}$ - cheksiz kichik miqdor deyiladi.

Masalan,

$$\alpha_n=\frac{1}{n}\quad \text{va}\quad \alpha_n=q^n,\ \ (|q|<1)$$

ketma-ketliklar cheksiz kichik miqdorlar bo'ladi.

Aytaylik, $\{x_n\}$ ketma-ketlik yaqinlashuvchi bo'lib, uning limiti a ga teng bo'lsin:

$$\lim_{n\to\infty} x_n=a.$$

U holda $\alpha_n=x_n-a$ cheksiz kichik miqdor bo'ladi. Keyingi tenglikdan topamiz: $x_n=a+\alpha_n$. Bundan esa quyidagi muhim xulosa kelib chiqadi:

$\{x_n\}$ ketma-ketlikning a $(a\in R)$ limitga ega bo'lishi uchun $\alpha_n=x_n-a$ ning cheksiz kichik miqdor bo'lishi zarur va yetarli.

Ketma-ketlikning limiti ta'rifidan foydalanib quyidagi ikkita lemmani isbotlash qiyin emas.

1-lemma. Chekli sondagi cheksiz kichik miqdorlar yigindisi cheksiz kichik miqdor bo'ladi.

2-lemma. Chegaralangan miqdor bilan cheksiz kichik miqdor ko'paytmasi cheksiz kichik miqdor bo'ladi.

3-ta'rif. Agar har qanday M soni olinganda ham shunday natural n_0 soni topilsaki, barcha $n > n_0$ uchun

$$|x_n| > M$$

tengsizlik bajarilsa, $\{x_n\}$ ketma-ketlikning **limiti cheksiz** deyiladi va

$$\lim_{n\to\infty} x_n = \infty$$

kabi belgilanadi.

Agar $\{x_n\}$ ketma-ketlikning limiti cheksiz bo'lsa, $\{x_n\}$cheksiz katta miqdor deyiladi.

Masalan,

$$x_n = (-1)^n \cdot n$$

ketma-ketlik cheksiz katta miqdor bo'ladi.

Endi cheksiz kichik va cheksiz katta miqdorlar orasidagi boђlanishni ifodalovchi tasdiqlarni keltiramiz:

1) Agar $\{x_n\}$ cheksiz kichik miqdor $(x_n \neq 0)$ bo'lsa, u holda $\left\{\frac{1}{x_n}\right\}$ cheksiz katta miqdor bo'ladi.

2) Agar $\{x_n\}$ cheksiz katta miqdor bo'lsa, u holda $\left\{\frac{1}{x_n}\right\}$ cheksiz kichik miqdor bo'ladi.

2.3. Monoton ketma-ketliklar va ularning limiti

1⁰. Monoton ketma-ketlik tushunchasi. Aytaylik, $\{x_n\}$:

$$x_1, x_2, ..., x_n, ... \qquad (1)$$

ketma-ketlik berilgan bo'lsin.

1-ta'rif. Agar (1) ketma-ketlikda $\forall n \in N$ uchun $x_n \le x_{n+1}$ tengsizlik bajarilsa, $\{x_n\}$ o'suvchi ketma-ketlik deyiladi. Agar (1) ketma-ketlikda $\forall n \in N$ uchun $x_n < x_{n+1}$ tengsizlik bajarilsa, $\{x_n\}$ **qat'iy o'suvchi** ketma-ketlik deyiladi.

2-ta'rif. Agar (1) ketma-ketlik $\forall n \in N$ uchun $x_n \ge x_{n+1}$ tengsizlik bajarilsa $\{x_n\}$ **kamayuvchi ketma-ketlik** deyiladi. Agar (1) ketma-ketlikda $\forall n \in N$ uchun $x_n > x_{n+1}$ tengsizlik bajarilsa, $\{x_n\}$ **qat'iy kamayuvchi ketma-ketlik** deyiladi.

1-misol. Ushbu $x_n = \frac{n+1}{n}$: $\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, ...$ ketma-ketlik qat'iy kamayuvchi ketma-ketlik bo'ladi.

Haqiqatdan ham, berilgan ketma-ketlik uchun

$$x_n = \frac{n+1}{n}, \quad x_{n+1} = \frac{n+2}{n+1}$$

bo'lib, $\forall n \in N$ uchun

$$x_{n+1} - x_n = \frac{n+2}{n+1} - \frac{n+1}{n} = \frac{-1}{n(n+1)} < 0$$

bo'ladi. Unda $x_{n+1} < x_n$ bo'lishi kelib chiqadi.

Yuqoridagi ta'riflardan quyidagi xulosalar kelib chiqadi:

1) agar $\{x_n\}$ ketma-ketlik o'suvchi bo'lsa, u quyidan chegaralangan bo'ladi:

2) agar $\{x_n\}$ ketma-ketlik kamayuvchi bo'lsa, u yuqoridan chegaralangan bo'ladi.

O'suvchi hamda kamayuvchi ketma-ketliklar umumiy nom bilan monoton ketma-ketliklar deyiladi.

2-misol. Ushbu $x_n = \frac{n^2}{n^2+1}$ $\quad (n = 1,2,3,...)$ ketma-ketlikning qat'iy o'suvchi ekanligi isbotlansin.

Bu ketma-ketlikning $n-$hamda $(n+1)-$hadlari uchun

$$x_n = \frac{n^2}{n^2+1} = 1 - \frac{1}{n^2+1},$$

$$x_{n+1} = \frac{(n+1)^2}{(n+1)^2+1} = 1 - \frac{1}{(n+1)^2+1}$$

bo'ladi. Ravshanki,

$$\frac{1}{(n+1)^2} < \frac{1}{n^2}.$$

Shu tengsizlikni e'tiborga olib, topamiz:

$$x_{n+1} = 1 - \frac{1}{(n+1)^2+1} > 1 - \frac{1}{n^2+1} = x_n.$$

Demak, $\forall n \in N$ uchun $x_n < x_{n+1}$. Demak, uchun . Bu esa qaralayotgan ketma-ketlikning qat'iy o'suvchi bo'lishini bildiradi.

2^0. Monoton ketma-ketlikninshg limiti. Quyida mono-ton ketma-ketliklarning limiti haqidagi teoremalarni keltiramiz.

1-teorema. Agar $\{x_n\}$ ketma-ketlik

1) o'suvchi,

2) yuqoridan chegaralangan bo'lsa, u chekli limitga ega bo'ladi.

Aytaylik, $\{x_n\}$ ketma-ketlik teoremaning ikkala shartlarini bajarsin. Bu ketma-ketlikning barcha hadlaridan iborat to'plamni E bilan belgilaymiz:

$$E = \{x_1, x_2, ..., x_n, ...\}.$$

Ravshanki, E yuqoridan chegaralangan to'plam bo'lib, $E \neq \varnothing$. Unda to'plamning aniq chegarasining mavjudligi haqidagi teoremaga muvofiq, $\sup E$ mavjud bo'ladi. Uni a bilan belgilaylik:

$$\sup E = a.$$

Ixtiyoriy $\varepsilon > 0$ sonini olaylik. To'plamning aniq yuqori chegarasi ta'rifiga binoan:

1) $\forall n \in N$ uchun $x_n \le a$

2) $\exists x_{n_0} \in E, \quad x_{n_0} > a - \varepsilon$

bo'ladi. Ayni paytda $\forall n > n_0$ uchun $x_n \ge x_{n_0}$ tengsizlik bajarilib, $x_n > a - \varepsilon$ bo'ladi.

Natijada $\forall n > n_0$ uchun $a - \varepsilon < x_n < a + \varepsilon$ ya'ni $|a - x_n| < \varepsilon$ bo'lishini topamiz.

Demak $\{x_n\}$ ketma-ketlik chekli limitga ega va

$$\lim_{n\to\infty} x_n = a = \sup E.$$

2-teorema. Agar $\{x_n\}$ ketma-ketlik

1) kamayuvchi,

2) quyidan chegaralangan bo'lsa, u chekli limitga ega bo'ladi.

Bu teorema yuqorida keltirilgan teoremaning isboti kabi isbotlanadi.

3-misol. Ushbu

$$x_n = \frac{n!}{n^n}$$

ketma-ketlikning limiti topilsin.

Ravshanki, $\forall n \ge 1$ uchun $x_n > 0$ bo'ladi. Bu ketma-ketlikning x_{n+1} va x_n hadlarining nisbatini qaraymiz:

$$\frac{x_{n+1}}{x_n} = \frac{(n+1)!}{(n+1)^{n+1}} : \frac{n!}{n^n} = \frac{n+1}{(n+1)^{n+1}} \cdot n^n = \left(\frac{n}{n+1}\right)^n < 1.$$

Demak, $x_{n+1} < x_n$. Bundan esa berilgan ketma-ketlikning kamayuvchi ekanligi kelib chiqadi.

Ayni paytda $\forall n \ge 1$ da

$$0 < x_n \le x_1$$

munosabat o'rinli bo'ladi. Demak berilgan ketma-ketlik chegaralangan. 1-teoremaga ko'ra $\{x_n\}$ ketma-ketlik chekli limitga ega. Uni a bilan belgilaymiz:

$$\lim_{n\to\infty} \frac{n!}{n^n} = a. \qquad (a \ge 0)$$

Endi ushbu $x_n - x_{n+1}$ ayirmani qaraymiz. Bu ayirma uchun

$$x_n - x_{n+1} = x_n - x_n \cdot \frac{n^n}{(n+1)^n} = x_n \frac{(n+1)^n - n^n}{(n+1)^n} \ge$$

$$\ge x_n \cdot \frac{2n^n - n^n}{(n+1)^n} = x_n \cdot \frac{n^n}{(n+1)^n} = x_{n+1}$$

bo'lib,unda

$$x_n \ge 2x_{n+1}$$

bo'lishi kelib chiqadi. Keyingi munosabatlardan topamiz:

$$\lim_{n\to\infty} x_n \ge 2 \lim_{n\to\infty} x_{n+1}, \qquad a \ge 2a.$$

Ravshanki, bu holda $a = 0$ bo'ladi.

Demak,

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0.$$

3^0. e soni. Ushbu

$$x_n = \left(1 + \frac{1}{n}\right)^n, \qquad (n = 1, 2, 3, ...) \qquad (1)$$

ketma-ketlikni qaraymiz.

Tasdiq. (1) ketma-ketlik o'suvchi bo'ladi.

Berilgan ketma-ketlikning x_{n+1} hamda x_n hadlarining nisbatini topamiz:

$$\frac{x_{n+1}}{x_n}=\left(1+\frac{1}{n+1}\right)^{n+1}:\left(1+\frac{1}{n}\right)^n=\frac{(n+2)^{n+1}\cdot n^n}{(n+1)^{n+1}\cdot(n+1)^n}=$$

$$=\left[\frac{n^2+2n}{(n+1)^2}\right]^{n+1}\cdot\frac{n+1}{n}=\left[1-\frac{1}{(n+1)^2}\right]^{n+1}\cdot\frac{n+1}{n}.$$

Bernulli tengsizligiga ko'ra:

$$\left[1-\frac{1}{(n+1)^2}\right]^{n+1}>1-\frac{n+1}{(n+1)^2}=\frac{n}{n+1}$$

bo'ladi.

Natijada $\forall n\in N$ uchun

$$\frac{x_{n+1}}{x_n}>\frac{n}{n+1}\cdot\frac{n+1}{n}=1$$

ya'ni, $x_{n+1}>x_n$ ya'ni, bo'lishi kelib chiqadi.

Tasdiq. (1) ketma-ketlik chegaralangan bo'ladi.

Ravshanki, $k\geq 1$ uchun

$$k!=1\cdot 2\cdot 3\cdot\ldots\cdot(k-1)\cdot k\geq 2\cdot 2\cdot 2\cdot\ldots\cdot 2=2^{k-1}$$

bo'ladi.

Endi Nyuton binomi formulasidan foydalanib topamiz:

$$x_n=\left(1+\frac{1}{n}\right)^n=1+C_n^1\cdot\frac{1}{n}+C_n^2\cdot\frac{1}{n^2}+\ldots+C_n^k\cdot\frac{1}{n^k}+\ldots+C_n^n\cdot\frac{1}{n^n}=2+\sum_{k=2}^{n}\frac{1}{n^k}C_n^k=$$

$$=2+\sum_{k=2}^{n}\frac{1}{k!}\cdot\frac{n(n-1)\ldots(n-k+1)}{n^k}=2+\sum_{k=2}^{n}\frac{1}{k!}\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(\frac{n-1}{n}\right)\leq$$

$$\leq 2+\sum_{k=2}^{n}\frac{1}{2^{k-1}}=3-\frac{1}{2^{n-1}}<3$$

Demak, $\forall n\in N$ uchun $0<x_n<3$ bo'ladi.

Monoton ketma-ketlikning limiti haqidagi teoremaga ko'ra

$$x_n=\left(1+\frac{1}{n}\right)^n$$

ketma-ketlik chekli limitga ega.

3-ta'rif. (1) ketma-ketlikning limiti e soni deyiladi:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e.$$

Bu e soni irratsional son bo'lib,

$$e=2{,}7182818284\ 59045\ldots$$

bo'ladi.

2.4. Fundamental ketma-ketliklar. Koshi teoremasi

1°. Qismiy ketma-ketliklar. Boltsano- Veyershtrass teoremasi. Aytaylik,

$$\{x_n\}:\ x_1,x_2,x_3,\ldots,x_n,\ldots \qquad (1)$$

ketma-ketlik berilgan bo'lsin. Bu (1) ketma-ketlikning biror n_1 nomerli x_{n_1} hadini olamiz. So'nra nomeri n_1 dan katta bo'lgan n_2 nomerli x_{n_2} hadini olamiz. Shu usul bilan x_{n_3}, x_{n_4} va h.k. hadlarni tanlab olamiz. Natijada nomerlari

$$n_1<n_2<n_3<\ldots<n_k<\ldots$$

tengsizliklarni qanoatlantiruvchi (1) ketma-ketlikning hadlari ushbu

$$x_{n_1},\ x_{n_2},\ \ldots,\ x_{n_k},\ \ldots \qquad (2)$$

ketma-ketlikni hosil qiladi.

(2) ketma-ketlik (1) ketma-ketlikning qismiy ketma-ketligi deyiladi va $\{x_{n_k}\}$ kabi belgilanadi.

Masalan,

$$2,\ 4,\ 6,\ 8,\ \ldots,$$
$$1,\ 3,\ 5,\ 7,\ \ldots,$$
$$1,\ 4,\ 9,\ 16,\ \ldots$$

ketma-ketlik $1,2,3,4,\ldots,n,\ldots$ ketma-ketlikning qismiy ketma-ketliklari,

$$1,\ 1,\ 1,\ \ldots,\ 1,\ \ldots,$$
$$-1,\ -1,\ \ldots,\ -1,\ \ldots$$

ketma-ketliklar $1,-1,1,-1,\ldots,(-1)^{n+1},\ldots$ ketma-ketliklar ketma-ketlikning qismiy ketma-ketliklari bo'ladi.

Keltirilgan tushuncha va misollardan bitta ketma-ketlikning turli qismiy ketma-ketliklari bo'lishi mumkinligi ko'rinadi.

1-teorema. Agar $\{x_n\}$ ketma-ketlik limitga ega bo'lsa, uning har qanday qismiy ketma-ketligi ham shu limitga ega bo'ladi.

Bu teoremaning isboti ketma-ketlik limiti ta'rifidan kelib chiqadi.

Eslatma. Ketma-ketlik qismiy ketma-ketliklarining limiti mavjud bo'lishidan berilgan ketma-ketlikning limiti mavjud bo'lishi har doim ham kelib chiqavermaydi.

Masalan, $1,-1,1,-1,\ldots,(-1)^{n+1},\ldots$ ketma-ketlikning qismiy ketma-ketliklari

$$1, 1, 1, \ldots, 1, \ldots,$$
$$-1, -1, \ldots, -1, \ldots$$
larning limiti bo'lgan holda ketma-ketlikning o'zining limiti mavjud emas.

2-teorema (Boltsano-Veyershtrass teoremasi). Har qan-day chegaralangan ketma-ketlikdan chekli songa intiluvchi qismiy ketma-ketlik ajratish mumkin.

$\{x_n\}$ ketma-ketlik berilgan bo'lib, u chegaralangan bo'lsin: Bu holda $\{x_n\}$ ketma-ketlikning barcha hadlari $[a,b]$ da joylashgan deb qarash mumkin:
$$x_n \in [a, b], \quad n=1, 2, 3, \ldots$$
$[a, b]$ segmenti
$$\left[a, \ \frac{a+b}{2}\right], \quad \left[\frac{a+b}{2}, \ b\right]$$
segmentlarga ajratamiz. $\{x_n\}$ ketma-ketlikning cheksiz ko'p hadlari joylashganini $[a_1, b_1]$ deymiz.Ravshanki, $[a_1, b_1]$ ning uzunligi $\frac{b-a}{2}$ ga teng bo'ladi. Yuqoridagiga o'xshash $[a_1, b_1]$ segmentni
$$\left[a_1, \ \frac{a_1+b_1}{2}\right], \quad \left[\frac{a_1+b_1}{2}, \ b_1\right]$$
segmentlarga ajratamiz. Berilgan ketma-ketlikning cheksiz ko'p sondagi hadlari bo'lganini $[a_2, b_2]$ deymiz.Bunda $[a_2, b_2]$ ning uzunligi $\frac{b-a}{2^2}$ ga teng bo'ladi.

Bu jarayonni davom ettirish natijasida ushbu
$$[a_1, b_1], [a_2, b_2], \ \ldots, \ [a_k, b_k], \ldots$$
segmentlar ketma-ketligi hosil bo'ladi. Bu segmentlar ketma-ketligi uchun $[a_1, b_1] \supset [a_2, b_2] \supset \ldots \supset [a_k, b_k] \supset \ldots$ bo'lib, $k \to \infty$ da
$$b_k - a_k = \frac{b-a}{2^k} \to 0$$
bo'ladi.

Ichma-ich joylashgan segmentlar printsipiga ko'ra
$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = C \qquad (C \in R)$$
bo'ladi.

Endi $\{x_n\}$ ketma-ketlikning $[a_1, b_1]$ dagi birorta x_{n_1} hadini, $[a_2, b_2]$ dagi birorta x_{n_2} hadinini va h.k. $[a_k, b_k]$ dagi birorta x_{n_k} hadini va h.k. hadlarini olamiz. Natijada $\{x_n\}$ ketma-ketlikning hadlaridan tashkil topgan ushbu
$$x_{n_1}, \ x_{n_2}, \ldots, \ x_{n_k}, \ldots \qquad (n_1 < n_2 < \ldots < n_k < \ldots)$$

qismiy ketma-ketlik hosil bo'ladi. Bu ketma-ketlik uchun
$$a_k \le x_{n_k} \le b_k \qquad (k=1, 2, \ldots)$$
bo'lib,unda $k \to \infty$ da $x_{n_k} \to C$ ya'ni $\lim_{k\to\infty} x_{n_k} = C$ bo'lishi kelib chiqadi.

2°. Fundamental ketma-ketliklar. Koshi teoremasi. $\{x_n\}$ ketma-ketlik berilgan bo'lsin.

1-ta'rif. Agar har qanday $\varepsilon > 0$ olinganda ham shunday natural n_0 soni topilsaki, barcha $n > n_0$ va $m > n_0$ uchun
$$|x_n - x_m| < \varepsilon$$
tengsizlik bajarilsa (ya'ni $\forall\varepsilon > 0, \ \exists n_0 \in N, \ \forall n > n_0, \quad \forall m > n_0 : |x_n - x_m| < \varepsilon$ bo'lsa), $\{x_n\}$ fundamental ketma-ketlik deyiladi.

Masalan, $x_n = \frac{n}{n+1} \qquad (n=1,2,\ldots)$

fundamental ketma-ketlik bo'ladi.

Haqiqatdan ham, berilgan ketma-ketlik uchun
$$|x_n - x_m| = \left|\frac{n}{n+1} - \frac{m}{m+1}\right| < \frac{n+m}{nm} = \frac{1}{n} + \frac{1}{m}$$
bo'lib, $\forall\varepsilon > 0$ ga ko'ra $n_0 = \left[\frac{2}{\varepsilon}\right] + 1$ deyilsa, $\forall n > n_0, \ \ \forall m > m_0$ bo'lganda
$$|x_n - x_m| < \frac{1}{n_0} + \frac{1}{n_0} < \varepsilon$$
bo'ladi.

3-teorema. (Koshi teoremasi). Ketma-ketlikning yaqinlashuvchi bo'lishi uchun uning fundamental bo'lishi zarur va yetarli.

Zarurligi. $\{x_n\}$ ketma-ketlik yaqinlashuvchi bo'lib, $\lim_{n\to\infty} x_n = a$ bo'lsin. Limit ta'rifiga binoan
$$\forall\varepsilon > 0, \ \exists n_0 \in N, \ \forall n > n_0 : |x_n - a| < \frac{\varepsilon}{2}.$$
Shuningdek, $\forall m > n_0 : |x_m - a| < \frac{\varepsilon}{2}$ bo'ladi.Natijada $\forall n > n_0, \ \ \forall m > n_0$ uchun
$$|x_n - x_m| = |x_n - a + a - x_m| \le |x_n - a| + |x_m - a| < \varepsilon$$
bo'lishi kelib chiqadi. Demak, $\{x_n\}$ fundamental ketma-ketlik.

Yetarliligi. $\{x_n\}$ fundamental ketma-ketlik bo'lsin:
$$\forall\varepsilon > 0, \ \exists n_0 \in N, \ \forall n > n_0, \forall m > n_0 : |x_n - x_m| < \varepsilon.$$

Agar $m > n_0$ shartini qanotlantiruvchi m tayinlansa,unda

$$|x_n - x_m| < \varepsilon \Leftrightarrow x_m - \varepsilon < x_n < x_m + \varepsilon$$

bo'lib, $\{x_n\}$ ketma-ketlikning chegaralanganligi kelib chiqadi.

Boltsano-Veyershrass teoremasiga binoan bu ketma-ketlikdan yaqinlashuvchi qismiy $\{x_{n_k}\}$ ketma-ketlikni ajratish mumkin: $\lim\limits_{n\to\infty} x_{n_k} = a.$

Demak,

$$\forall \varepsilon > 0,\ \exists k_0 \in N,\ \forall k > k_0 : |x_{n_k} - a| < \varepsilon$$

bo'ladi.

Agar $m = n_k$ deyilsa,unda

$$|x_n - x_{n_k}| < \varepsilon$$

bo'ladi. Keyingi ikki tengsizliklardan

$$|x_n - a| = |x_n - x_{n_k} + x_{n_k} - a| \le |x_n - x_{n_k}| + |x_{n_k} - a| < 2\varepsilon$$

bo'lishi kelib chiqadi. Demak, $\lim\limits_{n\to\infty} x_n = a.$

3^0. Ketma-ketlikning quyi hamda yuqori limitlari. $\{x_n\}$ ketma-ketlik berilgan bo'lsin. Bu ketma-ketlikning qismiy ketma-ketligining limiti $\{x_n\}$ ning qismiy limiti deyiladi.

2-ta'rif. $\{x_n\}$ ketma-ketlik qismiy limitlarining eng kattasi berilgan **ketma-ketlikning yuqori limiti** deyiladi va

$$\overline{\lim_{n\to\infty}} x_n$$

kabi belgilanadi.

$\{x_n\}$ ketma-ketlik qismiy limitlarining eng kattasi berilgan **ketma-ketlikning quyi limiti** deyiladi va

$$\underline{\lim_{n\to\infty}} x_n$$

kabi belgilanadi.

Masalan, ushbu $\{x_n\}$: 1, 2, 3, 1, 2, 3, 1, 2, 3,... ketma-ketlikning yuqori limiti

$$\overline{\lim_{n\to\infty}} x_n = 3,$$

quyi limiti esa

$$\underline{\lim_{n\to\infty}} x_n = 1$$

bo'ladi.Umuman, $\{x_n\}$ ketma-ketlikning quyi hamda yuqori limitlari quyidagicha ham kiritilishi mumkin.

Aytaylik, $\{x_n\}$ ketma-ketlik berilgan bo'lib, A bu ketma-ketlikning qismiy limitlaridan iborat to'plam bo'lsin. Unda bu ketma-ketlikning quyi limitini

$$\underline{\lim_{n\to\infty}} x_n = \lim_{n\to\infty}\inf x_n = \begin{cases} -\infty; \{x_n\} \text{ agar chegaralanmagan bo'lsa} \\ \inf A; \text{quydagi chegaralangan bo'lsa va } A \ne (-\infty) \\ +\infty;\ A = \{+\infty\} \text{ bo'lsa} \end{cases}$$

deb olishimiz mumkin

$\{x_n\}$ ketma-ketlikning yuqori limitini esa

$$\overline{\lim_{n\to\infty}} x_n = \lim_{n\to\infty}\sup x_n = \begin{cases} -\infty; \{x_n\} \text{ agar chegaralanmagan bo'lsa} \\ \sup A; \text{quydagi chegaralangan bo'lsa va } A \ne (-\infty) \\ +\infty;\ A = \{+\infty\} \text{ bo'lsa} \end{cases}$$

deb qarash mumkin.

Endi quyi hamda yuqori limitlarning xossalarini keltiramiz.

Biror $\{x_n\}$ ketma-ketlik uchun $\overline{\lim\limits_{n\to\infty}} x_n = a$ bo'lsin.U holda $\forall \varepsilon > 0$ olinganda ham:

1) shunday $n_0 \in N$ topiladiki, $\forall n > n_0$ da $x_n < a + \varepsilon$

2) $\forall n_1 \in N$ uchun ε va n_1 larga bog'liq shunday $n' > n_1$ topiladiki, $x_{n'} > a - \varepsilon$ bo'ladi.

Bu xossalar quyidagilarni anglatadi: $\forall \varepsilon > 0$ tayin olganda, birinchi xossa $\{x_n\}$ ketma-ketlikning faqatgina chekli sondagi hadlarigina

$$x_n < a + \varepsilon$$

tengsizlikni qanoatlantirishini, ikkinchi xossa esa bu ketma-ketlikning

$$x_n > a - \varepsilon$$

tengsizlikni qanoatlantiruvchi hadlarining soni cheksiz ko'p bo'lishini ifodalaydi.

Agar $\{x_n\}$ ning cheksiz ko'p sondagi hadlari $a + \varepsilon$ dan katta bo'lsa, u holda $a + \varepsilon$ sonidan kichik bo'lmagan $b\ (b \ge a + \varepsilon)$ ga intiluvchi $\{x_n\}$ ketma-ketlikning qismiy ketma-ketligi mavjud va bu $\overline{\lim\limits_{n\to\infty}} x_n = a$ ga zid.

Demak, $a + \varepsilon$ dan o'ngda ketma-ketlikning ko'pi bilan chekli sondagi hadlari yotadi.

Modomiki,

$$\overline{\lim_{n\to\infty}} x_n = a$$

ekan,unda $\{x_n\}$ unda ning qismiy limitlaridan biri a ga teng:

$$\lim_{k\to\infty} x_{n_k} = a$$

Limit ta'rifiga ko'ra bu $\{x_{n_k}\}$ ketma-ketlikning, demak, $\{x_n\}$ ning ham cheksiz ko'p sondagi hadlari $a-\varepsilon$ dan katta bo'ladi.

Eslatma. Biror a soni yuqoridagi ikki shartni qanoat-lantirsa, u $\{x_n\}$ ketma-ketlikning yuqori limiti bo'ladi.

Faraz qilaylik, biror $\{x_n\}$ ketma-ketlik uchun

$$\underline{\lim}_{n\to\infty} x_n = b$$

bo'lsin.U holda $\forall\varepsilon > 0$ olinganda ham:

1') shunday $n_0 \in N$ topiladiki, $\forall n > n_0$ da $x_n > b-\varepsilon$

2') $\forall n_1 \in N$ uchun ε va n_1 larga bog'liq shunday $n' > n_1$ topiladiki, $x_{n'} < b+\varepsilon$ bo'ladi.

Quyi limitning bu xossasi yuqoridagidek isbotlanadi.

Ketma-ketlikning quyi hamda yuqori limitlari xossalaridan foydalanib, quyidagi teoremani isbotlash qiyin emas:

4-teorema. $\{x_n\}$ ketma-ketlik C limitga ega bo'lishi uchun

$$\underline{\lim}_{n\to\infty} x_n = \overline{\lim}_{n\to\infty} x_n = C$$

bo'lishi zarur va yetarlidir.

Berilgan ketma-ketliklarning yuqori va quyi limitini toping(1-15). Limitni hisoblang (16-30).

1. $x_n = 1-\frac{1}{n}$
2. $x_n = \frac{(-1)^n}{n} + \frac{1+(-1)^n}{2}$
3. $x_n = 1+\frac{n}{n+1}\cos\frac{n\pi}{2}$
4. $x_n = 1+2(-1)^{n+1}+3(-1)^{\frac{n(n-1)}{2}}$
5. $x_n = \frac{n-1}{n+1}\cos\frac{2n\pi}{3}$
6. $x_n = (-1)^n n$
7. $x_n = 1+n\sin\frac{n\pi}{2}$
8. $x_n = -n[2+(-1)^n]$

16. $\lim\limits_{n\to\infty}\frac{10000n}{n^2+1}$
17. $\lim\limits_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)$
18. $\lim\limits_{n\to\infty}\frac{\sqrt[3]{n^2}\sin n!}{n+1}$
19. $\lim\limits_{n\to\infty}\frac{(-2)^n+3^n}{(-2)^{n+1}+3^{n+1}}$
20. $\lim\limits_{n\to\infty}\frac{1+a+a^2+...+a^n}{1+b+b^2+...+b^n}, (|a|<1, |b|<1)$
21. $\lim\limits_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+...+\frac{n-1}{n^2}\right)$
22. $\lim\limits_{n\to\infty}\left|\frac{1}{n}-\frac{2}{n}+\frac{3}{n}-....+\frac{(-1)^{n-1}n}{n}\right|$
23. $\lim\limits_{n\to\infty}\left[\frac{1^2}{n^3}+\frac{2^2}{n^3}+....+\frac{(n-1)^2}{n^3}\right]$

9. $x_n = n^{(-1)^n}$
10. $x_n = \frac{1}{n-10,2}$
11. $x_n = \frac{n^2}{1+n}\cos\frac{2n\pi}{3}$
12. $x_n = \left(1+\frac{1}{n}\right)^n\cdot(-1)^n+\sin\frac{\pi n}{4}$
13. $x_n = \frac{n}{n+1}\sin^2\frac{n\pi}{4}$
14. $x_n = \sqrt[n]{1+2^{n\cdot(-1)^n}}$
15. $x_n = \cos^n\frac{2n\pi}{3}$

24. $\lim\limits_{n\to\infty}\left[\frac{1^2}{n^3}+\frac{3^2}{n^3}+....+\frac{(2n-1)^2}{n^3}\right]$
25. $\lim\limits_{n\to\infty}\left(\frac{1}{2}+\frac{3}{2^2}+\frac{5}{2^3}+....+\frac{2n-1}{2^n}\right)$
26. $\lim\limits_{n\to\infty}\left[\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+....+\frac{1}{n(n+1)}\right]$
27. $\lim\limits_{n\to\infty}\left(\sqrt{2}\cdot\sqrt[4]{2}\cdot\sqrt[8]{2}\cdot.....\sqrt[2n]{2}\right)$
28. $\lim\limits_{n\to\infty}\frac{\sin 1}{2}+\frac{\sin 2}{2^2}+....+\frac{\sin n}{2^n}$
29. $\lim\limits_{n\to\infty}\frac{\cos 1!}{1\cdot 2}+\frac{\cos 2!}{2\cdot 3}+......+\frac{\cos n!}{n(n+1)}$
30. $\lim\limits_{n\to\infty}1+\frac{1}{2^2}+\frac{1}{3^2}+.....+\frac{1}{n^2}$

III BOB
FUNKSIYA VA UNING LIMITI

3.1. Funksiya tushunchasi

1^0. Funksiya ta'rifi, berilish usullari. Biz yuqorida E to'plamni F to'plamga akslantirish

$$f:\ E \to F$$

ni o'rgangan edik.

Endi $E=F$, $F=R$ deb olamiz. Unda har bir haqiqiy x songa biror haqiqiy y sonni mos qo'yuvchi

$$f:\ F \to R \quad (x \overset{f}{\to} y)$$

akslantirishga kelamiz. Bu esa funktsiya tushunchasiga olib keladi.

Funksiya tushunchasi o'quvchiga o'rta maktab matematika kursidan ma'lum. Shuni e'tiborga olib funksiya haqidagi dastlabki ma'lumotlarni qisqaroq bayon etishni lozim topdik.

Aytaylik, $X \subset R, Y \subset R$ to'plamlar berilgan bo'lib, x va y va o'zgaruvchilar mos ravishda shu to'plamlarda o'zgarsin: $x \in X$, $y \in Y$.

1-ta`rif. Agar X to'plamdagi har bir x songa biror f qoidaga ko'ra Y to'plamdan bitta y son mos qo'yilgan bo'lsa, X to'plamda **funksiya berilgan (aniqlangan)** deyiladi va $f: x \to y$ yoki $y=f(x)$ kabi belgilanadi. Bunda X - funksiyaning aniqlanish to'plami (sohasi), Y - **funksiyaning o'zgarish to'plami (sohasi) deyiladi.** x - **erkli o'zgaruvchi** yoki **funksiya argumenti**, y esa **erksiz o'zgaruvchi** yoki **funksiya** deyiladi.

Misollar. 1. $X=(-\infty,+\infty)$, $Y=(0,+\infty)$ bo'lib, f qoida

$$f: x \to y = x^2+1$$

bo'lsin. Bu holda har bir $x \in X$ ga bitta $x^2+1 \in Y$ mos qo'yilib,

$$y = x^2+1$$

funksiyaga ega bo'lamiz.

2. Har bir ratsional songa 1 ni, har bir irratsional songa 0 ni mos qo'yish natijasida funksiya hosil bo'ladi. Odatda, bu **Dirixle funksiyasi** deyilib, u $D(x)$ kabi belgilanadi:

$$D(x)=\begin{cases} 1, \text{ agar } x \text{ ratsional son bo'lsa} \\ 0, \text{ agar } x \text{ irratsional son bo'lsa} \end{cases}$$

Shunday qilib, $y=f(x)$ funksiya uchta: X to'plam, Y to'plam har bir $x \in X$ ga bitta $y \in Y$ ni mos qo'yuvchi f qoidaning berilishi bilan aniqlanar ekan.

Faraz qilaylik, $y=f(x)$ funksiya $X \subset R$ to'plamda berilgan bo'lsin. $x_0 \in X$ nuqtaga mos keluvchi y_0 miqdor $y=f(x)$ **funksiyaning $x=x_0$ nuqtadagi xususiy qiymati** deyiladi va $f(x_0)=y_0$ kabi belgilanadi.

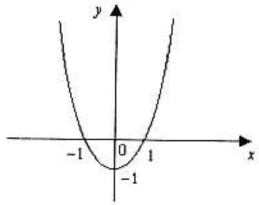
Tekislikda dekart koordinatalar sistemasini olamiz. Tekislikdagi $(x,f(x))$ nuqtalardan iborat ushbu

$$\{(x,f(x))\}=\{(x,f(x)) \mid x\in X, f(x)\in Y\}$$

to'plam $y=f(x)$ to'plam funksiyaning grafigi deyiladi. Masalan,

$$y=x^2-1 \quad (x \in X = [-2,2])$$

funksiyaning grafigi 1-chizmada tasvirlangan.



1-chizma.

Funksiya ta`rifidagi f qoida turlicha bo'lishi mumkin.

a) Ko'pincha x va y o'zgaruvchilar orasidagi bog'lanish formulalar yordamida ifodalanadi. Bu funksiyaning analitik usulda berilishi deyiladi. Masalan,

$$y=\sqrt{1-x^2}$$

funksiya analitik usulda berilgan bo'lib, uning aniqlanish to'plami

$$X=\{x \in R \mid -1 \le x \le 1\}=[-1,1]$$

bo'ladi.

x va y o'zgaruvchilar orasidagi bog'lanish quyidagi formulalar yordamida berilgan bo'lsin:

$$y=f(x)=\begin{cases} 1, \text{ agar } x>0 \text{ bo'lsa,} \\ -1, \text{agar } x<0 \text{ bo'lsa.} \end{cases}$$

Bu funksiyaning aniqlanish to'plami $X=R\setminus\{0\}$ bo'lib, qiymatlar to'plami esa $Y=\{-1,1\}$ bo'ladi. Odatda bu funksiya $y=\operatorname{sign}x$ kabi belgilanadi.

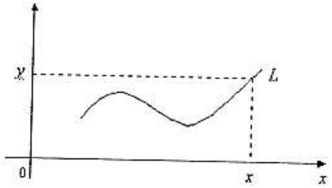
b) Ba'zi hollarda $x \in X$, $y \in Y$ o'zgaruvchilar orasidagi bog'lanish jadval orqali bo'lishi mumkin. Masalan, kun davomida havo haroratini kuzatganimizda, t_1

vaqtda havo harorati T_1, t_2 vaqtda havo harorati T_2 va h.k. bo'lsin. Natijada quyidagi jadval hosil bo'ladi.

$t-$ вақт	t_1	t_2	t_3	...	t_n
$T-$ ҳарорат	T_1	T_2	T_3	...	T_n

Bu jadval t vaqt bilan havo harorati T orasidagi bog'lanishni ifodalaydi, bunda t - argument, T esa t ning funksiyasi bo'ladi.

c) x va y o'zgaruvcgilar orasidagi bog'lanish tekislikdagi biror egri chiziq orqali ham ifodalash mumkin (2-chizma).



2-chizma.

Masalan, 2-chizmada tasvirlangan L egri chiziq berilgan bo'lsin. Aytaylik, $[a,b]$ segmentdagi har bir nuqtadan o'tkazilgan perpendikulyar L chiziqni faqat bitta nuqtada kessin. $\forall x \in [a,b]$ nuqtadan perpendikulyar chiqarib, uning L chiziq bilan kesishish nuqtasini topamiz. Olingan x nuqtaga kesishish nuqtasining ordinatasi y ni mos qo'yamiz. Natijada har bir $x \in [a,b]$ ga bitta mos qo'yilib, funktsiya hosil bo'ladi.

Bizga, $f_1(x)$ funksiya $X_1 \subset R$ to'plamda, $f_2(x)$ funksiya esa $X_2 \subset R$ to'plamda aniqlangan bo'lsin.

Agar

1) $X_1 = X_2$

2) $\forall x \in X_1$ da $f_1(x) = f_2(x)$

bo'lsa, $f_1(x)$ hamda $f_2(x)$ funksiyalar o'zaro teng deyiladi va $f_1(x) = f_2(x)$ kabi belgilanadi

2^0. Funksiyaning chegaralanganligi. $f(x)$ funksiya $X \subset R$ to'plamda berilgan bo'lsin.

2-ta'rif. Agar shunday o'zgarmas M son topilsaki, $\forall x \in X$ uchun $f(x) \leq M$ tengsizlik bajarilsa, $f(x)$ **funksiya** X **to'plamda yuqoridan chegaralangan** deyiladi. Agar shunday o'zgarmas m soni topilsaki, $\forall x \in X$ uchun $f(x) \geq m$ tengsizlik bajarilsa, $f(x)$ **funksiya** X **to'plamda quyidan chegaralangan** deyiladi.

3-ta'rif. Agar $f(x)$ funksiya X to'plamda ham yuqoridan, ham quyidan chegaralangan bo'lsa, $f(x)$ **funksiya** X **to'plamda chegaralangan** deyiladi.

1-misol. Ushbu $f(x) = \dfrac{1+x^2}{1+x^4}$ funksiyani qaraylik. Bu funksiya R da chegaralangan bo'ladi.

Ravshanki, $\forall x \in R$ da $f(x) = \dfrac{1+x^2}{1+x^4} > 0$.

Demak, berilgan funksiya R da quyidan chegaralangan.

Ayni paytda, $f(x)$ funksiya uchun

$$f(x) = \frac{1}{1+x^4} + \frac{x^2}{1+x^4} \leq 1 + \frac{x^2}{1+x^4}$$

bo'ladi.Endi

$$0 \leq (x^2-1)^2 = x^4 - 2x^2 + 1 \quad \Rightarrow \quad 2x^2 \leq x^4 + 1 \quad \Rightarrow \quad \frac{x^2}{x^4+1} \leq \frac{1}{2}$$

bo'lishini e'tiborga olib, topamiz: $f(x) \leq 1 + \dfrac{1}{2} = \dfrac{3}{2}$.

Bu esa $f(x)$ funksiyaning yuqoridan chegaralanganligini bildiradi. Demak, berilgan funksiya R da chegaralangan.

4-ta'rif. Agar har qanday $M > 0$ son olinganda ham shunday $x_0 \in X$ nuqta topilsaki,

$$f(x_0) > M$$

tengsizlik bajarilsa, $f(x)$ **funksiya** X **to'plamda yuqoridan chegaralanmagan** deyiladi.

3^0. Davriy funksiyalar. Juft va toq funksiyalar. $f(x)$ funksiya $X \subset R$ to'plamda berilgan bo'lsin.

5-ta'rif. Agar shunday o'zgarmas $T\ (T \neq 0)$ son mavjud bo'lsaki, $\forall x \in X$ uchun

1) $x - T \in X$, $x + T \in X$

2) $f(x+T) = f(x)$

bo'lsa, $f(x)$ **davriy funsiya** deyiladi, T son esa $f(x)$ **funksiyaning davri** deyiladi. Bunda $T \neq 0$ va $T \to x$ ga bog'liq bo'lmagan son.

Masalan, $f(x)=\sin x$, $f(x)=\cos x$ funksiyalar davriy funksiyalar bo'lib, ularning davri 2π ga, $f(x)=tgx$, $f(x)=ctgx$ funksiyalarning davri esa π ga teng.

Davriy funksiyalar quyidagi xossalarga ega:

a) Agar davriy $f(x)$ davriy funksiya bo'lib, uning davri $T\ (T\neq 0)$ bo'lsa,u holda

$$T_n = nT \quad (n=\pm 1, \pm 2,)$$

sonlar ham shu funksiyaning davri bo'ladi.

b) Agar T_1 va T_2 sonlar $f(x)$ funksiyaning davri bo'lsa, u holda $T_1+T_2\neq 0$ hamda T_1-T_2 $(T_1\neq T_2)$ sonlar ham $f(x)$ funksiyaning davri bo'ladi.

c) Agar $f(x)$ hamda $g(x)$ funksiyalar davriy funksiyalar bo'lib, ularning har birining davri T $(T\neq 0)$ bo'lsa,u holda

$$f(x)+g(x),\quad f(x)-g(x),\quad f(x)\cdot g(x),\quad \frac{f(x)}{g(x)}\quad (g(x)\neq 0)$$

funksiyalar ham davriy funktsiyalar bo'lib, T son ularning ham davri bo'ladi.

2-misol. Ixtiyoriy $T\ (T\neq 0)$ ratsional son Dirixle funksiyasi

$$D(x)=\begin{cases} 1, \text{ agar } x \text{ ratsional son bo'lsa,} \\ 0, \text{ agar } x \text{ irratsional son bo'lsa.}\end{cases}$$

ning davri bo'lishi ko'rsatilsin.

$T(T\neq 0)$ ratsional son bo'lsin. Ravshanki, $\forall\, x\in R$ irratsional son uchun $x+T-$ irratsional son, $\forall\, x\in R$ ratsional son uchun $x+T$ ratsional son bo'ladi. Demak,

$$D(x+T)=\begin{cases} 1, \text{ agar } x \text{ ratsional son bo'lsa,} \\ 0, \text{ agar } x \text{ irratsional son bo'lsa.}\end{cases}$$

Shunday qilib, $\forall\, x\in R$, T - ratsional son bo'lganda

$$D(x+T)=D(x)$$

bo'ladi.

Ma'lumki, $\forall\, x\in X$ $(X\subset R)$ uchun $-x\in X$ bo'lsa, X to'plam O nuqtaga nisbatan **simmetrik to'plam** deyiladi.

O nuqtaga nisbatan simmetrik bo'lgan X to'plamda $f(x)$ funksiya berilgan bo'lsin.

6-ta'rif. Agar $\forall\, x\in X$ uchun $f(-x)=f(x)$ tenglik bajarilsa, $f(x)$ **juft funksiya** deyiladi. Agar $\forall\, x\in X$ uchun $f(-x)=-f(x)$ tenglik bajarilsa, $f(x)$ **toq funksiya** deyiladi.

Masalan, $f(x)=x^2+1$ juft funksiya, $f(x)=x^3+x$ esa **toq funksiya** bo'ladi. Ushbu $f(x)=x^2-x$ funksiya juft ham emas, toq ham emas.

Agar $f(x)$ va $g(x)$ juft funksiyalar bo'lsa, u holda

$$f(x)+g(x),\quad f(x)-g(x),\quad f(x)\cdot g(x),\quad \frac{f(x)}{g(x)}\quad (g(x)\neq 0)$$

funksiyalar ham juft bo'ladi.

Agar $f(x)$ va $g(x)$ toq funksiyalar bo'lsa, u holda

$$f(x)+g(x),\quad f(x)-g(x)$$

funksiyalar toq bo'ladi,

$$f(x)\cdot g(x),\qquad \frac{f(x)}{g(x)}\quad (g(x)\neq 0)$$

funksiyalar esa juft bo'ladi.

Juft funksiyaning grafigi ordinatalar o'qiga nisbatan, toq funksiyaning grafigi esa kordinatalar boshiga nisbatan simmetrik joylashgan bo'ladi.

4°. Monoton funksiyalar. Faraz qilaylik, $f(x)$ funksiya $X\subset R$ to'plamda berilgan bo'lsin.

7-ta'rif. Agar $\forall x_1,x_2\in R$ uchun $x_1<x_2$ bo'lganda $f(x_1)\leq f(x_2)$ tengsizlik bajarilsa, $f(x)$ **funksiya** X **to'plamda o'suvchi** deyiladi. Agar $\forall x_1,x_2\in X$ uchun $x_1<x_2$ bo'lganda $f(x_1)<f(x_2)$ tengsizlik bajarilsa, $f(x)$ **funksiya** X **to'plamda qat'iy o'suvchi** deyiladi.

8-ta'rif. Agar $\forall x_1,x_2\in X$ uchun $x_1<x_2$ bo'lganda $f(x_1)\geq f(x_2)$ tengsizlik bajarilsin, $f(x)$ **funksiya** X **to'plamda kamayuvchi** deyiladi. Agar $\forall x_1,x_2\in X$ uchun $x_1<x_2$ bo'lganda $f(x_1)>f(x_2)$ tengsizlik bajarilsa, $f(x)$ **funksiya** X **to'plamda qat'iy kamayuvchi** deyiladi.

O'suvchi hamda kamayuvchi funksiyalar umumiy nom bilan **monoton funksiyalar** deyiladi.

3-misol. Ushbu $f(x)=\dfrac{x}{1+x^2}$ funksiyaning $X=[1,+\infty)$ to'plamda kamayuvchi ekanligi isbotlansin.

$[1,+\infty)$ da ixtiyoriy x_1 va x_2 nuqtalarni olib, $x_1 < x_2$ bo'lsin deylik.

Unda

$$f(x_1)-f(x_2)=\frac{x_1}{1+x_1^2}-\frac{x_2}{1+x_2^2}=\frac{x_1+x_1x_2^2-x_2-x_2x_1^2}{(1+x_1^2)(1+x_2^2)}=$$

$$=\frac{x_1-x_2+x_1\cdot x_2(x_2-x_1)}{(1+x_1^2)(1+x_2^2)}=\frac{(x_1-x_2)(1-x_1\cdot x_2)}{(1+x_1^2)(1+x_2^2)}$$

bo'ladi. Keyingi tenglikda

$$x_1-x_2<0, \qquad 1-x_1\cdot x_2<0$$

bo'lishini e`tiborga olib,

$$f(x_1)-f(x_2)>0$$

ya`ni, $f(x_1)>f(x_2)$ ekanligini topamiz. Demak,

$$x_1<x_2 \;\Rightarrow\; f(x_1)>f(x_2).$$

Aytaylik, $f(x)$ va $g(x)$ funksiyalar $X\subset R$ to'plamda o'suvchi (kamayuvchi) bo'lib, $C=const$ bo'lsin. U holda

a) $f(x)+C$ funksiya o'suvchi (kamayuvchi) bo'ladi.

b) $C>0$ bo'lganda $C\cdot f(x)$ o'suvchi, $C<0$ bo'lganda $C\cdot f(x)$ kamayuvchi bo'ladi.

d) $f(x)+g(x)$ funksiya o'suvchi (kamayuvchi) bo'ladi.

5^0. Teskari funksiya. Murakkab funksiyalar. $y=f(x)$ funksiya $X\subset R$ to'plamda berilgan bo'lib, bu funksiyaning qiymatlaridan iborat to'plam

$$Y_f=\{f(x)\,|\,x\in X\}$$

bo'lsin.

Faraz qilaylik, biror qoidaga ko'ra Y_f to'plamdan olingan har bir y ga X to'plamdagi bitta x mos qo'yilgan bo'lsin. Bunday moslik natijasida funksiya hosil bo'ladi. Odatda bu funksiya $y=f(x)$ ga nisbatan *teskari funksiya* deyiladi va $x=f^{-1}(y)$ kabi belgilanadi.

Masalan, $y=\frac{1}{2}x+1$ funksiyaga nisbatan teskari funksiya $x=2y-1$ bo'ladi.

Yuqorida aytilganlardan $y=f(x)$ da x argument, y esa x ning funsiyasi, teskari $x=f^{-1}(y)$ funksiyada y argument, x esa y ning funksiyasi bo'lishi ko'rinadi.

Qulaylik uchun teskari funksiya argumenti ham, uning funksiyasi bilan belgilanadi: $y=g(x)$.

$y=f(x)$ ga nisbatan teskari $g(x)$ funksiya grafigi $f(x)$ funksiya grafigini I va III choraklar bissektrisasi atrofiida 180° ga aylantirish natijasida hosil bo'ladi.

Aytaylik, Y_f to'plamda $u=F(y)$ funksiya berilgan bo'lsin. Natijada X to'plamdan olingan har bir x ga Y_f to'plamda bitta y:

$$f:x\to y \quad (y=f(x)),$$

va Y_f to'plamdagi bunday y songa bitta u:

$$F:y\to u \quad (u=F(y))$$

son mos qo'yiladi. Demak, X to'plamdan olingan har bir x songa bitta u son mos qo'yilib, yangi funksiya hosil bo'ladi: $u=F(f(x))$. Odatda bunday funksiyalar murakkab funksiya deyiladi.

3.2. Elementar funksiyalar

Elementar funksiyalar kitobxonga o'rta maktab matematika kursidan ma'lum. Biz quyida elementar funksiyalar haqidagi asosiy ma'lumotlarni bayon etamiz.

1^0. Butun ratsional funksiyalar. Ushbu

$$y=a_0+a_1x+a_2x^2+\ldots+a_{n-1}x^{n-1}+a_nx^n$$

ko'rinishdagi funksiya butun ratsional funksiya deyiladi. Bunda $a_0,a_1,\ldots a_n$ – o'zgarmas sonlar, $n\in N$. Bu funksiya $R=(-\infty,+\infty)$ da aniqlangan.

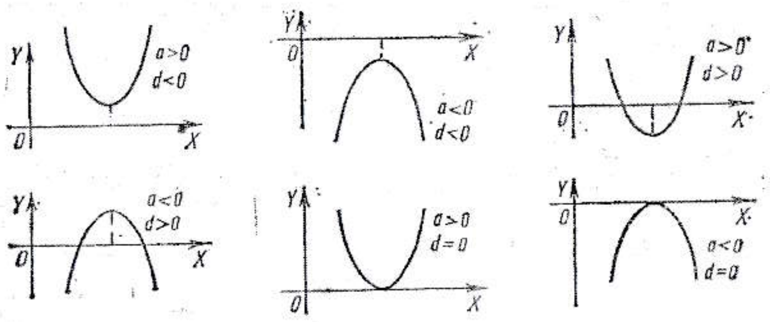
Butun ratsional funksiyaning ba'zi xususiy hollari:

a) Chiziqli funksiya. Bu funksiya

$$y=ax+b \quad (a\neq 0)$$

ko'rinishga ega, bunda a,b - o'zgarmas sonlar.

Chiziqli funksiya $(-\infty,\infty)$ da aniqlangan $a>0$ bo'lganda o'suvchi, $a<0$ bo'lganda kamayuvchi: grafigi tekislikdagi to'g'ri chiziqdan iborat.



2-chizma

b) Kvadrat funksiya. Bu funksiya

$$y = ax^2 + bx + c \quad (a \neq 0)$$

ko'rinishga ega, bunda a, b, c – o'zgarmas sonlar.

Kvadrat funksiya R da aniqlangan bo'lib, uning grafigi parabolani ifodalaydi.

Ravshanki,

$$y = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}.$$

Parabolaning tekislikda joylashishi a hamda $D = b^2 - 4ac$ larning ishorasiga bog'liq bo'ladi. Masalan $a > 0,\ D > 0$ va $a < 0,\ D < 0$ bo'lganda uning grafigi 2-chizmada tasvirlangan parabolalar ko'rinishida bo'ladi.

2⁰. Kasr ratsional funksiyalar. Ushbu

$$y = \frac{a_0 + a_1x + a_2x^2 + \ldots + a_nx^n}{b_0 + b_1x + b_2x^2 + \ldots + b_nx^n}$$

ko'rinishdagi funksiya kasr ratsional funksiya deyiladi. Bunda $a_0, a_1, \ldots, a_n$ va $b_0, b_1, \ldots, b_m$ - o'zgarmas sonlar $n \in N,\ m \in N$. Bu funksiya

$$X = (-\infty, +\infty) \setminus \{x \mid b_0 + b_1x + \ldots + b_mx^m = 0\}$$

to'plamda aniqlangan.

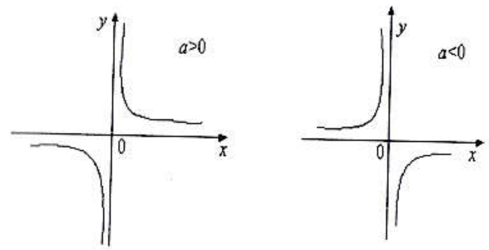
Kasr ratsional funksiyaning ba'zi xususiy hollari:

a) **Teskari proportsional bog'lanish.** U

$$y = \frac{a}{x} \qquad (x \neq 0 \quad a = const)$$

ko'rinishga ega. Bu funksiya

$$X = (-\infty, 0) \cup (0, +\infty) = R \setminus \{0\}$$



3-chizma

to'plamda aniqlangan, toq funksiya, a ning ishorasiga qarab funksiya $(-\infty, 0)$ va $(0, \infty)$ oraliqlarning har birida kamayuvchi yoki o'suvchi bo'ladi (3-chizma)

b) Kasr chiziqli funksiya. U ushbu

$$y = \frac{ax + b}{cx + d}$$

ko'rinishga ega. Bu funksiya

$$X = R \setminus \left\{-\frac{d}{c}\right\} \quad (c \neq 0)$$

to'plamda aniqlangan.

Ravshanki,

$$y = \frac{ax + b}{cx + d} = \frac{bc - ad}{c^2} \cdot \frac{1}{x + \frac{d}{c}} + \frac{a}{c}.$$

Demak, $y = \dfrac{\alpha}{x + \beta} + \gamma, \quad \left(\alpha = \dfrac{bc - ad}{c^2}, \beta = \dfrac{d}{c}, \gamma = \dfrac{a}{c}\right).$

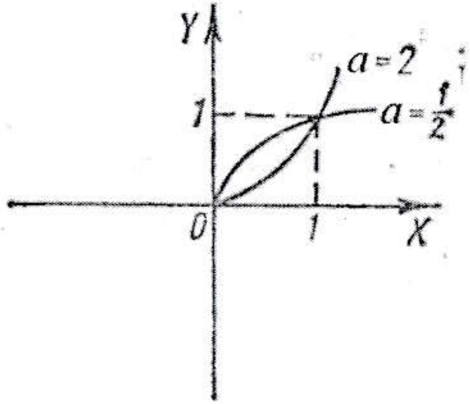
Uning grafigini $y = \dfrac{a}{x}$ funksiya grafigi yordamida chizish mumkin.

3⁰. Darajali funksiya. Ushbu

$$y = x^a, \quad (x \geq 0)$$

ko'rinishdagi funksiya darajali funksiya deyiladi.

Bu funksiyaning aniqlanish to'plami a ga bog'liq. Darajali funksiya $(0, \infty)$ da $a > 0$, bo'lganda o'suvchi, $a < 0$ bo'lganda kamayuvchi bo'ladi. $y = x^a$ funksiya grafigi tekislikning $(0,0)$ va $(1,1)$ nuqtalaridan o'tadi.



4-chizma

4⁰. Ko'rsatkichli funksiya. Ushbu

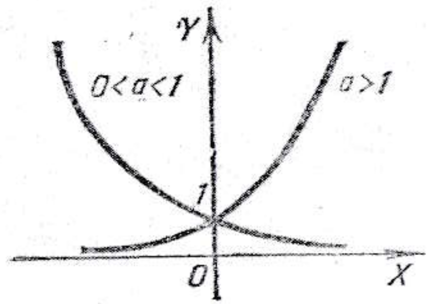
$$y = a^x$$

ko'rinishdagi funksiya ko'rsatkichli funksiya deyiladi. Bunda $a \in R,\ a > 0,\ a \neq 1$.

Ko'rsatkichli funksiya $(-\infty, \infty)$ aniqlangan, $\forall x \in R$ da $a^x > 0; a > 1$ bo'lganda o'suvchi; $0 < a < 1$ bo'lganda kamayuvchi bo'ladi.

Xususan, $a = e$ bo'lsa, matematikada muhim rol o'ynaydigan $y = e^x$ funksiya hosil bo'ladi.

Ko'rsatkichli funksiyaning grafigi Ox o'qidan yuqorida joylashgan va tekislikning $(0,1)$ nuqtasidan o'tadi.



5-chizma

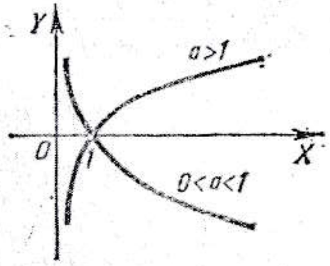
5⁰. Logarifmik funksiya. Ushbu

$$y = \log_a x$$

ko'rinishdagi funksiya *logarifmik funksiya* deyiladi. Bunda $a > 0,\ a \neq 1$.

Logarifmik funksiya $(0, +\infty)$ da aniqlangan, $y = a^x$ funksiyaga nisbatan teskari; $a > 1$ bo'lganda o'suvchi, $0 < a < 1$ bo'lganda kamayuvchi bo'ladi.

Logarifmik funksiyaning grafigi Oy o'qining o'ng tomonida joylashgan va tekislikning (1,0) nuqtasidan o'tadi.



6-chizma

6⁰. Trigonometrik funksiyalar. Ushbu

$$y = \sin x,\ y = \cos x,\ y = tgx,\ y = ctgx,$$

$$y = \sec x,\ y = \mathrm{cosec} x$$

funksiyalar trigonometrik funksiyalar deyiladi.

$y = \sin x, y = \cos x$ funksiyalar $R = (-\infty, +\infty)$ da aniqlangan, davriy funksiyalar va ularning eng kichik musbat davri 2π ga teng. Bu funksiyalar $\forall x \in R$ da

$$-1 \leq \sin x \leq 1,\ \ -1 \leq \cos x \leq 1$$

bo'ladi.

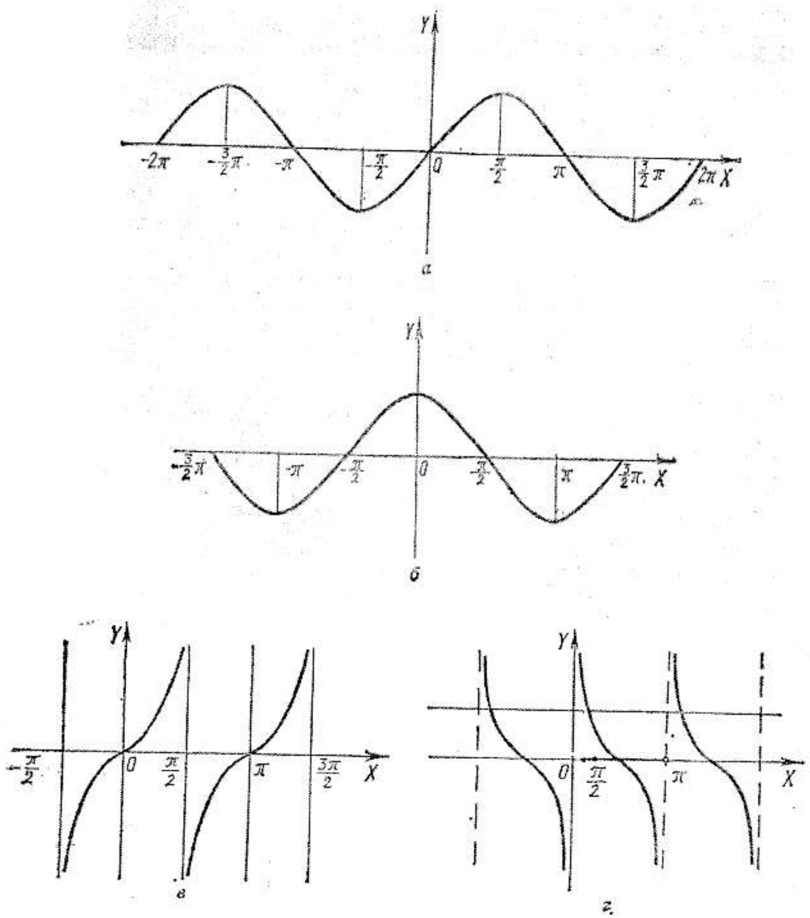
Ushbu $y = tgx$, funksiya

$$X = R \setminus \left\{ x \in R \mid x = (2k+1)\frac{\pi}{2};\ k = 0, \pm 1, \pm 2, \ldots \right\}$$

to'plamda aniqlangan dvriy funksiya va uning eng kichik musbat davri π ga teng.

Ushbu $y = ctgx$ funksiya $X = R \setminus \{x \in R | x = \pi k;\ k = 0, \pm 1, \pm 2, \ldots.\}$ to'plamda aniqlangan davriy funksiya va uning eng kichik musbat davri ham π ga teng.

$ctgx, \sec x, \operatorname{cosec} x$ funksiyalar $\sin x, \cos x, tgx$ lar orqali quyidagicha ifodalanadi: $ctgx = \frac{1}{tgx}$, $\sec x = \frac{1}{\cos x}$, $\operatorname{cosec} x = \frac{1}{\sin x}$



7-chizma

7^0. **Giperbolik funksiyalar.** Ko'rsatkichli $y = e^x$ funksiya yordamida tuzilgan ushbu

$$\frac{e^x - e^{-x}}{2}, \quad \frac{e^x + e^{-x}}{2}, \quad \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

funksiyalar *giperbolik* (mos ravishda *giperbolik sinus, giperbolik kosinus, giperbolik tangens, giperbolik katangens*) funksiyalar deyiladi va ular quyidagicha

$$shx = \frac{e^x - e^{-x}}{2}, \quad chx = \frac{e^x + e^{-x}}{2}, \quad thx = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad cthx = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

belgilanadi.

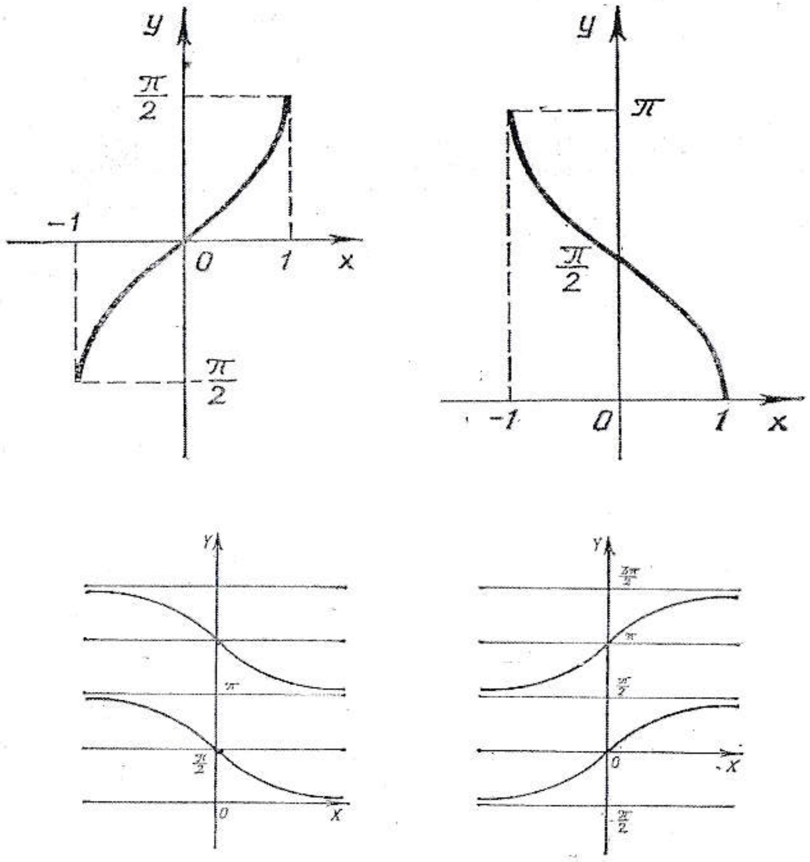
8^0. **Teskari trigonometrik funksiyalar.** Ma'lumki, $y = \sin x$ funksiya R da aniqlangan va uning qiymatlari to'plami

$$Y_f = [-1,1]$$

bo'ladi.

Agar $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ bo'lsa, u holda $X = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ va $Y_f = [-1,1]$ to'plamlarning elementlari o'zaro bir qiymatli moslikda bo'ladi.

$y = \sin x$ funksiyaga nisbatan teskari funksiya $y = \arcsin x$ kabi belgilanadi.



8-chizma

Shunga o'xshash $y=\cos x$, $y=tgx$, $y=ctgx$ funksiyalarga nisbatan teskari funksiyalar mos ravishda

$$y=\arccos x,\ y=arctgx,\ y=arcctgx,$$

kabi belgilanadi.

Ushbu $y=\arcsin x$, $y=\arccos x$, $y=arctgx$, $y=arcctgx$ funksiyalar teskari trigonometrik funksiyalar deyiladi.

Yuqoridagi 8-chizmada barcha teskari trigonametrik funksiyalarni grafiklari tasvirlangan.

Berilgan y=f(x) funksiyani aniqlanish va qiymatlar sohasini toping.So'ng uning grafigini chizing.

1. $y=(x-2)\sqrt{\frac{1+x}{1-x}}$
2. $y=\sqrt{3x-x^3}$
3. $y=\lg(x^2-4)$
4. $y=\sqrt{\sin(\sqrt{x})}$
5. $y=\sqrt{\cos x^2}$
6. $y=\lg\left(\sin\frac{\pi}{x}\right)$
7. $y=\frac{\sqrt{x}}{\sin \pi x}$
8. $y=\arcsin\frac{2x}{1+x}$
9. $y=\arccos(2\sin x)$
10. $y=\lg[\cos(\lg x)]$
11. $y=(x+|x|)\sqrt{x\sin^2 \pi x}$
12. $y=ctg\pi x+\arccos(2^x)$
13. $y=\arcsin(1-x)+\lg(\lg x)$
14. $y=(2x)!$
15. $y=\log_2\log_3\log_4 x$
16. $y=\sqrt[4]{\lg tgx}$
17. $y=\sqrt{\sin 2x}+\sqrt{\sin 3x},\ (0\le x\le 2\pi)$
18. $y=\sqrt{2+x-x^2}$
19. $y=\lg(1-2\cos x)$
20. $y=\arccos\frac{2x}{1+x^2}$
21. $y=\arcsin\left(\lg\frac{x}{10}\right)$
22. $y=(-1)^x$
23. $y=x+[2x]$
24. $y=\sqrt{x-x^2}$
25. $y=\frac{x}{2x-1}$
26. $y=ctg\pi x$
27. $y=(x+|x|)(1-x)$
28. $y=\pm 2^{-x}\sqrt{\sin\pi x}$
29. $y=|1-x|-|1+x|$
30. $y=x+\sqrt{x}\,\mathrm{sgn}(\sin\pi x)$

3.3. Funksiya limiti

1°. To'plamning limit nuqtasi. Aytaylik, biror $X\subset R$ to'plam va $x_0\in R$ nuqta berilgan bo'lsin.

1-ta'rif. Agar x_0 nuqtaning ixtiyoriy

$$U_\varepsilon(x_0)=(x_0-\varepsilon, x_0+\varepsilon)\quad (\forall\varepsilon>0)$$

atrofida X to'plamning x_0 nuqtadan farqli kamida bitta nuqtasi bo'lsa, ya'ni

$$\forall\varepsilon>0,\ \exists x\in X,\ x\ne x_0: |x-x_0|<\varepsilon$$

bo'lsa, x_0 nuqta X to'plamning *limit nuqtasi* deyiladi.

Misollar. 1. $X=[0,1]$ to'plamning har bir nuqtasi shu to'plamning limit nuqtasi bo'ladi.

2. $X=(0,1)$ to'plamning har bir nuqtasi va $x=0, x=1$ nuqtalar shu to'plamning limit nuqtalari bo'ladi.

3. $X=\left\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},.....\right\}$ to'plamning limit nuqtasi $x_0=0$ bo'ladi.

4. $X=N=\{1,2,3...\}$ to'plam limit nuqtaga ega emas.

2- ta'rif. Agar x_0 nuqtaning ixtiyoriy

$$U_\varepsilon^+(x_0)=(x_0,x_0+\varepsilon) \quad (U_\varepsilon^-(x_0)=(x_0-\varepsilon,x_0)) \ (\forall\varepsilon>0)$$

o'ng atrofida (chap atrofida) X to'plamning kamida bitta nuqtasi bo'lsa, x_0 nuqta X to'plamning *o'ng (chap) limit nuqtasi* deyiladi.

3-ta'rif. Agar ixtiyoriy $c\in R$ uchun

$$U_c(+\infty)=\{x\in R \mid x>c\}$$

to'plamda X to'plamning kamida bitta nuqtasi bo'lsa, "$+\infty$" X to'plamning limit «nuqta» si deyiladi.

Agar ixtiyoriy $c\in R$ uchun

$$U_c(-\infty)=\{x\in R \mid x<c\}$$

to'plamda X to'plamning kamida bitta nuqtasi bo'lsa, "$-\infty$" X to'plamning limit «nuqta» si deyiladi.

Keltirilgan ta'rif va misollardan ko'rinadiki, to'plamning limit nuqtasi shu to'plamga tegishli bo'lishi ham, bo'lmasligi ham mumkin ekan.

1-teorema. Agar $x_0\in R$ nuqta $X\subset R$ to'plamning limit nuqtasi bo'lsa, u holda x_0 nuqtaning har qanday

$$U_\varepsilon(x_0)=(x_0-\varepsilon,x_0+\varepsilon) \quad (\forall\varepsilon>0)$$

atrofi X to'plamning cheksiz ko'p nuqtalari bo'ladi.

Isbot. x_0 nuqta X to'plamning limit nuqtasi bo'lsin. Teorema tasdig'ining teskarisini faraz qilaylik: x_0 nuqtaning biror $U_{\delta_0}(x_0)$ atrofida X to'plamning chekli sondagi $x_1, x_2,..., x_n$ nuqtalarigina bo'lsin. U holda

$$\min\{|x_0-x_1|, |x_0-x_2|, ..., |x_0-x_n|, \delta_0\}=\delta$$

deb olinsa, x_0 nuqtaning $U_\delta(x_0)$ atrofida X to'plamning x_0 dan farqli bitta ham nuqtasi bo'lmaydi. Bu esa x_0 nuqta X to'plamning limit nuqtasi bo'lishiga ziddir.

2-teorema. Agar x_0 nuqta $X\subset R$ to'plamning limit nuqtasi bo'lsa, u holda shunday sonlar ketma-ketligi $\{x_n\}$ topiladiki,

1) $\forall n\in N$ da $x_n\in X,\ x_n\neq x_0$;

2) $n\to\infty$ da $x_n\to x_0$ bo'ladi.

Isbot. $x_0\in R$ nuqta $X\subset R$ to'plamning limit nuqtasi bo'lsin. Unda 1-ta'rifga binoan

$$\forall\varepsilon>0,\ \exists x_n\in X,\ x_n\neq x_0:\ |x_n-x_0|<\varepsilon$$

bo'ladi.Jumladan,

$$\varepsilon=1 \text{ uchun } \exists x_1\in X,\ x_1\neq x_0:\ |x_1-x_0|<1,$$

$$\varepsilon=\frac{1}{2} \text{ uchun } \exists x_2\in X,\ x_2\neq x_0:\ |x_2-x_0|<\frac{1}{2},$$

$$\varepsilon=\frac{1}{3} \text{ uchun } \exists x_3\in X,\ x_3\neq x_0:\ |x_3-x_0|<\frac{1}{3},$$

....

$$\varepsilon=\frac{1}{n} \text{ uchun } \exists x_n\in X,\ x_n\neq x_0:\ |x_n-x_0|<\frac{1}{n},$$

....

bo'ladi.

Natijada qaralayotgan teoremaning 1) shartini qanoat-lantiruvchi $\{x_n\}$ ketma-ketlik hosil bo'lib, uning uchun $\forall n\in N$ da $|x_n-x_0|<1/n$ tengsizlik o'rinli bo'ladi. Keyingi munosa-batdan esa $n\to\infty$ da $x_n\to x_0$ kelib chiqadi.

Shuni ta'kidlash lozimki, 2-teoremaning shartlarini qanoatlantiruvchi ketma-ketliklar ko'plab topiladi.

2⁰. Funksiya limiti ta'riflari. Faraz qilaylik, $f(x)$ funksiya $X\subset R$ to'plamda berilgan bo'lib, x_0 nuqta X to'plamning limit nuqtasi bo'lsin. x_0 nuqtaga intiluvchi ixtiyoriy $\{x_n\}$:

$$x_1,x_2,...,x_n,... \quad (x_n\in X, x_n\neq x_0)$$

ketma-ketlikni olib, funksiya qiymatlaridan iborat $\{f(x_n)\}$:

$$f(x_1), f(x_2),.....f(x_n),....$$

ketma-ketlikni hosil qilamiz.

4-ta'rif. (Geyne ta'rifi). Agar $n \to \infty$ da $x_n \to x_0$ $(x_n \in X, x_n \neq x_0)$ da bo'ladigan ixtoyoriy $\{x_n\}$ ketma-ketlik uchun $n \to \infty$ da $f(x_n) \to b$ bo'lsa, b ga *$f(x)$ funksiyaning x_0 nuqtadagi limiti* deyiladi va $x \to x_0$ da $f(x) \to b$ yoki

$$\lim_{x \to x_0} f(x) = b$$

kabi belgilanadi.

Eslatma. Agar $n \to \infty$ da

$$x_n \to x_0 \ \ (x_n \in X,\ x_n \neq x_0) \ \text{ va } \ y_n \to x_0 \ \ (y_n \in X,\ y_n \neq x_0)$$

bo'ladigan turli $\{x_n\}, \{y_n\}$ ketma-ketliklar uchun $n \to \infty$ da $f(x_n) \to b_1$, $f(y_n) \to b_2$ bo'lib, $b_1 \neq b_2$ bo'lsa *$f(x)$ funksiya $x \to x_0$ da limitga ega emas* deyiladi.

1-misol. Ushbu $f(x) = \dfrac{x^2 - 16}{x^2 - 4x}$

funksiyaning $x_0 = 4$ nuqtadagi limiti topilsin.

Quyidagi $\{x_n\}$:

$$\lim_{n \to \infty} x_n = 4 \quad (x_n \neq 4,\ n = 1, 2, \ldots)$$

ketma-ketlikni olaylik. Unda

$$f(x) = \frac{x_n^2 - 16}{x_n^2 - 4x_n} = \frac{x_n + 4}{x_n}$$

bo'lib, $n \to \infty$ da $f(x_n) \to 2$ bo'ladi. Demak,

$$\lim_{n \to \infty} \frac{x^2 - 16}{x^2 - 4x} = 2.$$

2-misol. Ushbu $f(x) = \sin\dfrac{1}{x}$ funksiyaning $x \to 0$ dagi limiti mavjud bo'lmasligi ko'rsatilsin.

Ravshanki, $n \to \infty$ da

$$x_n' = \frac{2}{(4n-1)\pi} \to 0, \quad x_n'' = \frac{2}{(4n+1)\pi} \to 0$$

bo'ladi.

Bu ketma-ketliklar uchun

$$f(x_n') = \frac{4n-1}{2}\pi = -1, \quad f(x_n'') = \frac{4n+1}{2}\pi = 1$$

bo'lib, $n \to \infty$ da

$$f(x_n') \to -1, \quad f(x_n'') \to 1$$

bo'ladi. Demak, berilgan funksiya $x_0 = 0$ nuqtada limitga ega emas.

5-ta'rif. (Koshi ta'rifi). Agar $\forall \varepsilon > 0$ son olinganda ham shunday $\delta = \delta(\varepsilon) > 0$ topilsaki, $\forall x \in X \cap \left(U_\delta(x_0) \setminus \{x_0\}\right)$ uchun

$$|f(x) - b| < \varepsilon$$

tengsizlik bajarilsa, b soni *$f(x)$ funksiyaning x_0 nuqtadagi limiti* deyiladi:

$$\lim_{x \to x_0} f(x) = b.$$

Bu ta'rifni qisqacha quyidagicha ham aytish mumkin:

$$\forall \varepsilon > 0,\ \ \delta = \delta(\varepsilon) > 0,\ \forall x \in X \cap \left(U_\delta(x_0) \setminus \{x_0\}\right):\ \ |f(x) - b| < \varepsilon$$

bo'lsa, $\lim\limits_{x \to x_0} f(x) = b$.

3-misol. $f(x) = C = const$ $(C \in R)$ bo'lsin. Bu funksiya uchun

$$\lim_{x \to x_0} f(x) = C$$

bo'ladi.

4-misol. Ushbu $f(x) = \dfrac{x^2 - 1}{x - 1}$ funksiyaning $x_0 = 1$ nuqtadagi limiti 2 ga teng ekani ko'rsatilsin.

$\forall \varepsilon > 0$ soniga ko'ra $\delta = \varepsilon$ deb olsak,u holda $|x - 1| < \delta$ $(x \neq 1)$ tengsizlikni qanoatlantiruvchi ixtiyoriy x da

$$\left|\frac{x^2 - 1}{x - 1} - 2\right| = |x + 1 - 2| = |x - 1| < \delta = \varepsilon$$

bo'ladi. Demak, $\lim\limits_{x \to x_0} \dfrac{x^2 - 1}{x - 1} = 2$.

5-misol. Faraz qilaylik, $X = R \setminus \{0\}$ da $f(x) = \dfrac{\sin x}{x}$ bo'lsin. Bu funksiya uchun

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

bo'ladi.

Ma'lumki, $x \in \left(0, \frac{\pi}{2}\right)$ uchun

$$\frac{1}{2}\sin x < \frac{1}{2}x < \frac{1}{2}tgx$$

bo'ladi. Bu tengsizliklardan

$$0 < |x| < \frac{\pi}{2} \text{ da}$$

$$\cos x < \frac{\sin x}{x} < 1$$

bo'lishini topamiz. Keyingi tengsizliklardan esa

$$0 < 1 - \frac{\sin x}{x} < 1 - \cos x = 2\sin^2\frac{x}{2} < 2\cdot\frac{x^2}{4} = \frac{x^2}{2}$$

bo'lishi kelib chiqadi.

Endi $\forall\varepsilon > 0$ ni olib, $\delta = \min\{\varepsilon; 1\}$ deyilsa,unda $\forall x, |x| < \delta,\ x \neq 0$ uchun

$$0 < 1 - \frac{\sin x}{x} < \varepsilon$$

bo'ladi. Demak,

$$\lim_{x\to 0}\frac{\sin x}{x} = 1.$$

6-misol. Ushbu $f(x) = a^x,\ a > 0,\ x \in R,\ x_0 = 0$

funksiya uchun

$$\lim_{x\to 0} a^x = 1$$

bo'lishi isbotlansin.

$a > 1$ bo'lgan xolni qaraylik. Bu holda $f(x)$ funksiya qat'iy o'suvchi bo'ladi:

$$\forall x_1, x_2 \in R,\ x_1 < x_2 \Rightarrow f(x_1) < f(x_2) : a^{x_1} < a^{x_2}.$$

$\forall\varepsilon > 0$ sonni olaylik. Ma'lumki, $n \to \infty$ da

$$a^{\frac{1}{n}} \to 1,\ a^{-\frac{1}{n}} \to 1$$

bo'lib, ketma-ketlik limiti ta'rifiga binoan

$$\exists n_1 \in N,\ \forall n > n_1 :\ a^{\frac{1}{n}} < 1 + \varepsilon,$$

$$\exists n_1 \in N,\ \forall n > n_1 :\ a^{\frac{1}{n}} < 1 + \varepsilon$$

bo'ladi. Endi $n_0 = \max\{n_1, n_2\},\ \delta = \frac{1}{n_0}$ deyilsa, unda

$$\forall x, |x - 0| < \delta \Leftrightarrow -\frac{1}{n} < x < \frac{1}{n_0}$$

bo'lganda

$$a^{\frac{1}{n_0}} < a^x < a^{\frac{1}{n}} \Rightarrow 1 - \varepsilon < a^x < 1 + \varepsilon \Leftrightarrow |a^x - 1| < \varepsilon$$

bo'ladi. Demak, $\lim_{x\to 0} a^x = 1$.

$0 < a < 1$ bo'lganda $\lim_{x\to 0} a^x = 1$ bo'lishini isbotlash o'quvchiga havola etiladi.

6-ta'rif. Agar $\forall\varepsilon > 0$ son olinganda ham shunday $\delta > 0$ son topilsaki, $\forall x \in X \cap (U_\delta(x_0) \setminus \{x_0\})$ uchun $f(x) > \varepsilon$ tengsizlik bajarilsa, $f(x)$ funksiyaning x_0 nuqtadagi limiti $+\infty$ deb ataladi va

$$\lim_{x\to x_0} f(x) = +\infty$$

kabi belgilanadi.

Masalan,

$$f(x) = \frac{1}{x^2},\ (x \neq 0)$$

funksiya uchun

$$\lim_{x\to 0}\frac{1}{x^2} = +\infty$$

bo'ladi.

Aytaylik, $f(x)$ funksiya $X \subset R$ to'plamda berilgan bo'lib, $x_0 = +\infty$ nuqta X to'plamning limit nuqtasi bo'lsin.

7-ta'rif. Agar $\forall\varepsilon > 0$ son olinganda ham shunday $\delta > 0$ topilsaki, $\forall x \in X,\ x > \delta$ uchun

$$|f(x) - b| < \varepsilon$$

tengsizlik bajarilsa, b soni *$f(x)$ funksiyaning $x_0 = +\infty$ dagi limiti deyiladi* va

$$\lim_{x\to +\infty} f(x) = b$$

kabi belgilanadi.

7-misol. $X=(0,+\infty)$, $x_0=+\infty$, $f(x)=\frac{1}{x}$ bo'lsin. U holda

$$\lim_{x\to+\infty}\frac{1}{x}=0$$

bo'ladi.

Haqiqatan ham, $\forall\varepsilon>0$ sonnni olaylik. Ravshanki, $\forall x>0$ uchun

$$\left|\frac{1}{x}-0\right|=\frac{1}{x}<\varepsilon \Leftrightarrow x>\frac{1}{\varepsilon}.$$

Demak, $\delta=\frac{1}{\varepsilon}$ deyilsa, u holda $\forall x>\delta$ uchun

$$\left|\frac{1}{x}-0\right|=\frac{1}{x}<\frac{1}{\delta}=\varepsilon$$

bo'ladi.

8-misol. Faraz qilaylik,

$$f(x)=\frac{x^m}{a^x},\ a>1,\ m\in N,\ X=R$$

bo'lsin. Unda

$$\lim_{x\to+\infty}\frac{x^m}{a^x}=0$$

bo'lishini isbotlaymiz.

$\varepsilon>0$ sonni olaylik. Ma'lumki, $n\to\infty$ da

$$\frac{(n+1)^m}{a^n}\to 0$$

bo'ladi. U holda $\forall\varepsilon>0$, $\exists n_0$, $\forall n>n_0:\ \frac{(n+1)^m}{a^n}<\varepsilon$ bo'ladi.

Agar $C=n_0$ deyilsa, unda $\forall x>C$ uchun

$$\left|\frac{x^m}{a^x}-0\right|=\frac{x^m}{a^x}<\frac{([x]+1)^m}{a^{[x]}}<\varepsilon$$

bo'ladi $([x]\ge n_0=C)$. Demak, $\lim_{x\to+\infty}\frac{x^m}{a^x}=0$.

9-misol. Ushbu $\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x=e$

munosabat isbotlansin.

$\varepsilon>0$ sonni olamiz. Ma'lumki, $n\to\infty$ da

$$\left(1+\frac{1}{n}\right)^n\to e,$$

$$\left(1+\frac{1}{n+1}\right)^n=\left(1+\frac{1}{n+1}\right)^{n+1}\cdot\frac{n+1}{n+2}\to e$$

Limit ta'rifiga binoan,

$$\forall\varepsilon>0,\ \exists n_0\in N,\ \forall n>n_0:$$

$$e-\varepsilon<\left(1+\frac{1}{n+1}\right)^n,\left(1+\frac{1}{n}\right)^{n+1}<e+\varepsilon$$

bo'ladi.

Endi $C=n_0$ desak, unda $\forall x>C$ uchun

$$e-\varepsilon<\left(1+\frac{1}{[x]+1}\right)^{[x]}<\left(1+\frac{1}{x}\right)^x<\left(1+\frac{1}{[x]}\right)^{[x]+1}<e+\varepsilon$$

bo'lib, $\left|\left(1+\frac{1}{x}\right)^x-e\right|<\varepsilon$

bo'ladi. Demak, $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$.►

3⁰. Funksiya limiti ta'riflarining ekvivalentligi.

3-teorema. Funksiya limitining Koshi hamda Geyne ta'riflari ekvivalent ta'riflardir.

Koshi ta'rifiga ko'ra b soni $f(x)$ funksiyaning x_0 nuqtadagi limiti bo'lsin:

$$\lim_{x\to x_0}f(x)=b$$

Unda

$$\forall\varepsilon>0,\ \exists\delta>0,\ \forall x\in X,\ |x-x_0|<\delta,\ x\ne x_0$$

bo'lganda

$$|f(x)-b|<\varepsilon \qquad (1)$$

bo'ladi. x_0 nuqta X to'plamning limit nuqtasi. Unda 2-teoremaga ko'ra $\{x_n\}$ ketma-ketlik topiladiki, $n\to\infty$ da $x_n\to x_0$ $(x_n\neq x_0,\ n=1,2,\ldots)$ bo'ladi. Ketma-ketlik limiti ta'rifiga binoan

$$\delta>0,\ \exists n_0\in N,\ \forall n>n_0: |x_n-x_0|<\delta, \qquad (2)$$

bo'ladi. (1) va (2) munosabatlardan $\forall n>n_0$ uchun

$$|f(x_n)-b|<\varepsilon$$

bo'lishi kelib chiqadi. Bu esa b sonini Geyne ta'rifi bo'yicha $f(x)$ funksiyaning x_0 nuqtadagi limiti ekanini bildiradi.

Endi b soni Geyne ta'rifi bo'yicha $f(x)$ funksiyaning x_0 nuqtadagi limiti bo'lsin.

Teskarisini faraz qilamiz, ya'ni $f(x)$ funksiyaning x_0 nuqtadagi limiti Geyne ta'rifi bo'yicha b ga teng bo'lsa ham, Koshi ta'rifi bo'yicha limiti bo'lmasin. Unda biror $\varepsilon_0>0$ uchun ixtiyoriy $\delta>0$ son olganimizda ham $0<|x-x_0|<\delta$ ni qanotlantiruvchi biror x' da

$$|f(x')-b|\geq\varepsilon_0$$

bo'ladi.

Nolga intiluvchi musbat sonlar ketma-ketligi $\{\delta_n\}$ ni olaylik:

$$n\to\infty \text{ da } \delta_n\to 0 \quad (\delta_n>0,\ n=1,2,\ldots).$$

U holda

$$0<|x_n-x_0|<\delta_n \ \Rightarrow\ |f(x_n)-b|\geq\varepsilon_0 \qquad (3)$$

bo'ladi.Ammo $\delta_n\to 0$, da $x_n\to x_0$, demak, Geyne ta'rifiga asosan

$$f(x_n)\to b$$

bo'ladi. Bu (3) ga ziddir. Demak, b soni Koshi ta'rifi bo'yi-cha ham, $f(x)$ funksiyaning x_0 nuqtadagi limitini bildiradi.

4⁰. Funksiyaning o'ng va chap limitlari. Aytaylik, $f(x)$ funksiya $X\subset R$ to'plamda berilgan, x_0 nuqta X ning chap limit nuqtasi bo'lib,

$$(x_0-\gamma,x_0)\subset X \quad (\gamma>0)$$

bo'lsin.

8-ta'rif. Agar

$$\forall\varepsilon>0,\ \exists\delta>0,\ \forall x\in(x_0-\delta,x_0): |f(x)-b|<\varepsilon$$

bo'lsa, b son $f(x)$ **funksiyaning x_0 nuqtadagi chap limiti** deyiladi va

$$b=\lim_{x\to x_0-0} f(x)=f(x_0-0)$$

kabi belgilanadi.

Faraz qilaylik, $f(x)$ funksiya $X\subset R$ to'plamda berilgan, x_0 nuqta X ning o'ng limit nuqtasi va

$$(x_0+\gamma,x_0)\subset X \quad (\gamma>0)$$

bo'lsin.

9-ta'rif. Agar $\forall\varepsilon>0,\exists\delta>0,\forall x\in(x_0,x_0+\delta): |f(x)-b|<\varepsilon$

bo'lsa, b son $f(x)$ funksiyaning x_0 nuqtadagi o'ng limiti deyiladi va

$$b=\lim_{x\to x_0+0} f(x)=f(x_0+0)$$

kabi belgilanadi.

Masalan,

$$f(x)=\begin{cases}1 & \text{agar } x>0 \text{ bo'lsa,}\\ 0 & \text{agar } x=0 \text{ bo'lsa,}\\ -1 & \text{agar } x<0 \text{ bo'lsa}\end{cases}$$

funksiyaning 0 nuqtadagi o'ng limiti 1 , chap limiti -1 bo'ladi.

Limitni hisoblang.

1. $\lim\limits_{x\to\infty}\left(\dfrac{x^2+4}{x^2-4}\right)^{x^2}$
2. $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{1+3x}-\sqrt[5]{1+2x}}{\sqrt{1+5x}-\sqrt[4]{1+2x}}$
3. $\lim\limits_{x\to 0}\dfrac{e^{\sin 5x}-e^{\sin x}}{\sin^2 x}$
4. $\lim\limits_{x\to 0}\dfrac{e^{7x}-e^{2x}}{tgx}$
5. $\lim\limits_{x\to+\infty}x^2\left(4^{\frac{1}{x}}-4^{\frac{1}{x+1}}\right)$
6. $\lim\limits_{x\to+\infty}x\left(3^{\frac{1}{x}}-1\right)$
7. $\lim\limits_{x\to 0}\dfrac{10^x-1}{2^x-1}$
8. $\lim\limits_{x\to\infty}x^2\ln\cos\dfrac{\pi}{x}$
9. $\lim\limits_{x\to\frac{1}{4}}\dfrac{1-ctg\pi x}{\ln tg\pi x}$

16. $\lim\limits_{x\to 0}\left(\sqrt{1+x}-x\right)^{\frac{1}{x}}$
17. $\lim\limits_{x\to 0}\left(1+3x^4\right)^{\frac{1}{\sin^2 x}}$
18. $\lim\limits_{x\to\frac{\pi}{2}}(1+ctgx)^{tgx}$
19. $\lim\limits_{x\to 0}(\cos x)^{-\frac{1}{x^2}}$
20. $\lim\limits_{x\to\frac{\pi}{2}}(\sin x)^{tg^2 x}$
21. $\lim\limits_{x\to 0}(\cos 6x)^{ctg^2 x}$
22. $\lim\limits_{x\to 0}(\ln(e+x))^{ctgx}$
23. $\lim\limits_{x\to 0}\left(\dfrac{xe^x+1}{x\pi^x+1}\right)^{\frac{1}{x^2}}$
24. $\lim\limits_{x\to 0}\left(\cos x+arctg^2 x\right)^{\frac{1}{arctgx^2}}$

10. $\lim\limits_{x\to 0}\frac{\ln\cos 5x}{\ln\cos 4x}$

11. $\lim\limits_{x\to 0}\frac{\lg tg\left(\frac{\pi}{4}+4x\right)}{x}$

12. $\lim\limits_{x\to 0}\frac{\sqrt[3]{1+x}-1-\sin x}{\ln(1+x)}$

13. $\lim\limits_{x\to\infty} x\log_2\frac{10+x}{5+x}$

14. $\lim\limits_{x\to 0}\frac{\ln(1+3x+x^2)+\ln(1-3x+x^2)}{x^2}$

15. $\lim\limits_{x\to 10}\frac{\lg x-10}{x-10}$

25. $\lim\limits_{x\to 0}\frac{shx}{x}$

26. $\lim\limits_{x\to 0}\frac{ch2x-1}{\cos x-1}$

27. $\lim\limits_{x\to 0}\frac{\ln ch5x}{x^2}$

28. $\lim\limits_{x\to 0}\frac{e^{sh3x}-e^{shx}}{tgx}$

29. $\lim\limits_{x\to\infty}(x^2-\ln chx^2)$

30. $\lim\limits_{x\to 0}\left(\frac{ch2x}{chx}\right)^{\frac{1}{x^2}}$

3.4. Limitga ega bo'lgan funksiyalarning xossalari. Limitning mavjudligi

1^0. Limitga ega bo'lgan funksiyalarning xossalari. Chekli limitga ega bo'lgan funksiyalar ham yaqinlashuvchi ketma-ketlik singari qator xossalarga ega.

Faraz qilaylik, $f(x)$ funksiya $X\subset R$ to'plamda berilgan bo'lib, $x_0\in R$ nuqta X to'plamning limit nuqtasi bo'lsin.

1-xossa. Agar $x\to x_0$ da $f(x)$ funksiya limitga ega bo'lsa, u yagona bo'ladi.

Bu xossaning isboti limit ta'riflarining ekvivalentligi hamda ketma-ketlik limitining yagonaligidan kelib chiqadi.

2-xossa. Agar

$$\lim_{x\to x_0} f(x)=b,\ \ (b-\text{chekli son})$$

bo'lsa, u holda x_0 nuqtaning shunday $U_\delta(x_0)$ $(\delta>0)$ atrofi topiladiki, bu atrofda $f(x)$ funksiya chegaralangan bo'ladi.

Isbot. Aytaylik,

$$\lim_{x\to x_0} f(x)=b$$

bo'lsin. Funksiya limiti ta'rifga binoan

$$\forall\varepsilon>0,\ \exists\delta>0,\ \forall x\in X\cap(U_\delta(x_0)\setminus\{x_0\})\ \text{da}\quad |f(x)-b|<\varepsilon,$$

ya'ni $b-\varepsilon<f(x)<b+\varepsilon$ bo'ladi. Keyingi tengsizliklardan $f(x)$ funksiyaning x_0 nuqtaning $U_\delta(x_0)$ atrofida chegaralanganligi kelib chiqadi.

3-xossa. Agar

$$\lim_{x\to x_0} f(x)=b,$$

bo'lib, $b<p$ bo'lsa,u holda x_0 nuqtaning shunday $U_\delta(x_0)$ atrofi topiladiki, bu atrofda

$$f(x)<p$$

bo'ladi.

Isbot. Shartga ko'ra

$$\lim_{x\to x_0} f(x)=b.$$

Funksiyaning limiti ta'rifiga ko'ra $\varepsilon=p-b>0$ uchun shunday $\delta>0$ son topiladiki, $\forall x\in X$, $|x-x_0|<\delta$, $x\neq x_0$ uchun

$$|f(x)-b|<\varepsilon\ \Rightarrow\ f(x)<b+\varepsilon=p$$

bo'ladi.Bu esa $\forall x\in U_\delta(x_0)$ da $f(x)<p$ bo'lishini bildiradi.

Faraz qilaylik, $f(x)$ va $g(x)$ funksiyalar $X\subset R$ to'plamda berilgan bo'lib, $x_0\in R$ nuqta X to'plamning limit nuqtasi bo'lsin.

4-xossa. Agar

$$\lim_{x\to x_0} f(x)=b_1,\quad \lim_{x\to x_0} g(x)=b_2$$

bo'lib, $\forall x\in X$ da $f(x)\le g(x)$ tengsizlik bajarilsa, u holda $b_1\le b_2$, ya'ni

$$\lim_{x\to x_0} f(x)\le \lim_{x\to x_0} g(x)$$

bo'ladi.

Isbot. Aytaylik,

$$\lim_{x\to x_0} f(x)=b_1,\ \lim_{x\to x_0} g(x)=b_2$$

bo'lsin.

Funksiya limitining Geyne ta'rifiga ko'ra x_0 ga intiluvchi ixtiyoriy

$$x_n\to x_0\qquad (x_n\in X,\ \ x_n\neq x_0)$$

Ketma-ketlik uchun

$$n\to\infty\ \text{da}\ f(x_n)\to b_1,\quad g(x_n)\to b_2 \tag{1}$$

bo'ladi.

Ravshanki, $\forall\, n\in N$ da

$$f(x_n)\le g(x_n) \tag{2}$$

Yaqinlashuvchi ketma-ketlikning xossalaridan foydala-nib, (1) va (2) munosabatlardan $\lim\limits_{x\to x_0} f(x_n)\ \le \lim\limits_{x\to x_0} g(x_n)$, ya'ni $b_1\le b_2$ bo'lishini topamiz.

5-xossa. Faraz qilaylik,

$$\lim_{x\to x_0} f(x)=b_1,\ \ \lim_{x\to x_0} g(x)=b_2,\ \ (b_1,b_2\in R)$$

limitlar mavjud bo‘lsin. U holda

a) $\forall c\in R$ da $\lim\limits_{x\to x_0}(c\cdot f(x))=c\cdot\lim\limits_{x\to x_0} f(x)$;

b) $\lim\limits_{x\to x_0}(f(x)+g(x))=\lim\limits_{x\to x_0} f(x)+\lim\limits_{x\to x_0} g(x)$;

d) $\lim\limits_{x\to x_0}(f(x)\cdot g(x))=\lim\limits_{x\to x_0} f(x)\cdot\lim\limits_{x\to x_0} g(x)$;

e) Agar $b_2\neq 0$ bo'lsa, $\lim\limits_{x\to x_0}\dfrac{f(x)}{g(x)}=\dfrac{\lim\limits_{x\to x_0} f(x)}{\lim\limits_{x\to x_0} g(x)}$

bo‘ladi.

Bu tasdiqlarning isboti sonlar ketma-ketliklari ustida arifmetik amallar bajarilishi haqidagi ma’lumotlardan kelib chiqadi.

1-misol. Ushbu $\lim\limits_{x\to 1}\dfrac{x+x^2+x^3+...+x^n-n}{x-1}$

limit hisoblansin.

◀ Bu limitni yuqoridagi xossalardan foydalanib hisoblaymiz:

$$\lim_{x\to 1}\frac{x+x^2+x^3+...+x^n-n}{x-1}=\lim_{x\to 1}\frac{(x-1)+(x^2-1)+\left(x^3-1\right)+...+(x^n-1)}{x-1}=$$

$$\lim_{x\to 1}\frac{(x-1)\left[1+(x+1)+(x^2+x+1)+...+(x^{n-1}+x^{n-2}+x+1)\right]}{x-1}=$$

$$=1+2+3+....+n=\frac{n(n+1)}{2}$$

2-misol. Ushbu $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}$

limit hisoblansin.

Ma’lumki, $1-\cos x=2\sin^2\dfrac{x}{2}$. Shuni hisobga olib topamiz:

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2}=\lim_{x\to 0}\frac{1}{2}\cdot\left[\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right]^2=$$

$$=\frac{1}{2}\cdot\left[\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\cdot\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right]=\frac{1}{2}$$

2⁰. Funksiya limitining mavjudligi. Faraz qilaylik, $f(x)$ funksiya $X\subset R$ to'plamda berilgan bo‘lib, $(x_0-\gamma,x_0)\subset X$ bo’lsin $(\gamma>0)$. Ravshanki, $x_0\in R$ nuqta X to‘plamning limit nuqtasi bo‘ladi.

1-teorema. Agar $f(x)$ funksiya X to‘plamda o‘suvchi bo‘lib, u yuqoridan chegaralangan bo‘lsa, funksiya x_0 nuqtada

$$\lim_{x\to x_0-0} f(x)$$

limitga ega bo’ladi.

Isbot. $f(x)$ funksiya qiymatlaridan iborat bo‘lgan ushbu

$$F=\{f(x)|\,x\in X\cap\{x<x_0\}\}$$

to'plamni qaraymiz. Teoremaning shartiga ko‘ra bu to‘plam yuqoridan chegaralangan bo'ladi. U holda to‘plamning aniq chegarasining mavjudligi haqidagi teoremaga ko'ra F to'plam aniq yuqori chegaraga ega. Uni b bilan belgilaymiz:

$$\sup F=b.$$

Endi, $\lim\limits_{x\to x_0-0} f(x)=b$ bo‘lishini isbotlaymiz. Aniq yuqori chegara ta’rifiga ko'ra:

1) $\forall x\in X\cap\{x<x_0\}$ uchun $f(x)\le b$;

2) $\exists x^*\in X\cap\{x<x_0\}$, $x^*<x_0$: $f(x^*)>b-\varepsilon$, $(\forall\varepsilon>0)$bo’ladi.

Agar $\delta=x_0-x^*>0$ deyilsa,unda $\forall x\in(x_0-\delta,\,x_0)\cap(x_0-\gamma,x_0)$ uchun

$$b-\varepsilon<f(x^*)\le f(x)\le b<b+\varepsilon$$

bo'lib,

$$|f(x)-b|<\varepsilon$$

tengsizlik bajariladi. Bu esa

$$\lim_{x \to x_0 - 0} f(x) = b$$

ekanini bildiradi.

Xuddi shunga o‘xshash quyida keltiriladigan teorema isbotlanadi.

Aytaylik, $f(x)$ funksiya $X \subset R$ to‘plamda berilgan bo‘lib, $(x_0, x_0 + \gamma) \subset X$ bo’lsin $(\gamma > 0)$. Ravshanki, $x_0 \in R$ nuqta X to‘plamning limit nuqtasi bo‘ladi.

2-teorema. Agar $f(x)$ funksiya X to‘plamda kamayuvchi bo‘lib, u quyidan chegaralangan bo‘lsa, funksiya x_0 nuqtada

$$\lim_{x \to x_0 + 0} f(x)$$

limitga ega bo‘ladi.

Endi funksiya limitining mavjudligi haqidagi umumiy teoremani keltiramiz.

Faraz qilaylik, $f(x)$ funksiya $X \subset R$ to‘plamda berilgan bo‘lib, $x_0 \in R$ nuqta X to‘plamning limit nuqtasi bo‘lsin.

1-ta’rif. Agar $\forall \varepsilon > 0$ olinganda ham shunday $\delta > 0$ son topilsaki,

$$\forall x \in X \cap (U_\delta(x_0) \setminus \{x_0\}),\ \forall y \in X \cap (U_\delta(x_0) \setminus \{x_0\})$$

lar uchun

$$|f(x) - f(y)| < \varepsilon \tag{1}$$

tengsizlik bajarilsa, $f(x)$ uchun x_0 nuqtada **Koshi sharti bajariladi** deyiladi.

3-misol. Ushbu $f(x) = x\sin\frac{1}{x}$ funksiya uchun $x_0 = 0$ nuqtada Koshi sharti bajariladi.

Haqiqatan ham, $\forall \varepsilon > 0$ songa ko’ra $\delta = \frac{\varepsilon}{2}$ deyilsa,u holda

$$\forall x \in X \cap (U_{\frac{\varepsilon}{2}}(0) \setminus \{0\}),\ \forall y \in X \cap (U_{\frac{\varepsilon}{2}}(0) \setminus \{0\})$$

lar uchun (ya`ni $|x| < \delta, |y| < \delta$ uchun)

$$|f(x) - f(y)| = \left|x\sin\frac{1}{x} - y\sin\frac{1}{y}\right| \le \left|x\sin\frac{1}{x}\right| + \left|y\sin\frac{1}{y}\right| \le$$

$$\le |x| + |y| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

bo’ladi.

3-teorema (Koshi teoremasi). $f(x)$ funksiya x_0 nuqtada chekli limitga ega bo‘lishi uchun bu funksiya x_0 nuqtada Koshi shartining bajarishi zarur va yetarli.

Zarurligi. $f(x)$ funksiya x_0 nuqtada chekli limitga ega bo‘lsin:

$$\lim_{x \to x_0} f(x) = b.$$

Limit ta'rifiga binoan:

$$\forall \varepsilon > 0,\ \exists \delta > 0, \forall x \in X \cap (U_\delta(x_0) \setminus \{x_0\}) \text{ uchun}$$

$$| f(x) - b) | < \frac{\varepsilon}{2} \tag{2}$$

bo'ladi. Shuningdek, $\forall y \in X \cap (U_\delta(x_0) \setminus \le \{x_0\})$ uchun ham

$$| f(y) - b) | < \frac{\varepsilon}{2} \tag{3}$$

bo'ladi. (2) va (3) munosabatlardan

$$|f(x) - f(y)| \le |f(x) - b| + |b - f(y)| < \varepsilon$$

bo'lishi kelib chiqadi.

Yetarliligi. Aytaylik $f(x)$ funksiya uchun (1) shart bajarilsin. x_0 nuqtaga intiluvchi ikkita

$$x_n \to x_0\ (x_n \ne x_0,\ n = 1, 2, \ldots),\ x_n \in X,$$

$$y_n \to x_0\ (y_n \ne x_0,\ n = 1, 2, \ldots),\ y_n \in X,$$

ketma-ketliklarni olamiz. Bu ketma-ketliklardan foydalanib, ushbu

$$x_1,\ y_1,\ x_2,\ y_2,\ \ldots,\ x_n,\ y_n,\ \ldots$$

ketma-ketlikni hosil qilamiz. Uni z_n bilan belgilaymiz. Ravshanki, z_n ketma-ketlik uchun

$$z_n \to x_0\ (z_n \ne x_0,\ n = 1, 2, \ldots),\ z_n \in X$$

bo'ladi. Teorema shartiga binoan $\forall \varepsilon > 0$ soniga ko’ra $\delta > 0$ sonni olamiz. Modomiki, $n \to \infty$ da $z_n \to x_0$ ekan, unda limit ta’rifiga ko‘ra:

$$\delta > 0,\ \exists n_0 \in N,\ \forall n > n_0 : | z_n - x_0 | < \varepsilon$$

bo'ladi. Unda $\forall m > n_0, \forall n > n_0$ uchun

$$| f(z_m) - f(z_n) | < \varepsilon$$

tengsizlik bajariladi. Bundan $f(z_n)$ ketma-ketlikning fundamental ekanligi kelib chiqadi. Demak $f(z_n)$ ketma-ketlik yaqinlashyvchi:

$$n \to \infty \text{ da } f(z_n) \to b.$$

Unda

$$f(x_n)\to b, \quad f(y_n)\to b$$

bo'lib, funksiya limitining Geyne ta'rifiga binoan

$$\lim_{x\to x_0} f(x)=b.$$

bo'ladi.

3^0. Cheksiz katta va cheksiz kichik funksiyalar. Aytaylik, $\alpha(x)$ hamda $\beta(x)$ funksiyalar $X\subset R$ to'plamda berilgan bo'lib, $x_0\in R$ nuqta X to'plamning limit nuqtasi bo'lsin.

2-ta'rif. Agar

$$\lim_{x\to x_0}\alpha(x)=0$$

bo'lsa, $\alpha(x)$ funksiya $x\to x_0$ da **cheksiz kichik funksiya** deyiladi.

Masalan, $x\to 0$ da $\alpha(x)=\sin x$ funksiya cheksiz kichik funksiya bo'ladi.

3-ta'rif. Agar

$$\lim_{x\to x_0}\beta(x)=\infty$$

bo'lsa, $\beta(x)$ funksiya $x\to x_0$ da **cheksiz katta funksiya** deyiladi.

Masalan, $x\to 0$ da $\beta(x)=\dfrac{1}{x}$ funksiya cheksiz katta funksiya bo'ladi.

Cheksiz kichik hamda cheksiz katta funksiyalar cheksiz kichik hamda cheksiz katta miqdorlar kabi xossalarga ega bo'ladi:

1) Chekli sondagi cheksiz kichik funksiyalar yig'indisi cheksiz kichik funksiya bo'ladi;

2) Chegaralangan funksiyaning cheksiz kichik funksiya bilan ko'paytmasi cheksiz kichik funksiya bo'ladi;

3) Agar $\alpha(x)$ $(\alpha(x)\neq 0)$ cheksiz kichik funksiya bo'lsa, $\dfrac{1}{\alpha(x)}$ funksiya bo'ladi.

4) Agar $\beta(x)$ cheksiz katta funksiya bo'lsa, $\dfrac{1}{\beta(x)}$ cheksiz kichik funksiya bo'ladi.

Limitni hisoblang:

1. $\lim\limits_{x\to 1}\dfrac{x^n-1}{x^k-1}, n,k\in N$

2. $\lim\limits_{x\to 1}\left(\dfrac{n}{1-x^n}-\dfrac{k}{1-x^k}\right), n,k\in N$

3. $\lim\limits_{x\to 6}\dfrac{\sqrt{x-2}-2}{x-6}$

4. $\lim\limits_{x\to 0}\dfrac{x}{\sqrt{1+x}-1}$

5. $\lim\limits_{x\to 1}\dfrac{1-\sqrt[3]{x}}{1-\sqrt[4]{x}}$

6. $\lim\limits_{x\to 0}\dfrac{2\sqrt{x^2+x+1}-2-x}{x^2}$

7. $\lim\limits_{x\to 5}\dfrac{\sqrt{6-x}-1}{3-\sqrt{4+x}}$

8. $\lim\limits_{x\to 2}\dfrac{\sqrt{7+2x-x^2}-\sqrt{1+x+x^2}}{2x-x^2}$

9. $\lim\limits_{x\to 3}\dfrac{\sqrt{x^2-2x+6}-\sqrt{x^2+2x-6}}{x^2-4x+3}$

10. $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{x+8}-2}{\sqrt{1+2x}-1}$

11. $\lim\limits_{x\to -8}\dfrac{\sqrt[3]{9+x}+x+7}{\sqrt{15+x}+1}$

12. $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{2x^2+10x+1}-\sqrt[7]{x^2+10x+1}}{x}$

13. $\lim\limits_{x\to \infty}\dfrac{1+14x}{2x+\sqrt[3]{x^2}}$

14. $\lim\limits_{x\to \infty}\dfrac{5x^6-1}{\sqrt{x^2+5x^5-1}}$

15. $\lim\limits_{x\to \infty}\dfrac{\sqrt{4x^2+\sqrt{x^3+x^4}}}{\sqrt{x^2+4}}$

16. $\lim\limits_{x\to \infty}\dfrac{\sqrt{9x^2+1}-\sqrt[3]{x^2-1}}{\sqrt[4]{x^4+1}-\sqrt[5]{x^4-1}}$

17. $\lim\limits_{x\to \infty}\dfrac{\sqrt{x^2+6}+|x|}{\sqrt[6]{x^4+2}-|x|}$

18. $\lim\limits_{x\to \infty}\dfrac{\sqrt[3]{1+\frac{4}{x}}-\sqrt[4]{1+\frac{3}{x}}}{1-\sqrt[5]{1-\frac{5}{x}}}$

19. $\lim\limits_{x\to \infty}\left(\sqrt{x^2-1}-\sqrt{x^2+1}\right)$

20. $\lim\limits_{x\to \infty}\left(\sqrt{x^4+2x^2-1}-\sqrt{x^4-2x^2-1}\right)$

21. $\lim\limits_{x\to \infty}\left(\sqrt{4x^4+13x^2-7}-2x^2\right)$

22. $\lim\limits_{x\to \infty}\left(\sqrt[3]{x^3+3x^2+4x}-\sqrt[3]{x^3-3x^2+4}\right)$

23. $\lim\limits_{x\to \infty}\left(\sqrt{x^4+x^2\sqrt{x^4+1}}-\sqrt{2x^4}\right)$

24. $\lim\limits_{x\to \infty}\left(\sqrt{x^2+\sqrt{x^2+\sqrt{x^2}}}-\sqrt{x^2}\right)$

25. $\lim\limits_{x\to 1}\dfrac{\sqrt[n]{x}-1}{\sqrt[k]{x}-1}, n,k\in N$

26. $\lim\limits_{x\to 0}\dfrac{\sqrt[n]{a+x}-\sqrt[n]{a-x}}{x}, \ n\in N, a>0$

27. $\lim\limits_{x\to 0}\dfrac{\sqrt[n]{1+ax}-\sqrt[k]{1+bx}}{x}, \ n,k\in N$

28. $\lim\limits_{x\to 0}\dfrac{x}{\sqrt[n]{1+ax}\cdot\sqrt[k]{1+bx}-1}, \ n,k\in N$

29. $\lim\limits_{x\to 0}\dfrac{\left(\sqrt{1+x^2}+x\right)^n-\left(\sqrt{1+x^2}-x\right)^n}{x}, n\in N$

30. $\lim\limits_{x\to \infty}\left(\sqrt[n]{(1+x^2)(2+x^2)\dots(n+x^2)}-x^2\right)$, $n\in N$

3.5. Funksiyalarni taqqoslash

1⁰. «O» va «o» belgilar, ularning xossalari. Faraz qilaylik, $f(x)$ va $g(x)$ funksiyalari $X \subset R$ to'plamda berilgan bo'lib, x_0 nuqta X to'plamning limit nuqtasi bo'lsin.

1-ta'rif. Agar shunday o'zgarmas $C > 0$ soni va shunday $\delta > 0$ son topilsaki, $\forall x \in X \cap (U_\delta(x_0) \setminus \{x_0\})$ uchun

$$|f(x)| \le C|g(x)|$$

tengsizlik bajarilsa, yoki

$$\exists C \in R_+, \exists \delta > 0, \forall x \in X \cap (U_\delta(x_0) \setminus \{x_0\}) : |f(x) \le C|g(x)||$$

bo'lsa, $x \to x_0$ da $f(x)$ funksiya $g(x)$ funksiyaga nisbatan *chegaralangan* deyiladi va $f(x) = O(g(x))$ kabi belgilanadi.

Agar

$$\exists C \in R,\ \exists d \in R_+, \forall x, |x| > d : |f(x)| \le C|g(x)|$$

bo'lsa, $x \to x_0 = \infty$ da $f(x)$ funksiya $g(x)$ funksiyaga nisbatan *chegaralangan* deyiladi va yuqoridagidek $f(x) = O(g(x))$ kabi belgilanadi.

Masalan, $x \to 0$ da $x^2 = O(x)$ bo'ladi chunki $x \in (-1,1)$ da $|x^2| \le |x|$.

Agar $f(x)$ funksiya x_0 nuqta atrofida chegaralangan bo'lsa, $x \to x_0$ da $f(x) = O(1)$ kabi yoziladi.

«O» ning xossalari:

1) Agar

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = b$$

bo'lsa, $x \to x_0$ da $f(x) = O(g(x))$ bo'ladi.

2) Agar $x \to x_0$ da $f(x) = O(g(x))$ va $g(x) = O(h(x))$ bo'lsa, u holda $x \to x_0$ da $f(x) = O(h(x))$ bo'ladi. Demak, $x \to x_0$ da $O(O(h(x))) = O(h(x))$

3) Agar $x \to x_0$ da $f(x) = O(g(x))$ va $h(x) = O(g(x))$ bo'lsa, u holda $x \to x_0$ da $f(x) + h(x) = O(g(x))$ bo'ladi.

4) Agar $x \to x_0$ da $f_1(x) = O(g_1(x))$ va $f_2(x) = O(g_2(x))$ bo'lsa, u holda $x \to x_0$ da $f_1(x) \cdot f_2(x) = O(g_1(x) \cdot g_2(x))$ bo'ladi.

2-ta'rif. Agar har qanday $\varepsilon > 0$ son olinganda ham shunday $\delta > 0$ son topilsaki,

$$\forall x \in X \cap (U_\delta(x_0) \setminus \{x_0\})$$

uchun

$$|f(x)| \le \varepsilon |g(x)|$$

tengsizlik bajarilsa, ya'ni

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in X \cap (U_\delta(x_0) \setminus \{x_0\}) : |f(x) \le \varepsilon |g(x)||$$ bo'lsa,

$x \to x_0$ da $f(x)$ funksiya $g(x)$ funksiyaga nisbatan **yuqori tartibli cheksiz kichik funksiya** deyiladi va $f(x) = o(g(x))$ yoki $f = o(g)$ kabi belgilanadi.

«o» ning xossalari:

1) Agar $x \to x_0$ da $f = o(g)$ bo'lsa, u holda $x \to x_0$ da $f = O(g)$ bo'ladi.

2) Agar $x \to x_0$ da $f = o(g)$, $g = o(h)$ bo'lsa,u holda $x \to x_0$ da $f = o(h)$ bo'ladi. Demak, $o(o(h)) = o(h)$.

3) Agar $x \to x_0$ da $f_1 = o(g), f_2 = o(g)$ bo'lsa, u holda $x \to x_0$ da $f_1 + f_2 = o(g)$ bo'ladi.

4) Agar $x \to x_0$ da $f_1 = o(g_1)$, $f_2 = o(g_2)$ bo'lsa, u holda $x \to x_0$ da $f_1 \cdot f_2 = o(g_1 \cdot g_2)$ bo'ladi. Demak, $o(g_1) \cdot o(g_2) = o(g_1 \cdot g_2)$.

2⁰. Funksiyalarning ekvivalentligi. Aytaylik, $f(x)$ va $g(x)$ funksiyalari $X \subset R$ to'plamda berilgan bo'lib, x_0 nuqta to'plamning limit nuqtasi bo'lsin.

3-ta'rif. $x \to x_0$ da $f(x)$ va $g(x)$ funksiyalar ($x \ne x_0$ da $g(x \ne 0)$ uchun

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$$

bo'lsa, $x \to x_0$ da $f(x)$ va $g(x)$ **ekvivalent funksiyalar** deyiladi va $f(x) \sim g(x)$ $(x \to x_0)$ kabi belgilanadi.

Masalan, $x \to 0$ da $f(x) = \sin x$ va $g(x) = x$ funksiyalar ekvivalent funksiyalar bo'ladi: $\sin x \sim x \quad (x \to 0)$.

1-teorema. $x \to x_0$ da $f(x)$ va $g(x)$ funksiyalar $x \ne x_0$ da $g(x) \ne 0$ ekvivalent bo'lishi uchun

$$g(x)-f(x)=o(g(x))$$

tenglikning o'rinli bo'lishi zarur va yetarli.

Zarurligi. $x\to x_0$ da $f(x)\sim g(x)$ bo'lsin. Ta'rifga binoan

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=1$$

bo'lib, undan

$$\lim_{x\to x_0}\left[1-\frac{f(x)}{g(x)}\right]=\lim_{x\to x_0}\frac{g(x)-f(x)}{g(x)}=0$$

bo'lishi kelib chiqadi. Demak, $g(x)-f(x)=o(g(x))$.

Yetarliligi. $x\to x_0$ da $g(x)-f(x)=o(g(x))$ bo'lsin.U holda $x\to x_0$ da

$$1-\frac{f(x)}{g(x)}=\frac{g(x)-f(x)}{g(x)}=\frac{o(g(x))}{g(x)}$$

bo'lib,unda

$$\lim_{x\to x_0}\left[1-\frac{f(x)}{g(x)}\right]=\lim_{x\to x_0}\frac{g(x)-f(x)}{g(x)}=0$$

bo'lishi kelib chiqadi. Bu esa

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=1$$

ya'ni $f(x)\sim g(x)$ ekanligini bildiradi.

« ~ » ning xossalari:

1) Har qanday funksiya uchun $x\to x_0$ da $f(x)\sim f(x)$ bo'ladi.

2) Agar $x\to x_0$ da $g(x)\sim h(x)$ bo'lsa, $x\to x_0$ da $h(x)\sim g(x)$ bo'ladi.

3) Agar $x\to x_0$ da $f_1(x)\sim g_1(x)$, $f_2(x)\sim g_2(x)$ bo'lsa, $x\to x_0$ da $f_1(x)\cdot f_2(x)\sim g_1(x)\cdot g_2(x)$ bo'ladi.

3⁰. Funksiyaning asimptotik yoyilmasi. Aytaylik,

$$\lim_{x\to x_0}\frac{f(x)}{g_1(x)}=c_1=const\neq 0$$

bo'lsin.Unda $x\to x_0$ da $f(x)\sim c\,g_1(x)$ bo'lib,

$$f(x)=c_1g_1(x)+o(g_1(x))$$

bo'ladi.Bu holda $c_1g_1(x)$ funksiya $x\to x_0$ da $f(x)$ funksiyaning **bosh qismi** deyiladi.

Faraz qilaylik, $x\to x_0$ da $c_2g_2(x)$ $(c_2=const\neq 0)$ funksiya $f(x)-c_1g_1(x)$ ning bosh qismi bo'lsin. U holda $x\to x_0$ da

$$f(x)-c_1g_1(x)\sim c_2g_2(x)$$

bo'lib,

$$f(x)=c_1g_1(x)+c_2g_2(x)+o(g_2(x))$$

bo'ladi.

Bu jarayonni n marta takrorlab, $x\to x_0$ da $f(x)$ funksiyani quyidagicha yozish mumkin:

$$f(x)=c_1g_1(x)+c_2g_2(x)+\dots+c_ng_n(x)+o(g_n(x)) \qquad (1)$$

bunda $c_i\neq 0$ va

$$g_{i+1}(x)=o(g_i(x)) \qquad (i=1,2,\dots,n).$$

Odatda, (1) formula $x\to x_0$ da $f(x)$ funksiyaning **asimptotik yoyilmasi** deyiladi.

4⁰. Ekvivalentlikdan foydalanib, funksiyalarning limitini topish. Endi funksiyalarning ekvivalentligiga asoslangan holda funksiyalarning limitini hisoblashda foydalaniladigan teoremani keltiramiz.

2-teorema. Agar $x\to x_0$ da $f_1(x)\sim f_2(x)$, $g_1(x)\sim g_2(x)$ bo'lib, ushbu

$$\lim_{x\to x_0}\frac{f_1(x)}{g_1(x)}$$

limit mavjud bo'lsa, u holda

$$\lim_{x\to x_0}\frac{f_2(x)}{g_2(x)}$$

limit ham mavjud va

$$\lim_{x\to x_0}\frac{f_2(x)}{g_2(x)}=\lim\frac{f_1(x)}{g_1(x)}$$

bo'ladi.

Aytaylik, $x\to x_0$ da $f_1(x)\sim f_2(x)$, $g_1(x)\sim g_2(x)$ bo'lsin. Unda ravshanki, $x\to x_0$ da

$$f_2(x)=f_1(x)+o(f_1(x)),$$

$$g_2(x)=g_1(x)+o(g_1(x))$$

bo'ladi. Bu munosabatlardan foydalanib topamiz:

$$\lim_{x\to x_0}\frac{f_2(x)}{g_2(x)}=\lim_{x\to x_0}\frac{f_1(x)+o(f_1(x))}{g_1(x)+o(g_1(x))}=\lim_{x\to x_0}\frac{f_1(x)}{g_1(x)}.$$

Misol. Ushbu $\lim\limits_{x\to 0}\dfrac{\cos x-\cos 2x}{x^2}$

limit hisoblansin.

Ravshanki,

$$\lim_{x\to 0}\frac{\cos x-\cos 2x}{x^2}=\lim_{x\to 0}\frac{2\sin\frac{3x}{2}\cdot\sin\frac{x}{2}}{x^2}$$

Endi $\sin\frac{3x}{2}=\frac{3x}{2}+o(x)$ va $\sin\frac{x}{2}=\frac{x}{2}+o(x)$ bo'lishini e'tiborga olib, topamiz:

$$\lim_{x\to 0}\frac{2\sin\frac{3x}{2}\cdot\sin\frac{x}{2}}{x^2}=2\lim_{x\to 0}\frac{\left(\frac{3}{2}x+o(x)\right)\left(\frac{1}{2}x+o(x)\right)}{x^2}=2\lim_{x\to 0}\frac{\frac{3}{4}x^2+o(x^2)}{x^2}=\frac{3}{2}.$$

Demak, $\lim\limits_{x\to 0}\dfrac{\cos x-\sin 2x}{x^2}=\dfrac{3}{2}.$

IV BOB
FUNKSIYANING UZLUKSIZLIGI VA TEKIS UZLUKSIZLIGI

4.1. *Funksiyaning uzluksizligi tushunchasi*

1°. Funksiyaning uzluksizligi ta'riflari. Faraz qilaylik, $f(x)$ funksiya $X\subset R$ to'plamda berilgan bo'lib, $x_0\in X$ nuqta X to'plamning limit nuqtasi bo'lsin.

1-ta'rif. Agar

$$\lim_{x\to x_0}f(x)=f(x_0) \qquad (1)$$

bo'lsa, $f(x)$ **funksiya** x_0 **nuqtada uzluksiz** deyiladi.

Demak, $f(x)$ funksiyaning x_0 nuqtada uzluksizligi ushbu

1) $\lim\limits_{x\to x_0}f(x)=b$ ning mavjudligi,

2) $b=f(x_0)$ bo'lishi

shartlarining bajarilishi bilan ifodalanadi.

Misollar. 1. Ushbu $f(x)=x^4+x^2+1$

funksiya $\forall x_0\in R$ nuqtada uzluksiz bo'ladi, chunki

$$\lim_{x\to x_0}f(x)=\lim_{x\to x_0}(x^4+x^2+1)=x_0^4+x_0^2+1=f(x_0).$$

2. Ushbu $f(x)=(signx)^2=\begin{cases}1 & \text{agar } x\neq 0 \text{ bo'lsa,}\\ 0 & \text{agar } x=0 \text{ bo'lsa,}\end{cases}$

funksiyani qaraylik. Ravshanki, $\forall x_0\in R$ nuqtada $\lim\limits_{x\to x_0}f(x)=1$ bo'ladi. Demak, qaralayotgan funksiya $\forall x_0\in R$, $x_0\neq 0$ nuqtada uzluksiz bo'ladi.Ammo $f(0)=0$ bo'lganligi sababli

$$\lim_{x\to 0}f(x)\neq f(0)$$

bo'ladi. Demak, $f(x)$ funksiya $x_0=0$ nuqtada uzluksiz bo'lmaydi.

Funksiya limitining Geyne va Koshi ta'riflariga binoan funksiyaning x_0 nuqtadagi uzluksizligini quyidagicha ta'riflash mumkin.

2-ta'rif. Agar

$$n\to\infty \text{ da } x_n\to x_0 \quad (x_n\in X,\ n=1,2,....)$$

bo'ladigan ixtiyoriy $\{x_n\}$ ketma-ketlik uchun

$$n\to\infty \text{ da } f(x_n)\to f(x_0)$$

bo'lsa, $f(x)$ funksiya x_0 funksiya nuqtada uzluksiz deyiladi.

3-ta'rif. Agar $\forall\varepsilon>0$ son olinganda ham shunday $\delta=\delta(\varepsilon)>0$ son topilsaki,

$$\forall x\in X\cap U_{\delta}(x_0)$$

uchun

$$|f(x)-f(x_0)|<\varepsilon$$

tengsizlik bajarilsa, $f(x)$ funksiya x_0 nuqtada uzluksiz deyiladi.

Odatda, $x-x_0$ ayirma **argument orttirmasi**, $f(x)-f(x_0)$ esa **funksiya orttirmasi** deyilib, ular mos ravishda Δx va Δf kabi belgilanadi:

$$\Delta x=x-x_0,\qquad \Delta f=f(x)-f(x_0)=f(x_0+\Delta x)-f(x_0).$$

Unda funksiya uzluksizligining 1-ta'rifidagi (1) munosabat ushbu

$$\lim_{\Delta x\to 0}\Delta f=0 \tag{2}$$

ko'rinishga keladi.

Demak, (2) munosabatni funksiyaning x_0 nuqtada uzluksizligi ta'rifi sifatida qarash mumkin.

Aytaylik, $f(x)$ funksiya $X\subset R$ to'plamda berilgan bo'lib, $x_0\in X$ nuqta X to'plamning o'ng (chap) limit nuqtasi bo'lsin.

4-ta'rif. Agar $\lim\limits_{x\to x_0+0}f(x)=f(x_0)$ $\quad(\lim\limits_{x\to x_0-0}f(x)=f(x_0))$

bo'lsa, $f(x)$ **funksiya** x_0 **nuqtada o'ngdan (chapdan) uzluksiz** deyiladi.

Demak, $f(x)$ funksiya x_0 nuqtada o'ngdan (chapdan) uzluk-siz bo'lganda funksiyaning o'ng (chap) limiti uning x_0 nuqtadagi qiymatiga teng bo'ladi:

$$f(x_0+0)=f(x_0)\qquad (f(x_0-0)=f(x_0)).$$

Keltirilgan ta'riflardan, $f(x)$ funksiya x_0 nuqtada ham o'ngdan, ham chapdan bir vaqtda uzluksiz bo'lsa, funksiya shu nuqtada uzluksiz bo'lishini topamiz.

Umuman, $f(x)$ funksiyaning x_0 nuqtada uzluksiz bo'lishi, $\forall\varepsilon>0$ berilganda ham unga ko'ra shunday $\delta=\delta(\varepsilon)>0$ topilib,

$$\forall x\in U_{\delta}(x_0)\subset X\Rightarrow f(x)\in U_{\varepsilon}(f(x_0))$$

bo'lishini bildiradi.

5-ta'rif. Agar $f(x)$ funksiya $X\subset R$ to'plamning har bir nuqtasida uzluksiz bo'lsa, $f(x)$ funksiya X to'plamda uzluksiz deyiladi.

6-ta'rif. $X\subset R$ to'plamda uzluksiz bo'lgan funksiyalar-dan iborat to'plam uzluksiz funksiyalar to'plami deyiladi va $C(X)$ kabi belgilanadi.

Masalan, $f(x)\in C[a,b]$ bo'lishi, $f(x)$ funksiyaning $[a,b]$ segmentining har bir nuqtasida uzluksiz, ya'ni $f(x)$ funksiya (a,b) intervalning har bir nuqtasida uzluksiz, a nuqtada o'ngdan, b nuqtada esa chapdan uzluksiz bo'lishini bildiradi.

2⁰. Uzluksiz funksiyalar ustida amallar. Uzluksiz funksiyalarning yig'indisi, ko'paytmasi va nisbatining uzluksiz funksiya bo'lishi haqidagi tasdiqlarini keltiramiz.

1-teorema. $f(x)$ va $g(x)$ funksiyalari $X\subset R$ to'plamda berilgan bo'lib, $x_0\in X$ nuqtada uzluksiz bo'lsin. U holda

a) $\forall c\in R$ da $c\cdot f(x)$ funksiya x_0 nuqtada uzluksiz bo'ladi;

b) $f(x)+g(x)$ funksiya x_0 nuqtada uzluksiz bo'ladi;

c) $f(x)\cdot g(x)$ funksiya x_0 nuqtada uzluksiz bo'ladi;

d) $\dfrac{f(x)}{g(x)}$ $(g(x)\neq 0)$ funksiya x_0 nuqtada uzluksiz bo'ladi.

Teoremaning tasdiqlari uzluksizlik ta'rifi hamda limitga ega bo'lgan funksiyalar ustida arifmetik amallar haqidagi teoremadan kelib chiqadi. Masalan, teoremaning c) tasdig'i quyidagicha isbotlanadi:

$$\lim_{x\to x_0}f(x)=f(x_0),\quad \lim_{x\to x_0}g(x)=g(x_0)\quad\Rightarrow$$

$$\Rightarrow\ \lim_{x\to x_0}(f(x)\cdot g(x))=\lim_{x\to x_0}f(x)\cdot\lim_{x\to x_0}g(x)=f(x_0)\cdot g(x_0).$$

1-misol. $f(x)=c,\ c\in R$ bo'lsin. Unda $f(x)\in C(R)$ bo'ladi.

Haqiqatan ham, $\forall\varepsilon>0$ ga ko'ra $\delta=\varepsilon$ deyilsa,u holda

$$\forall x,\ |x-x_0|<\delta:\quad |f(x)-f(x_0)|=|c-c|=0<\varepsilon$$

bo'ladi.

2-misol. $f(x)=x,\ x\in R$ bo'lsa,u holda $f(x)\in C(R)$ bo'ladi.

Haqiqatan ham, $\forall\varepsilon>0$ ga ko'ra $\delta=\varepsilon$ deyilsa,u holda

$$\forall x,\ |x-x_0|<\delta:\quad |f(x)-f(x_0)|=|x-x_0|<\delta=\varepsilon$$

bo'ladi.

3-misol. $f(x)=a_0x^m+a_1x^{m-1}+\ldots+a_{m-1}x+a_m;\ m\in N,\ a_0,a_1,\ldots,a_m\in R$ bo'lsin.U holda $f(x)\in C(R)$ bo'ladi.

Bu tasdiqning isboti 1- va 2-misollar hamda 1-teoremadan kelib chiqadi.

Shunga o'xshash ushbu

$$f(x)=\frac{a_0x^m+a_1x^{m-1}+\ldots+a_{m-1}x+a_m}{b_0x^n+b_1x^{n-1}+\ldots+b_{n-1}x+b_n}$$

funksiyani, (bunda $m,n\in N;\ a_0,a_1,\ldots,a_m,\ b_0,b_1,\ldots,b_n\in R$)

$$\{x\in R\setminus b_0x^n+b_1x^{n-1}+\ldots+b_{n-1}x+b_n=0\}$$

to'plamda uzluksiz bo'lishi ko'rsatiladi.

4-misol. $f(x)=\sin x$ bo'lsin. U holda $f(x)\in C(R)$ bo'ladi.

$x_0 \in R$ nuqtani olib, $\forall \varepsilon > 0$ ga ko'ra $\delta = \varepsilon$ deymiz.

Unda $\forall x,\ |x - x_0| < \delta$:

$$|\sin x - \sin x_0| = 2|\cos\frac{x + x_0}{2} \cdot \sin\frac{x - x_0}{2}| \le |x - x_0| < \delta = \varepsilon$$

bo'ladi.

Xuddi shunga o'xshash $f(x) = \cos x$ funksiya R da, $f(x) = tgx$ va $f(x) = ctgx$ funksiyalarning esa o'z aniqlanish to'plamlarida uzluksiz bo'lishi ko'rsatiladi.

5-misol. $f(x) = a^x$, $a > 0$ bo'lsin.U holda $f(x) \in C(R)$ bo'ladi.

Ravshanki,

$$\lim_{x - x_0 \to 0} (a^{x - x_0} - 1) = 0.$$

Unda

$$0 = \lim_{x - x_0 \to 0} (a^{x - x_0} - 1) \Leftrightarrow \lim_{x - x_0 \to 0} a^{-x_0}(a^x - a^{x_0}) = 0 \Leftrightarrow$$

$$\Leftrightarrow a^{-x_0} \lim_{x \to x_0} (a^x - a^{x_0}) = 0 \Leftrightarrow \lim_{x \to x_0} a^x = a^{x_0}$$

bo'ladi

6-misol. Aytaylik, $f(x) = \begin{cases} -1 & \text{agar} \quad x < 0 \quad \text{bo'lsa} \\ 0 & \text{agar} \quad x = 0 \quad \text{bo'lsa} \\ 1 & \text{agar} \quad x > 0 \quad \text{bo'lsa} \end{cases}$

bo'lsin. Bu funksiya uchun

$$f(+0) = 1, \quad f(-0) = -1$$

bo'lib, berilgan funksiya $X = R \setminus \{0\}$ to'plamda uzluksiz bo'ladi.

3^0. Funksiyaning uzilishi. Aytaylik, $f(x)$ funksiya (a, b) da $(-\infty \le a < b \le +\infty)$ berilgan bo'lib, $x_0 \in (a, b)$ bo'lsin.

Ma'lumki, $f(x)$ funksiyaning x_0 nuqtadagi o'ng va chap limitlari

$$f(x_0 + 0), \quad f(x_0 - 0) \tag{3}$$

mavjud bo'lib,

$$f(x_0 - 0) = f(x_0) = f(x_0 + 0) \tag{4}$$

tenglik o'rinli bo'lsa, u holda $f(x)$ funksiya x_0 nuqtada uzluksiz bo'lar edi.

Agar $f(x)$ funksiya x_0 nuqtada uzluksiz bo'lmasa, unda x_0 nuqta $f(x)$ funksiyaning **uzilish nuqtasi** deyiladi.

7-ta'rif. Agar (3) limitlar mavjud va chekli bo'lib, (4) tengliklarning birortasi o'rinli bo'lmasa, x_0 nuqta $f(x)$ funksiyaning **birinchi tur uzilish nuqtasi** deyiladi.

Bunda

$$f(x_0 + 0) - f(x_0 - 0)$$

ayirma **funksiyaning x_0 nuqtadagi sakrashi** deyiladi.

Masalan, $f(x) = [x]$ funksiya $x = p$ $(p \in Z)$ nuqtada birinchi tur uzilishga ega, chunki

$$f(p + 0) = p, \quad f(p_0 - 0) = p - 1$$

bo'lib,

$$f(p + 0) \ne f(p_0 - 0)$$

bo'ladi.

Agar hech bo'lmaganda (3) limitlarning birortasi mavjud bo'lmasa yoki cheksiz bo'lsa, x_0 nuqta $f(x)$ funksiyaning **ikkinchi tur uzilish nuqtasi** deyiladi.

Masalan, ushbu

$$f(x) = \begin{cases} \sin\frac{1}{x}, \text{agar } x \ne 0 \text{ bo'lsa}, \\ 0, \text{ agar } x = 0 \text{ bo'lsa} \end{cases}$$

funksiya $x = 0$ nuqtada ikkinchi tur uzilishga ega bo'ladi, chunki bu funksiyaning $x = 0$ nuqtadagi o'ng va chap limitlari mavjud emas.

4^0. Murakkab funksiyaning uzluksizligi. Faraz qilaylik, $y = f(x)$ funksiya $X \subset R$ to'plamda, $u = F(y)$ funksiya esa Y_f to'plamda aniqlangan bo'lib, ular yordamida $u = F(f(x))$ murakkab funksiya tuzilgan bo'lsin.

2-teorema. Agar $y = f(x)$ funksiya $x_0 \in X$ nuqtada, $u = F(y)$ funksiya esa $y_0 \in Y_f$ nuqtada $(y_0 = f(x_0))$ uzlukliksiz bo'lsa, $F(f(x))$ funksiya x_0 nuqtada uzluksiz bo'ladi.

$u = F(y)$ funksiya $y_0 \in Y_f$ nuqtada $(y_0 = f(x_0))$ uzluksiz bo'lgani uchun

$$\forall \varepsilon > 0,\ \exists \sigma > 0,\ \forall y,\ |y - y_0| < \sigma:\ |F(y) - F(y_0)| < \varepsilon \tag{5}$$

ya'ni, $|F(f(x)) - F(f(x_0))| < \varepsilon$ bo'ladi.

Shartga ko'ra $y = f(x)$ funksiya $x_0 \in X$ nuqtada uzluksiz. U holda yuqoridagi $\sigma > 0$ ga ko'ra

$$\exists \delta > 0,\ \forall x,\ |x - x_0| < \delta:\ |f(x) - f(x_0)| < \sigma$$

ya'ni,

$$|y - y_0| < \sigma \tag{6}$$

bo'ladi.

(5) va (6) munosabatlardan

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x,\ |x - x_0| < \delta:\ |F(f(x)) - F(f(x_0))| < \varepsilon$$

bo'lishi kelib chiqadi. Demak, $F(f(x))$ funksiya x_0 nuqtada uzluksiz.

5^0. Monoton funksiya uzilish nuqtasining xarakteri.

3-teorema. $[a,b]\subset R$ da monoton bo‘lgan $f(x)$ funksiya shu $[a,b]$ ning istalgan nuqtasida yoki uzluksiz bo‘ladi, yoki birinchi tur uzilishga ega bo‘ladi.

Isbot. $f(x)$ funksiya $[a,b]$ da o‘suvchi bo‘lsin. Aytaylik,

$$x_0\in[a,b],\ (x_0-\delta,\ x_0+\delta)\subset[a,b]\quad(\delta>0)$$

bo‘lsin. Monoton funksiyaning limiti haqidagi teoremaga ko‘ra

$$\lim_{x\to x_0-0} f(x)=f(x_0-0)\le f(x_0),$$

$$\lim_{x\to x_0+0} f(x)=f(x_0+0)\ge f(x_0)$$

bo’ladi.Agar

$$f(x_0-0)=f(x_0)=f(x_0+0)$$

bo’lsa, $f(x)$ funksiya x_0 nuqtada uzluksiz, agar

$$f(x_0-0)<f(x_0+0)$$

bo’lsa, $f(x)$ funksiya x_0 nuqtada birinchi tur uzilishiga ega bo‘ladi. Xuddi shunga o‘xshash $f(x)$ funksiya $[a,b]$ da kamayuvchi bo‘lganda ham tasdiq isbotlanadi.

4.2. Uzluksiz funksiyalarning xossalari

1^0.Nuqtada uzluksiz bo‘lgan funksiyaning xossalari (lokal xossalari).

Misollar. Faraz qilaylik, $f(x)$ funksiya $X\subset R$ to‘plamda berilgan bo‘lib, $x_0\in X$ bo’lsin.

1. Agar $f(x)$ funksiya $x_0\in X$ nuqtada uzluksiz bo‘lsa, u holda shunday $\delta>0$ va $M>0$ sonlari topiladiki, $\forall x\in X\bigcap U_\delta(x_0)$ da $|f(x)|<M$ bo’ladi,ya’ni $f(x)$ funksiya x_0 nuqtaning $U_\delta(x_0)$ atrofida chegaralangan bo‘ladi.

2. Agar $f(x)$ funksiya $x_0\in X$ nuqtada uzluksiz bo‘lib, $f(x_0)\ne 0$ bo‘lsa, u holda shunday $\delta>0$ son topiladiki, $\forall x\in X\bigcap U_\delta(x_0)$ da $\operatorname{sign} f(x)=\operatorname{sign} f(x_0)$ bo’ladi,ya’ni $f(x)$ funksiyaning $U_\delta(x_0)$ dagi ishorasi $f(x_0)$ ning ishorasi kabi bo‘ladi.

Bu tasdiqlarning isboti limitga ega bo‘lgan funksiyaning xossalaridan kelib chiqadi.

3. Aytaylik, $y=f(x)$ funksiya x_0 nuqtada

$$\lim_{x\to x_0} f(x)=b\qquad(b\in R)\tag{1}$$

ga teng bo’lib, $g(y)$ funksiya Y to’plamda berilgan $\{f(x)|x\in X\}\bigcup\{b\}\subset Y$ va $y=b$ nuqtada uzluksiz bo‘lsin. U holda

$$\lim_{x\to x_0} g(f(x))=g(b),$$

ya’ni

$$\lim_{x\to x_0} g(f(x))=g(\lim_{x\to x_0} f(x))\tag{2}$$

bo‘ladi.

$n\to\infty$ da $x_n\to x_0$ $(x_n\in X,\ x_n\ne x_0,\ n=1,2,...)$ bo‘ladigan ixtiyoriy $\{x_n\}$ ketma-ketlikni olaylik. Unda (1) munosabatga ko‘ra

$$n\to\infty \text{ da } f(x_n)\to b$$

bo‘ladi. Shartga ko‘ra $g(f(x))$ funksiya b nuqtada uzluksiz. Demak,

$$n\to\infty \text{ da } g(f(x_n))\to g(b)$$

bo‘ladi. Keyingi munosabatdan (2) tengliknig o‘rinli bo‘lishi kelib chiqadi.

1-misol. Ushbu

$$\lim_{x\to 0}\frac{\log_a(1+x)}{x}=\log_a e\qquad(a>0,\ \ a\ne 1)\tag{3}$$

munosabat isbotlansin.

(2) munosabatdan foydalanib topamiz:

$$\lim_{x\to 0}\frac{\log_a(1+x)}{x}=\lim_{x\to 0}\log_a(1+x)^{\frac{1}{x}}=\log_a\left[\lim_{x\to 0}(1+x)^{\frac{1}{x}}\right]=\log_a e.$$

Xullas, $a=e$ bo’lganda $\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{x}=1$ bo’ladi.

2-misol. Ushbu $\lim\limits_{x\to 0}\dfrac{a^x-1}{x}=\ln a$ $(a>0)$ munosabat isbotlansin.

Keltirilgan tenglikni isbotlash uchun $a^x-1=t$ deb olamiz. Unda $x\to 0$ da $t\to 0$ bo‘ladi. Shuni hamda (3) munosabatni e’tiborga olib topamiz:

$$\lim_{x\to 0}\frac{a^x-1}{x}=\lim_{t\to 0}\frac{t}{\log_a(1+t)}=\frac{1}{\log_a e}=\ln a.$$

3-misol. Ushbu $\lim\limits_{x\to 0}\dfrac{(1+x)^\alpha-1}{x}=\alpha$ $(\alpha\in R)$

munosabat isbotlansin

Ravshanki,

$$(1+x)^\alpha=e^{\alpha\ln(1+x)}$$

va $x\to 0$ da $\ln(1+x)\to 0$ bo’ladi.Unda

$$\frac{(1+x)^\alpha-1}{x}=\frac{(e^{\alpha\ln(1+x)}-1)\cdot\ln(1+x)\cdot\alpha}{\alpha\cdot\ln(1+x)\qquad x}$$

bo'lib,Unda

$$\lim_{x\to 0}\frac{(1+x)^{\alpha}-1}{x}=\lim_{x\to 0}\frac{e^{\alpha\ln(1+x)}-1}{\alpha\ln(1+x)}\cdot\lim_{x\to 0}\frac{\ln(1+x)}{x}\cdot\alpha=\alpha$$

bo'lishi kelib chiqadi.

2^0. Segmentda uzluksiz bo'lgan funksiyalarning xossalari (global xossalar). Aytaylik, $f(x)$ funksiya $[a, b]$ segmentda berilgan bo'lsin.

Ma'lumki, $f(x)$ funksiya (a, b) da uzluksiz, a nuqtada o'ngdan, b nuqtada chapdan uzluksiz bo'lsa, $f(x)$ funksiya $[a, b]$ segmentda uzluksiz bo'ladi.

Endi segmentda uzluksiz bo'lgan funksiyalarning xossalarini keltiramiz. Ular teoremalar orqali ifodalanadi.

1-teorema. (Veyershtrassning birinchi teoremasi). Agar $f(x)$ funksiya $[a, b]$ segmentda uzluksiz, ya'ni $f(x)\in C[a, b]$ bo'lsa,funksiya $[a, b]$ da chegaralangan bo'ladi.

Ma'lumki, $f(x)$ funksiyaning $[a, b]$ da chegaralanganligi quyidagini

$$\exists M\in(0,+\infty),\ \forall x\in[a, b]:\ |f(x)|\le M$$

anglatadi.

Isbot. Teskarisini faraz qilaylik, ya'ni $f(x)\in C[a, b]$ bo'lsa ham funksiya $[a, b]$ da chegaralanmagan bo'lsin. U holda

$$\forall n\in N,\ \exists x_n\in[a, b]:\ |f(x_n)|>n\ \ (n=1,2,...) \qquad (4)$$

bo'ladi. Ayni paytda, hosil bo'ladigan $\{x_n\}$ ketma-ketlik uchun $x_n\in[a, b]$ $(n=1,2,...)$ bo'lganligi sababli u chegaralangan bo'ladi. Unda Bolsano-Veyershtrass teoremasiga ko'ra bu $\{x_n\}$ bu ketma-ketlikdan yaqinlashuvchi qismiy $\{x_{n_k}\}$ ketma-ketlik ajratish mumkin:

$$k\to\infty \text{ da } x_{n_k}\to x_0 \qquad (x_0\in[a, b]).$$

Shartga ko'ra $f(x)$ funksiya $[a, b]$ da uzluksiz. Binobarin,

$$k\to\infty \text{ da } f(x_{n_k})\to f(x_0) \qquad (5)$$

bo'ladi. Bu (5) munosabat yuqorida qilingan farazga ziddir (chunki, faraz bo'yicha

$$\lim_{k\to\infty} f(x_{n_k})=+\infty$$

bo'lishi lozim edi). Demak, $f(x)$ funksiya $[a, b]$ da chegaralangan bo'ladi.

Aytaylik, $f(x)$ funksiya $X\subset R$ to'plamda berilgan bo'lsin.

Ta`rif. Agar X to'plamda shunday $x_0\in X$ nuqta topilsaki, $\forall x\in X$ uchun

$$f(x)\le f(x_0) \qquad (f(x)\ge f(x_0))$$

tengsizlik bajarilsa, $f(x)$ **funksiya x_0 nuqtada eng katta (eng kichik) qiymatga erishadi** deyiladi va

$$f(x_0)=\max_X f(x) \qquad (f(x_0)=\min_X f(x))$$

kabi belgilanadi.

2-teorema. (Veyershtrassning ikkinchi teoremasi). Agar $f(x)\in C[a, b]$ bo'lsa,bu funksiya $[a, b]$ segmentda eng katta hamda eng kichik qiymatlarga erishadi, ya'ni

$$\exists c_1\in[a, b],\ \forall x\in[a, b]:\quad f(x)\le f(c_1),$$
$$\exists c_2\in[a, b],\ \forall x\in[a, b]:\quad f(x)\ge f(c_2)$$

bo'ladi.

Isbot. Aytaylik, $f(x)\in C[a, b]$ bo'lsin. Veyershtrassning 1-teoremasiga ko'ra $f(x)$ funksiya $[a, b]$ segmentda chegaralangan, ya'ni ushbu

$$\{f(x)\mid x\in[a, b]\}$$

to'plam chegaralangan bo'ladi. Unda to'plamning aniq chegarasi haqidagi teoremaga ko'ra

$$\sup_{x\in[a, b]} f(x)=M \qquad (M\in R)$$

mavjud bo'ladi.

To'plamning aniq yuqori chegarasi ta'rifiga muvofiq:

$$\forall x\in[a, b]:\quad f(x)\le M,$$
$$\forall\varepsilon>0,\ \exists x(\varepsilon)\in[a, b]:\ f(x(\varepsilon))>M-\varepsilon$$

bo'ladi. Keyingi tengsizlikda

$$\varepsilon=1,\ \frac{1}{2},\ \frac{1}{3},\ ...,\ \frac{1}{n},\ ...$$

deb olinadigan bo'lsa,

$$x_n=x\left(\frac{1}{n}\right)\in[a, b]$$

ketma-ketlik hosil bo'lib, uning uchun

$$f(x_n)>M-\frac{1}{n}$$

tengsizlik bajariladi. Demak, $\forall n\in N$ da

$$M-\frac{1}{n}<f(x_n)\le M$$

bo'ladi. Bu munosabatdan

$$\lim_{n\to\infty} f(x_n)=M \qquad (6)$$

bo'lishi kelib chiqadi.

Yuqorida hosil qilingan $\{x_n\}$ ketma-ketlik chegaralangan. Undan yaqinlashuvchi qismiy ketma-ketlikni ajratish mumkin. Uni $\{x_{n_k}\}$ deylik:

$$k \to \infty \text{ da } x_{n_k} \to c_1 \qquad (c_1 \in [a, b]).$$

Berilgan $f(x)$ funksiyaning uzluksizligidan foydalanib topamiz:

$$k \to \infty \text{ da } f(x_{n_k}) \to f(c_1).$$

Ravshanki, $\{f(x_{n_k})\}$ ketma-ketlik $\{f(x_n)\}$ ketma-ketlikning qismiy ketma-ketligi.

Demak (6) munosabatga ko'ra

$$k \to \infty \text{ da } f(x_{n_k}) \to M$$

bo'lib, $f(c_1) = M$ bo'lishi kelib chiqadi. Xuddi shunga o'xshash, $f(x)$ funksiyaning eng kichik qiymatga erishishi ko'rsatiladi.

3-teorema. Faraz qilaylik, $f(x)$ funksiya $[a, b]$ segmentda berilgan bo'lib, quyidagi shartlarni bajarsin:

1) $f(x) \in C[a, b]$;

2) segmentning chetki nuqtalar a va b larda har xil ishorali qiymatlarga ega, ya'ni

$$f(a) < 0 < f(b) \text{ yoki } f(a) > 0 > f(b)$$

bo'lsin.

U holda (a, b) da shunday x_0 nuqta $(a < x_0 < b)$ topiladiki, $f(x_0) = 0$ bo'ladi.

Isbot. Aytaylik, $f(x) \in C[a, b]$ bo'lib, $f(a) < 0 < f(b)$ bo'lsin. $[a, b]$ segmentning $f(x)$ funksiyaga manfiy qiymatlar beradigan nuqtalaridan iborat to'plamini E deylik:

$$E = \{x \in [a, b] \mid f(x) < 0\}.$$

Ravshanki, $a \in E$, $E \subset [a, b]$. Demak, E to'plam chegaralangan va $E \neq \varnothing$.

To'plamning aniq yuqori chegarasi haqidagi teoremaga ko'ra

$$\sup E = x_0 \qquad (x_0 \in (a, b))$$

mavjud bo'ladi.

Aniq yuqori chegara ta'rifiga binoan,

$$\forall n \in N, \ \exists x_n \in E: \quad x_0 - \frac{1}{n} < x_n < x_0$$

bo'ladi. Demak,

$$f(x_n) < 0. \qquad (n = 1, 2, 3, \ldots)$$

$f(x)$ funksiyaning $[a, b]$ da uzluksiz bo'lganligini e'tiborga olib topamiz:

$$n \to \infty \text{ da } x_n \to x_0 \text{ bo'lib, } f(x_n) \to f(x_0).$$

Bir tomondan

$$\lim_{n\to\infty} f(x_n) \le 0,$$

ikkinchi tomondan

$$\lim_{n\to\infty} f(x_n) = f(x_0)$$

bo'lishidan

$$f(x_0) \le 0 \qquad (7)$$

bo'lishi kelib chiqadi.

Ravshanki, $x > x_0$ da $x \notin E$. Binobarin, $f(x) \ge 0$. Shuning uchun

$$\lim_{x \to x_0 + 0} f(x) \ge 0$$

bo'lib,

$$f(x_0) = \lim_{x \to x_0 + 0} f(x) \ge 0 \qquad (8)$$

bo'ladi. (7) va (8) munosabatlardan $f(x_0) = 0$ bo'lishi kelib chiqadi. Xuddi shunga o'xshash, $f(x) \in C[a, b]$ va $f(a) > 0 > f(b)$ bo'lgan holda teorema isbotlanadi.

4-teorema. Agar $f(x) \in C[a, b]$ bo'lsa, u holda chegaralari $f(a)$ va $f(b)$ bo'lgan segmentga tegishli ixtiyoriy l soni olinganda $[a, b]$ da shunday x_0 nuqta topiladiki, $f(x_0) = l$ bo'ladi.

Isbot. $f(a) < f(b)$ deb, $f(a) \le l \le f(b)$ ni olaylik. Ravshanki, $f(a) = l$ yoki $f(b) = l$ bo'lgan holda teorema isbotlangan hisoblanadi.

Endi $f(a) < l < f(b)$ bo'lsin. Ushbu

$$g(x) = f(x) - l \qquad (x \in [a, b])$$

funksiyani olaylik. Bu funksiya uchun:

1) $g(x) \in C[a, b]$;

2) $g(a) < 0 < g(b)$

bo'ladi. Unda 3-teoremaga ko'ra shunday $x_0 \in (a, b)$ topiladiki,

$$g(x_0) = 0,$$

ya'ni,

$$f(x_0) = l$$

bo'ladi.

Ushbu ma'ruzaning pirovardida berilgan funksiyaga teskari bo'lgan funksiyaning mavjudligi haqidagi toyeremani isbotsiz keltiramiz.

5-teorema (teskari funksiyaning mavjudligi). Agar $f(x)$ funksiya $X \subset R$ orliqda uzluksiz va qat'iy o'suvchi (qat'iy kamayuvchi) bo'lsa, u holda $Y_f = \{ f(x) | x \in X \}$ oraliqda teskari $f^{-1}(y)$ funksiya mavjud bo'lib, u uzluksiz qat'iy o'suvchi (qat'iy kamayuvchi) bo'ladi.

4.3 Funksiyaning tekis uzluksizligi. Kantor teoremasi

1^0. Funksiyaning tekis uzluksizligi tushunchasi. Faraz qilaylik $f(x)$ funksiya $X \subset R$ to'plamda berilgan bo'lsin.

1-ta'rif. Agar ixtiyriy $\varepsilon > 0$ son olinganda ham shunday $\delta > 0$ son topilsaki,

$$| x' - x'' | < \delta$$

tengsizlikni qanoatlantiruvchi ixtiyoriy $x', x'' \in X$ uchun

$$| f(x') - f(x'') | < \varepsilon$$

tengsizlik bajarilsa, ya'ni

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x', x'' \in X, |x' - x''| < \delta :$$

$$| f(x') - f(x'') | < \varepsilon$$

bo'lsa, $f(x)$ funksiya X to'plamda **tekis uzluksiz** deyiladi.

Keltirilgan ta'rifdan:

1) $\delta > 0$ sonning faqat $\varepsilon > 0$ ga bog'liqligi,

2) $f(x)$ funksiya X da tekis uzluksiz bo'lsa, u shu X to'plamda uzluksiz bo'lishi kelib chiqadi.

1-misol. $f(x) = x, \quad x \in R$ bo'lsin. Bu funksiya R da tekis uzluksiz bo'ladi.

Agar $\forall \varepsilon > 0$ ga ko'ra $\delta = \varepsilon$ deb olinsa,unda $\forall x', x'' \in X$, $|x' - x''| < \delta$ da

$$| f(x') - f(x'') | = | x' - x'' | < \delta = \varepsilon$$

bo'ladi.

2-misol. $f(x) = \sin x, \quad x \in R$ bo'lsin.Bu funksiya R da tekis uzluksiz bo'ladi.

Agar $\forall \varepsilon > 0$ ga ko'ra, $\delta = \varepsilon$ deyilsa,unda $\forall x', x'' \in R$, $|x' - x''| < \delta$ da

$$|\sin x' - \sin x''| = 2\left|\cos\frac{x' + x''}{2}\right| \cdot \left|\sin\frac{x' - x''}{2}\right| \le |x' - x''| < \delta = \varepsilon$$

bo'ladi.

3-misol. $f(x) = \frac{1}{x}, \quad x \in X = (0, 1]$ bo'lsin.Bu funksiya $X = (0, 1]$ da tekis uzluksiz bo'lmaydi.

$\forall \varepsilon > 0$ sonni,masalan, $\varepsilon = \frac{1}{2}$ deb olib, x' va x'' nuqtalar sifatida

$$x' = \frac{1}{n}, \quad x'' = \frac{1}{n+1} \quad (n \in N)$$

deb olinsa, u holda $|x' - x''|$ ayirma quydagicha

$$| x' - x'' | = \left| \frac{1}{n} - \frac{1}{n+1} \right| = \frac{1}{n(n+1)}$$

bo'ladi.Bunda $(|x' - x''| < \delta)$ δ ni har qancha kichik qilib olish mumkin bo'lsa ham

$$| f(x') - f(x'') | = \left| \frac{1}{x'} - \frac{1}{x''} \right| = | n - (n+1) | = 1 > \frac{1}{2} = \varepsilon$$

bo'ladi.Demak, $f(x) = \frac{1}{x}$ funksiya $X = (0, 1]$ da tekis uzluksiz emas.

2^0. 1-teorema (Kantor teoremasi). Agar $f(x) \in C[a, b]$ bo'lsa,u holda $f(x)$ funksiya $[a, b]$ da tekis uzluksiz bo'ladi.

Isbot. Aytaylik, $f(x) \in C[a, b]$ bo'lsa ham funksiya $[a, b]$ da tekis uzluksiz bo'lmasin. Unda biror $\varepsilon > 0$ va ixtiyoriy $\delta > 0$ uchun $[a, b]$ da shunday x' va x'' nuqtalar topiladiki,

$$| x' - x'' | < \delta \quad \Rightarrow \quad | f(x') - f(x'') | \ge \varepsilon$$

bo'ladi. $n \to +\infty$ da $\delta_n \to 0$ $(\delta_n > 0, \ n = 1, 2, ...)$ bo'ladigan ixtiyoriy $\{\delta_n\}$ ketma-ketlikni olamiz. Unda

$$|x_1' - x_1''| < \delta_1 \Rightarrow |f(x_1') - f(x_1'')| \ge \varepsilon,$$

$$|x_2' - x_2''| < \delta_2 \Rightarrow |f(x_2') - f(x_2'')| \ge \varepsilon,$$

....

$$|x_n' - x_n''| < \delta_n \Rightarrow |f(x_n') - f(x_n'')| \ge \varepsilon,$$

....

bo'ladi.

Ravshanki, $\{x_n'\}$ uchun $x_n' \in [a, b]$ $(n = 1, 2, 3, ...)$ bo'lib, undan

$$k \to +\infty \text{ da } x_{n_k}' \to x_0 \quad (x_0 \in [a, b])$$

bo'ladigan qismiy ketma-ketlik ajratish mumkin. Ayni paytda, x_{n_k}'' uchun ham

$$k \to +\infty \text{ da } x_{n_k}'' \to x_0$$

bo'ladi. $f(x) \in C[a, b]$ bo'lishidan

$k \to +\infty$ da $f(x_{n_k}') \to f(x_0)$, $f(x_{n_k}'') \to f(x_0)$ bo'lib,ulardan $k \to +\infty$ da $f(x_{n_k}') - f(x_{n_k}'') \to 0$ bo'lishi kelib chiqadi. Bu esa $\forall n \in N$ uchun

$$| f(x_n') - f(x_n'') | \ge \varepsilon$$

deb olingan farazga zid. Demak $f(x)$ funksiya $[a, b]$ da tekis uzluksiz.

2-ta'rif. Faraz qilaylik $f(x)$ funksiya $X\subset R$ to'plamda berilgan bo'lsin. Ushbu

$$\sup_{x\in X} f(x)-\inf_{x\in X} f(x)$$

ayirma $f(x)$ **funksiyaning** X **to'plamdagi tebranishi** deyiladi va u ω orqali belgilanadi:

$$\omega=\omega(f;\ X)=\sup_{x\in X} f(x)-\inf_{x\in X} f(x).$$

$f(x)$ funksiyaning X to'plamdagi tebranishi quyidagicha

$$\omega=\sup_{x',\ x''\in X}\{|f(x')-f(x'')|\}$$

ham ta'riflanishi mumkin.

Natija. Agar $f(x)\in C[a, b]$ bo'lsa,u holda $\forall\varepsilon>0$ uchun shunday $\delta>0$ topiladiki, $[a, b]$ segment uzunliklari δ dan kichik bo'laklarga ajratilganda har bir bo'lakdagi funksiyaning tebranishi ε dan kichik bo'ladi.

Shartga ko'ra $f(x)\in C[a, b]$. Demak, Kantor teoremasiga ko'ra u $[a, b]$ da tekis uzluksiz. Unda ta'rifga binoan

$$\forall\varepsilon>0,\ \exists\delta>0,\ \forall x',\ x''\in[a, b],\ |x'-x''|<\delta:\ |f(x')-f(x'')|<\varepsilon$$

bo'ladi.

Endi $[a, b]$ segmentni uzunligi δ dan kichik bo'lgan

$$[x_k,\ x_{k+1}]\quad(x_0<x_1<x_2<...<x_n,\ x_0=a,\ x_n=b)$$

bo'laklarga ajaratamiz. Unda

$$\forall x',\ x''\in[x_k,\ x_{k+1}],\ |x'-x''|<\delta:\ |f(x')-f(x'')|<\varepsilon$$

bo'ladi.Demak,

$$\omega=\sup_{x',\ x''\in[x_k,\ x_{k+1}]}\{|f(x')-f(x'')|\}\le\varepsilon$$

bo'ladi.

3^0. Funksiyaning uzluksizlik moduli. $f(x)$ funksiya $X\subset R$ to'plamda berilgan bo'lib, u shu to'plamda uzluksiz bo'lsin. Endi

$$\forall\delta>0,\ \forall x',\ x''\in X,\ |x'-x''|<\delta$$

uchun

$$|f(x')-f(x'')| \qquad (1)$$

ayirmani qaraymiz.

3-ta'rif. (1) ayirmaning aniq yuqori chegarasi

$$\sup\{|f(x')-f(x'')|\}$$

$f(x)$ **funksiyaning** $X\subset R$ **to'plamdagi uzluksizlik moduli** deyiladi va $\omega(\delta)$ kabi belgilanadi:

$$\omega(\delta)=\sup_{|x'-x''|\le\delta}\{|f(x')-f(x'')|\}.$$

Demak, $f(x)$ funksiyaning X to'plamdagi uzluksizlik moduli δ ning manfiy bo'lmagan funksiyasi bo'ladi.

Endi uzluksizlik modulining ba'zi xossalarini keltiramiz:

1) Funksiyaning uzluksizlik moduli δ ning o'suvchi funksiyasi bo'ladi.

Aytaylik, $\delta_1>0$, $\delta_2>0$ va $\delta_1>\delta_2$ bo'lsin.U holda

$$\{x',\ x''\in X:\ |x'-x''|\le\delta_1\},\qquad \{x',\ x''\in X:\ |x'-x''|\le\delta_2\}$$

to'plamlar uchun

$$\{x',\ x''\in X:\ |x'-x''|\le\delta_2\}\subset\{x',\ x''\in X:\ |x'-x''|\le\delta_1\}$$

bo'lib,unda

$$\omega(\delta_2)\le\omega(\delta_1)$$

bo'lishi kelib chiqadi. Demak, $\delta_1>\delta_2\ \Rightarrow\ \omega(\delta_1)\ge\omega(\delta_2)$.

Uzluksizlik modulining keyingi xossasini isbotsiz keltiramiz.

2) Funksiyaning uzluksizlik moduli uchun ushbu

$$\omega(\lambda\delta)\le(1+\lambda)\cdot\omega(\delta)$$

munosabat o'rinli bo'ladi, bunda $\lambda-$ musbat son.

4-misol. Ushbu $f(x)=ax+b\ \ (a,b\in R)$ funksiyaning $X=[\alpha,\ \beta]$ dagi uzluksizlik moduli topilsin.

Ta'rifga binoan,

$$\omega(\delta)=\sup_{|x'-x''|\le\delta}|(ax'+b)-(ax''+b)|=\sup_{|x'-x''|\le\delta}|a(x'-x'')|=|a|\cdot\delta$$

bo'ladi. Demak, $\omega(\delta)=|a|\cdot\delta$.

2-teorema. $f(x)$ funksiya X to'plamda tekis uzluksiz bo'lishi uchun $\lim_{\delta\to+0}\omega(\delta)=0$ tenglikning o'rinli bo'lishi zarur va yetarli.

Zarurligi. $f(x)$ funksiya X to'plamda tekis uzluksiz bo'lsin:

$$\forall\varepsilon>0,\ \exists\delta_\varepsilon>0,\ \forall x',\ x''\in X,\ |x'-x''|<\delta_\varepsilon:\ |f(x')-f(x'')|<\frac{\varepsilon}{2}.$$

U holda $0<\delta<\delta_\varepsilon$ tengsizliklarni qanoatlantiruvchi ixtiyoriy δ uchun

$$\sup_{|x'-x''|\le\delta}\{|f(x')-f(x'')|\}\le\sup_{|x'-x''|\le\delta_\varepsilon}\{|f(x')-f(x'')|\}\le\frac{\varepsilon}{2}<\varepsilon$$

bo'lib,unda $\omega(\delta)<\varepsilon$, ya'ni

$$\lim_{\delta \to +0} \omega(\delta) = 0$$

bo‘lishi kelib chiqadi.

Yetarliligi. Ushbu

$$\lim_{\delta \to +0} \omega(\delta) = 0$$

munosabat o‘rinli bo‘lsin. Demak, $\delta \to +0$ da

$$\omega(\delta) = \sup_{|x'-x''| \le \delta} \{|f(x') - f(x'')|\} \to 0.$$

U holda

$$\forall x',\ x'' \in X,\ \ |x' - x''| \le \delta < \delta_\varepsilon :\ |f(x') - f(x'')| < \varepsilon$$

bo’ladi.Demak, $f(x)$ funksiya X funksiya to‘plamda tekis uzluksiz bo‘ladi.

Funksiyaning uzluksizlik moduli funksiyalarni sinflarga ajratish imkonini beradi. Masalan, uzluksizlik moduli ushbu

$$\omega(\delta) \le M \cdot \delta^\alpha$$

(bunda $M = const$, $0 < \alpha \le 1$) tengsizlikni qanoatlantiruvchi funksiyalar to‘plami α tartibli Lirshits sinfi deyiladi va $Lip_M \alpha$ kabi belgilanadi.

Tekis uzluksizlikka tekshiring:

1. $f(x) = 3x - 5, X = (-\infty : \infty)$
2. $f(x) = x^2 - x + 1, X = (-3 : 4)$
3. $f(x) = \frac{1}{x}, X = [0.2 : 1]$
4. $f(x) = \sqrt{x}, X = [0 : \infty)$
5. $f(x) = \sqrt[3]{x}, X = [0 : 2]$
6. $f(x) = x^3 - 1, X = [-2 : 3]$
7. $f(x) = \frac{1}{x-2}, X = [3 : 4]$
8. $f(x) = 3\sin x + 2\cos x, X = (-\infty : \infty)$
9. $f(x) = \frac{1}{x}, X = (0 : 1)$
10. $f(x) = \sin\frac{1}{x}, X = (0 : 1)$
11. $f(x) = \cos\frac{1}{x}, X = (0 : 1)$
12. $f(x) = x^2, X = (-\infty : \infty)$
16. $f(x) = x^2, X = (-10 : 10)$
17. $f(x) = x\sin x, X = [0 : \infty)$
18. $f(x) = e^x, X = (-\infty : \infty)$
19. $f(x) = \cos x \cos\frac{\pi}{x}, X = (0 : 1)$
20. $f(x) = x\sin\frac{1}{x}, X = (0 : \pi)$
21. $f(x) = \sin\sqrt{x}, X = [1 : \infty)$
22. $f(x) = e^{-\arcsin x}, X = [-1 : 1]$
23. $f(x) = x + \sin x, X = (-\infty : \infty)$
24. $f(x) = 2x - 1, X = (-\infty : \infty)$
25. $f(x) = x^2, X = [-e : e]$
26. $f(x) = \frac{1}{x^2}, X = (0 : 1)$
27. $f(x) = \ln x, X = [1 : \infty)$

13. $f(x) = \frac{1}{x-3}, X = (3 : 5)$
14. $f(x) = \ln x, X = (0 : 1)$
15. $f(x) = \frac{\sin x}{x}, X = (0 : \pi)$
28. $f(x) = \cos x, X = (-\infty : \infty)$
29. $f(x) = \frac{1}{x}, X = [a : \infty), a > 0$
30. $f(x) = \begin{cases} x+1 & agar \quad x \le 0 \quad bo'lsa \\ e^{-x} & agar \quad x > 0 \quad bo'lsa \end{cases}, X = (-\infty : \infty)$

V BOB
FUNKSIYANING HOSILA VA DIFFERENSIALLARI

5.1. Funksiyaning hosilasi

1⁰. Funksiya hosilasining ta'rifi. Misollar. Faraz qilaylik, $f(x)$ funksiya $(a,b)\subset R$ da berilgan bo'lib, $x_0\in(a,b)$, $x_0+\Delta x\in(a,b)$ bo'lsin.

Ma`lumki ushbu

$$\Delta f(x_0)=f(x_0+\Delta x)-f(x_0)$$

ayirma $f(x)$ funksiyaning x_0 nuqtadagi orttirmasi deyiladi.

1-ta'rif. Agar ushbu

$$\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

limit mavjud va chekli bo'lsa, u $f(x)$ **funksiyaning** x_0 **nuqtadagi hosilasi** deyiladi va $\frac{df(x_0)}{dx}$, yoki $f'(x_0)$, yoki $(f(x))'_{x_0}$ kabi belgilanadi. Demak,

$$f'(x_0)=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}. \quad (1)$$

Agar $x_0+\Delta x=x$ deyilsa,unda $\Delta x=x-x_0$ va $\Delta x\to 0$ da $x\to x_0$ bo'lib, (1) munosabat quyidagi

$$f'(x_0)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} \quad (2)$$

ko'rinishga keladi.

1-misol. $f(x)=x$, $x_0\in R$ bo'lsin. Bu funksiya uchun

$$\frac{f(x)-f(x_0)}{x-x_0}=\frac{x-x_0}{x-x_0}=1$$

bo'lib,

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=1$$

bo'ladi.Demak, $f'(x)=(x)'=1$.

2-misol. $f(x)=|x|$, $x\in R$ bo'lsin.

Agar $x>0$ bo'lsa,u holda $f(x)=x$ bo'lib, $f'(x)=1$ bo'ladi.

Agar $x<0$ bo'lsa,u holda $f(x)=-x$ bo'lib, $f'(x)=-1$ bo'ladi.

Agar $x_0=0$ bo'lsa,u holda $\frac{f(x)-0}{x-0}=\frac{|x|}{x}$ bo'lib, $x\to 0$ da bu nisbatlarning limiti mavjud bo'lmaydi. Demak, berilgan funksiya $x_0=0$ nuqtada hosilaga ega bo'lmaydi.

3-misol. $f(x)=x|x|$, $x\in R$, $x_0\in R$ bo'lsin.

a) $x_0>0, x>0$, $x\neq x_0$ uchun

$$\frac{f(x)-f(x_0)}{x-x_0}=\frac{x|x|-x_0|x_0|}{x-x_0}=\frac{x^2-x_0^2}{x-x_0}=x+x_0$$

bo'lib,

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=2x_0=2|x_0|$$

bo'ladi.

b) $x_0<0, x<0$, $x\neq x_0$ uchun

$$\frac{f(x)-f(x_0)}{x-x_0}=\frac{-x^2+x_0^2}{x-x_0}=-x-x_0$$

bo'lib,

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=-2x_0=2|x_0|$$

bo'ladi.

v) $x_0=0$, $x\neq x_0$ uchun

$$\frac{f(x)-f(x_0)}{x-0}=\frac{x|x|}{x}=|x|$$

bo'lib,

$$\lim_{x\to x_0}\frac{f(x)-f(0)}{x-0}=0$$

bo'ladi.Demak, $\forall x\in R$ da $f'(x)=(x|x|)'=2|x|$.

4-misol. Aytaylik,

$$f(x)=\begin{cases} x\cdot\sin\frac{1}{x}, & agar\ x\neq 0\ bo'lsa,\\ 0, & agar\ x=0\ bo'lsa\end{cases}$$

bo'lib, $x_0=0$ bo'lsin.Unda

$$\frac{f(x)-f(x_0)}{x-x_0}=\frac{x\cdot\sin\frac{1}{x}-0}{x-0}=\sin\frac{1}{x}$$

bo'lib,unda $x \to 0$ dagi limiti mavjud emas. Demak, berilgan funksiya nuqtada $x_0 = 0$ nuqtada hosilaga ega emas.

2^0. Funksiyaning o'ng va chap hosilalari. Faraz qilaylik, $f(x)$ funksiya $X \subset R$ to'plamda berilgan bo'lib, $(x_0 - \delta,\ x_0) \subset X$ $(\delta > 0)$ bo'lsin.

2-ta'rif. Agar ushbu

$$\lim_{x \to x_0 - 0} \frac{f(x) - f(x_0)}{x - x_0}$$

limit mavjud bo'lsa, bu limit $f(x)$ **funksiyaning** x_0 **nuqtadagi chap hosilasi** deyiladi va $f'(x_0 - 0)$ kabi belgilanadi:

$$f'(x_0 - 0) = \lim_{x \to x_0 - 0} \frac{f(x) - f(x_0)}{x - x_0}.$$

Aytaylik, $f(x)$ funksiya $X \subset R$ to'plamda berilgan bo'lib, $(x_0,\ x_0 + \delta) \subset X$ $(\delta > 0)$ bo'lsin.

3-ta'rif. Agar ushbu

$$\lim_{x \to x_0 + 0} \frac{f(x) - f(x_0)}{x - x_0}$$

limit mavjud bo'lsa, bu limit $f(x)$ **funksiyaning** x_0 **nuqtadagi o'ng hosilasi** deyiladi va $f'(x_0 + 0)$ kabi belgilanadi:

$$f'(x_0 + 0) = \lim_{x \to x_0 + 0} \frac{f(x) - f(x_0)}{x - x_0}.$$

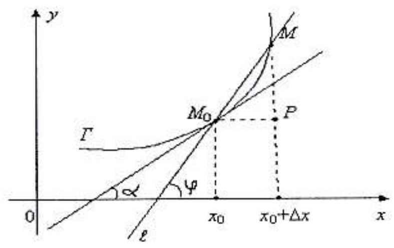
Masalan, $f(x) = |x|$ funksiyaning $x_0 = 0$ nuqtadagi o'ng hosilasi $f'(+0) = 1$, chap hosilasi $f'(-0) = -1$ bo'ladi.

Yuqorida keltirilgan ta'riflardan quyidagi xulosa-lar kelib chiqadi:

1. Agar $f(x)$ funksiya x_0 nuqtada $f'(x_0)$ hosilaga ega bo'lsa, u holda bu funksiya x_0 nuqtada o'ng $f'(x_0 + 0)$ hamda chap $f'(x_0 - 0)$ hosilalarga ega va $f' x_0 - 0) = f'(x_0) = f'(x_0 + 0)$ tengliklar o'rinli bo'ladi.

2. Agar $f(x)$ funksiya x_0 nuqtada o'ng $f'(x_0 + 0)$ hamda chap $f'(x_0 - 0)$ hosilalarga ega bo'lib, $f'(x_0 - 0) = f'(x_0 + 0)$ bo'lsa,u holda $f(x)$ funksiya x_0 nuqtada $f'(x_0)$ hosilaga ega va $f'(x_0 - 0) = f'(x_0) = f'(x_0 + 0)$ tengliklar o'rinli bo'ladi.

3^0. Hosilaning geometrik hamda mexanik ma'nolari. Faraz qilaylik, $f(x)$ funksiya (a,b) da berilgan bo'lib, $x_0 \in (a,b)$ nuqtada $f'(x_0)$ hosilaga ega bo'lsin. Bu $f(x)$ funksiyaning grafigi 5-chizmada tasvirlangan Γ egri chiziqni ifodalasin:



5-chizma.

Bu Γ chiziqda $M_0(x_0, y_0)$, $M(x, y)$ nuqtalarni olib, ular orqali o'tuvchi l kesuvchini qaraymiz.

$M_0(x_0, f(x_0)) \in \Gamma$, $M(x, f(x)) \in \Gamma$, $M \to M_0$ da l kesuvchi limit holati Γ chiziqqa M_0 nuqtada **o'tkazilgan urinma** deyiladi.

Ravshanki, φ burchak Δx ga bog'liq: $\varphi = \varphi(\Delta x)$. $f(x)$ funksiyaning grafigiga M_0 nuqtada o'tkazilgan urinmaning mavjud bo'lishi uchun

$$\lim_{\Delta x \to 0} \varphi(\Delta x) = \alpha$$

ning mavjud bo'lishi lozim. Bunda α–urinmanning OX o'qining musbat yo'nalishi bilan tashkil etgan burchak.

M_0MP uchburchakdan:

$$tg\varphi(\Delta x) = \frac{MP}{M_0P} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

bo'lib,unda

$$\varphi(\Delta x) = arctg \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

bo'lishi kelib chiqadi. Funksiya uzluksizligidan foydalanib topamiz:

$$\lim_{\Delta x \to 0} \varphi(\Delta x) = \lim_{\Delta x \to 0} arctg \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} =$$

$$= arctg\left[\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}\right] = arctgf'(x_0).$$

Demak, $\Delta x \to 0$ da $\varphi(\Delta x)$ ning limiti mavjud va

$$\alpha = arctgf'(x_0).$$

Keyingi tenglikdan

$$f'(x_0) = tg\alpha$$

bo'lishi kelib chiqadi.

Demak, funksiyaning x_0 nuqtadagi $f'(x_0)$ hosilasi urinmaning burchak koeffitsentini ifodalaydi. Bunda urinmaning tenglamasi

$$y = f(x_0) + f'(x_0)(x - x_0)$$

ko'rinishda bo'ladi.

Aytaylik, P nuqta to'g'ri chiziq bo'ylab $s = s(t)$ qonun bilan harakat qilsin, bunda t – vaqt, s – o'tilgan yo'l. Agar vaqtning t_1 va t_2 $(t_1 < t_2)$ qiymatlaridagi o'tilgan yo'l $s(t_1)$, $s(t_2)$ bo'lsa,unda ushbu nisbat

$$\frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

$[t_1, t_2]$ vaqt oralig'idagi o'rtacha tezlikni ifodalaydi.

Quyidagi

$$\lim_{t_2 \to t_1 + 0} \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

limit harakatdagi nuqtaning t_1 vaqtdagi oniy tezligini bildiradi.

Demak, harakatdagi P nuqtaning t vaqtdagi oniy tezligi $v(t)$, o'tilgan $s(t)$ yo'lning hosilasidan iborat bo'ladi:

$$v(t) = s'(t).$$

4⁰. Hosilaga ega bo'lgan funksiyaning uzluksizligi. Faraz qilaylik, $f(x)$ funksiya $(a, b) \subset R$ da berilgan bo'lsin.

Teorema. Agar $f(x)$ funksiya $x_0 \in (a, b)$ nuqtada chekli $f'(x_0)$ hosilaga ega bo'lsa, u holda $f(x)$ funksiya x_0 nuqtada uzluksiz bo'ladi.

Isbot. Aytaylik, $f(x)$ funksiya $x_0 \in (a, b)$ nuqtada chekli $f'(x_0)$ hosilaga ega bo'lsin. Ta'rifga binoan

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta f(x_0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

ya'ni

$$\Delta x \to 0 \text{ da } \frac{\Delta f(x_0)}{\Delta x} \to f'(x_0)$$

bo'ladi.

Endi

$$\alpha = \frac{\Delta f(x_0)}{\Delta x} - f'(x_0)$$

deb belgilaymiz.

Ravshanki,

$$\Delta x \to 0 \text{ da } \alpha \to 0.$$

Keyingi tengliklardan topamiz:

$$\Delta f(x_0) = f'(x_0) \cdot \Delta x + \alpha \Delta x.$$

Odatda, bu tenglik funksiya orttirmasining formulasi deyiladi. Undan

$$\lim_{\Delta x \to 0} \Delta f(x_0) = 0$$

bo'lishi kelib chiqadi. Bu $f(x)$ funksiyaning x_0 nuqtada uzluksiz ekanini bildiradi.

Eslatma. Funksiyaning biror nuqtada uzluksiz bo'lishi-dan uning shu nuqtada chekli hosilaga ega bo'lishi har doim ham kelib chiqavermaydi. Masalan, $f(x) = |x|$ funksiya $x = 0$ nuqtada uzluksiz, ammo u shu nuqtada hosilaga ega emas.

Funksiyaning hosilasini toping:

1. $y = (3x - 7)^{10}$
2. $y = (a + bx)^{\alpha}$
3. $y = (a\cos x + b\sin x)^{\alpha}$
4. $y = \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^{12}$
5. $y = \sqrt{2x^2 + \sqrt{x^2 + 1}}$
6. $y = \sqrt[13]{9 + 7\sqrt[5]{2x}}$
7. $y = \sqrt[3]{\frac{1 - x^3}{1 + x^3}}$
8. $y = \sqrt[n]{\frac{ax + b}{cx + d}}$
9. $y = \frac{1}{\sqrt{1 + x^4}\left(x^2 + \sqrt{1 + x^4}\right)}$
10. $y = \cos\frac{1}{x}$
11. $y = ctgx^2 - \frac{1}{3}tg^3 2x$
12. $y = \frac{1 - \cos(8x - 3\pi)}{tg2x - ctg2x}$
13. $y = e^{-\frac{x^2}{2}}$
14. $y = 2^{\sin 2x}$
15. $y = \frac{1}{2}arctg\frac{x}{2} - \frac{1}{3}arcctg\frac{x}{3}$
16. $y = x^3 + 2x$
17. $y = \frac{1}{x}$
18. $y = x\arcsin x$
19. $y = arctgx + x + arcctgx$
20. $y = 7arctg(x + 1)$
21. $y = \sqrt{x} + \sqrt[3]{x} + \sqrt[4]{x}$
22. $y = 5x\cos x$
23. $y = (x + 1)tgx$
24. $y = x^2 arctgx + 2$
25. $y = 2^x \ln|x|$
26. $y = e^x \log_2 x$
27. $y = \log_2 x \ln x \log_3 x$
28. $y = \log_x 2$
29. $y = \log_x 2^x$
30. $y = \frac{\arcsin x}{e^x}$

5.2. Hosilani hisoblash qoidalari

1^0. Ikki funksiya yig'indisi, ayirmasi, ko'paytmasi va nisbatining hosilasi. Aytaylik, $f(x)$ va $g(x)$ funksiyalari $(a,b)\subset R$ da berilgan bo'lib, $x_0\in(a,b)$ nuqtada $f'(x_0)$ va $g'(x_0)$ hosilalarga ega bo'lsin. Hosila ta'rifiga ko'ra

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=f'(x_0), \qquad (1)$$

$$\lim_{x\to x_0}\frac{g(x)-g(x_0)}{x-x_0}=g'(x_0) \qquad (2)$$

bo'ladi.

1) $f(x)\pm g(x)$ funksiya x_0 nuqtada hosilaga ega bo'lib,

$$(f(x)\pm g(x))'_{x_0}=f'(x_0)\pm g'(x_0)$$

bo'ladi.

$F(x)=f(x)\pm g(x)$ deb topamiz:

$$\frac{F(x)-F(x_0)}{x-x_0}=\frac{f(x)-f(x_0)}{x-x_0}\pm\frac{g(x)-g(x_0)}{x-x_0}.$$

Bu tenglikda $x\to x_0$ da limitga o'tib, yuqoridagi (1) va (2) munosabatlarni e'tiborga olsak, unda

$$\lim_{x\to x_0}\frac{F(x)-F(x_0)}{x-x_0}=\lim_{x\to x_0}\frac{f(x)+f(x_0)}{x-x_0}\pm$$

$$\pm\lim_{x\to x_0}\frac{g(x)-g(x_0)}{x-x_0}=f'(x_0)\pm g'(x_0)$$

bo'lishi kelib chiqadi. Demak,

$$F'(x_0)=(f(x)\pm g(x))'_{x_0}=f'(x_0)\pm g'(x_0).$$

2) $f(x)\cdot g(x)$ funksiya x_0 nuqtada hosilaga ega bo'lib,

$$(f(x)\cdot g(x))'_{x_0}=f'(x_0)\cdot g(x_0)\pm f(x_0)\cdot g'(x_0)$$

bo'ladi.

$\Phi(x)=f(x)\cdot g(x)$ deb

$$\frac{\Phi(x)-\Phi(x_0)}{x-x_0}$$

nisbatni quydagicha

$$\frac{\Phi(x)-\Phi(x_0)}{x-x_0}=\frac{f(x)-f(x_0)}{x-x_0}\cdot g(x_0)+\frac{g(x)-g(x_0)}{x-x_0}\cdot f(x)$$

yozib olamiz.So'ng $x\to x_0$ da limitga o'tib topamiz:

$$\lim_{x\to x_0}\frac{\Phi(x)-\Phi(x_0)}{x-x_0}=g(x_0)\cdot\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}+\lim_{x\to x_0}\frac{g(x)-g(x_0)}{x-x_0}\cdot f(x)=$$

$$=f'(x_0)\cdot g(x_0)+f(x_0)\cdot g'(x_0).$$

Demak,

$$\Phi'(x_0)=(f(x)\cdot g(x))'_{x_0}=f'(x_0)\cdot g(x_0)+f(x_0)\cdot g'(x_0)$$

3) $\dfrac{f(x)}{g(x)}$ funksiya $(g(x_0)\neq 0)$ x_0 nuqtada hosilaga ega bo'lib,

$$\left(\frac{f(x)}{g(x)}\right)'_{x_0}=\frac{f'(x_0)\cdot g(x_0)-f(x_0)\cdot g'(x_0)}{g^2(x_0)}$$

bo'ladi.

Modomiki, $g(x_0)\neq 0$ ekan,unda x_0 nuqtaning biror atrofidagi x larda $g(x)\neq 0$ bo'ladi. Shuni e'tiborga olib topamiz:

$$\frac{\dfrac{f(x)}{g(x)}-\dfrac{f(x_0)}{g(x_0)}}{x-x_0}=\frac{f(x)\cdot g(x_0)-f(x_0)\cdot g(x_0)+f(x_0)\cdot g(x_0)-f(x_0)\cdot g(x)}{g(x)\cdot g(x_0)\cdot(x-x_0)}=$$

$$=\frac{1}{g(x)\cdot g(x_0)}\left[\frac{f(x)-f(x_0)}{x-x_0}\cdot g(x_0)-f(x_0)\cdot\frac{g(x)-g(x_0)}{x-x_0}\right].$$

Bu tenglikda $x\to x_0$ da limitga o'tib ushbu

$$\left(\frac{f(x)}{g(x)}\right)'_{x_0}=\frac{f'(x_0)\cdot g(x_0)-f(x_0)\cdot g'(x_0)}{g^2(x_0)}$$

tenglikka kelamiz.

1-natija. Agar $f(x)$ funksiya x_0 nuqtada $f'(x_0)$ hosilaga ega bo'lsa, $c\cdot f(x)$ funksiya $(c=const)$ x_0 nuqtada hosilaga ega bo'lib,

$$(c\cdot f(x))'_{x_0}=c\cdot f'(x_0)$$

bo'ladi, ya'ni o'zgarmas sonni hosila ishorasidan tashqariga chiqarish mumkin.

2-natija. Agar $f_1(x),\ f_2(x),\ldots,\ f_n(x)$ funksiyalar x_0 nuqtada hosilalarga ega bo'lib, $c_1,\ c_2,\ldots,c_n$ o'zgarmas sonlar bo'lsa, u holda

$$(c_1f_1(x)+c_2f_2(x)+\ldots+c_nf_n(x))'_{x_0}=c_1f_1'(x_0)+c_2f_2'(x_0)+\ldots+c_nf_n'(x_0)$$

bo'ladi.

2^0. Murakkab funksiyaning hosilasi. Faraz qilaylik, $y=f(x)$ funksiya $X\subset R$ to'plamda, $g(y)$ funksiya $\{f(x)|x\in X\}$ to'plamda berilgan bo'lib, $x_0\in X$ nuqtada $f'(x_0)$ hosilaga, $y_0\in\{f(x)|\ x\in X\}$ nuqtada $(y_0=f(x_0))$ $g'(y_0)$

hosilaga ega bo'lsin. U holda $g(f(x))$ murakkab funksiya x_0 nuqtada hosilaga ega bo'lib,

$$(g(f(x)))'_{x_0} = g'(f(x_0))\cdot f'(x_0)$$

bo'ladi.

$g(y)$ funksiyaning y_0 nuqtada $g'(y_0)$ hosilaga ega bo'lganligidan

$$g(y)-g(y_0)=g'(y_0)\cdot(y-y_0)+\alpha\cdot(y-y_0)$$

bo'lishi kelib chiqadi, bunda

$$y=f(x),\ y_0=f(x_0) \text{ va } y\to y_0 \text{ da } \alpha\to 0.$$

Keyingi tenglikning har ikki tomonini $x-x_0$ ga bo'lib topamiz:

$$\frac{g(f(x))-g(f(x_0))}{x-x_0}=g'(f(x_0))\cdot\frac{f(x)-f(x_0)}{x-x_0}+\alpha\frac{f(x)-f(x_0)}{x-x_0}.$$

Bundan $x\to x_0$ da limitga o'tib,

$$(g(f(x)))'_{x_0} = g'(f(x_0))\cdot f'(x_0)$$

tenglikka kelamiz.

3⁰. Teskari funksiyaning hosilasi. Aytaylik, $y=f(x)$ funksiya (a,b) da berilgan, uzluksiz va qat'iy o'suvchi (qat'iy kamayuvchi) bo'lib, $x_0\in(a,b)$ nuqtada $f'(x_0)$ $(f'(x_0)\neq 0)$ hosilaga ega bo'lsin. U holda $x=f^{-1}(y)$ funksiya y_0 $(y_0=f(x_0))$ nuqtada hosilaga ega va

$$\left[f^{-1}(y)\right]'_{x_0}=\frac{1}{f'(x_0)}$$

bo'ladi.

Ravshanki,

$$f(x)-f(x_0)=f'(x_0)(x-x_0)+\alpha(x-x_0)$$

bo'lib, $x\to x_0$ da $\alpha\to 0$ bo'ladi.Bu tenglikdan

$$y-y_0=f'(x_0)\left[f^{-1}(y)-f^{-1}(y_0)\right]-\alpha\left[f^{-1}(y)-f^{-1}(y_0)\right]=$$
$$=\left[f^{-1}(y)-f^{-1}(y_0)\right]\cdot\left[f'(x_0)+\alpha\right]$$

ifodaga kelamiz. Bundan esa

$$\frac{f^{-1}(y)-f^{-1}(y_0)}{y-y_0}=\frac{1}{f'(x_0)+\alpha}$$

bo'lishi kelib chiqadi.

Keyingi tenglikda $y\to y_0$ da limitga o'tib topamiz:

$$\left[f^{-1}(y)\right]'_{y_0}=\frac{1}{f'(x_0)}.$$

4⁰. Misollar. 1-misol. $(x^\alpha)'=\alpha x^{\alpha-1}$ bo'ladi, $\alpha\in R$, $x>0$.

Aytaylik, $x>0$ bo'lsin.Unda $f(x)=x^\alpha$ funksiya uchun

$$\frac{(x+\Delta x)^\alpha-x^\alpha}{\Delta x}=x^{\alpha-1}\cdot\frac{\left(1+\frac{\Delta x}{x}\right)^\alpha-1}{\frac{\Delta x}{x}}$$

bo'lib, $\Delta x\to 0$ da $(x^\alpha)'=\alpha x^{\alpha-1}$ bo'ladi.

2-misol. $(a^x)'=a^x\ln a$ bo'ladi, $a>0$, $x\in R$.

$f(x)=a^x$ funksiya uchun

$$\frac{a^{x+\Delta x}-a^x}{\Delta x}=a^x\cdot\frac{a^{\Delta x}-1}{\Delta x}$$

bo'lib, $\Delta x\to 0$ da $(a^x)'=a^x\ln a$ bo'ladi.

3-misol. $(\sin x)'=\cos x$, $(\cos x)'=-\sin x$ bo'ladi, $x\in R$.

$f(x)=\sin x$ funksiya uchun

$$\frac{\sin(x+\Delta x)-\sin x}{\Delta x}=2\cdot\frac{1}{\Delta x}\cdot\sin\frac{\Delta x}{2}\cos\left(x+\frac{\Delta x}{2}\right)=\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}\cos\left(x+\frac{\Delta x}{2}\right)$$

bo'lib, $\Delta x\to 0$ da $(\sin x)'=\cos x$ bo'ladi. Xuddi shunga o'xshash $(\cos x)'=-\sin x$ bo'lishi topiladi.

4-misol. $(\log_a x)'=\dfrac{1}{x\ln a}$ bo'ladi, $a>0$, $a\neq 1$, $x>0$.

$f(x)=\log_a x$ funksiya uchun

$$\frac{\log_a(x+\Delta x)-\log_a(x)}{\Delta x}=\frac{1}{\Delta x}\log_a\left(1+\frac{\Delta x}{x}\right)=\frac{1}{x}\log_a\left(1+\frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}$$

bo'lib, $\Delta x\to 0$ da

$$(\log_a x)'=\frac{1}{x}\log_a e=\frac{1}{x\ln a}$$

bo'ladi.Xususan, $(\ln x)'=\dfrac{1}{x}$ bo'ladi.

5-misol. $(arctgx)'=\frac{1}{1+x^2}$ bo'ladi.

Teskari funksiya hosilasini hisoblash formulasiga asosan ($y=arctgx,\ x=tgy$)

$$y'=(arctgx)'=\frac{1}{(tgy)'}=\cos^2 y=\frac{1}{1+tg^2y}=\frac{1}{1+x^2}$$

bo'ladi.

Xuddi shunga o'xshash,

$$(\arcsin x)'=\frac{1}{\sqrt{1-x^2}} \qquad (x\in(-1,\ 1)),$$

$$(\arccos x)'=-\frac{1}{\sqrt{1-x^2}} \qquad (x\in(-1,\ 1)),$$

$$(arcctgx)'=-\frac{1}{1+x^2}$$

bo'ladi.

6-misol. Faraz qilaylik,

$$y=[u(x)]^{v(x)} \qquad (u(x)>0)$$

bo'lib, $u'(x)$ va $v'(x)$ lar mavjud bo'lsin. U holda

$$\left([u(x)]^{v(x)}\right)'=[u(x)]^{v(x)}\cdot\left[v'(x)\ln u(x)+\frac{v(x)}{u(x)}u'(x)\right]$$

bo'ladi.

Ushbu $y=[u(x)]^{V(x)}$ ni logarifmlab,

$$\ln y=v(x)\ln u(x),$$

so'ng murakkab funksiyaning hosilasini hisoblash qoidasidan foydalanib topamiz:

$$\frac{1}{y}y'=v'(x)\cdot\ln u(x)+v(x)\cdot\frac{1}{u(x)}\cdot u'(x),$$

$$y'=y\left[v'(x)\cdot\ln u(x)+v(x)\cdot\frac{v(x)}{u(x)}\cdot u'(x)\right]=$$

$$=[u(x)]^{v(x)}\cdot\left[v'(x)\ln u(x)+\frac{v(x)}{u(x)}u'(x)\right].$$

Bu,

$$\left(u^v\right)'=u^v\cdot\ln u\cdot v'+v\cdot u^{v-1}\cdot u'. \qquad (3)$$

tenglikdan, $y=u^v$ funksiya hosilasini hisoblashning quyidagi qoidasi kelib chiqadi: $y=u^v$ funksiyaning hosilasi ikki qo'shiluvchidan iborat bo'lib, birinchi qo'shiluvchi u^v ni ko'rsatkichli funksiya deb olingan hosilasiga (bunda asos $u(x)$ o'zgarmas deb qaraladi) ikkinchi qo'shiluvchi esa u^v ni darajali funksiya deb olingan hosilasiga (bunda daraja ko'rsatkich $v(x)$ o'zgarmas deb qaraladi) teng bo'ladi.

7-misol. Ushbu

$$f(x)=x^x, \quad g(x)=x^{x^x}$$

funksiyalarning hosilalari topilsin.

(3) formuladan foydalanib topamiz:

$$f'(x)=\left(x^x\right)'=x^x\cdot\ln x+x\cdot x^{x-1}=x^x(\ln x+1),$$

$$g'(x)=\left(x^{x^x}\right)'=\left(x^{f(x)}\right)'=x^{f(x)}\cdot\ln x\cdot f'(x)+f(x)\cdot x^{f(x)-1}=$$

$$=x^{x^x}\cdot\ln x\cdot(x^x(\ln x+1))+x^{x^x}\cdot x^{x^x-1}=$$

$$=x^{x^x+x-1}(x^x\ln x(\ln x+1)+1).$$

5⁰. Hosilalar jadvali. Quyida sodda funksiyalarning hosilalarini ifodalovchi formulalarni keltiramiz:

1. $(C)'=0, \quad C=const.$
2. $(x^\alpha)'=\alpha\cdot x^{\alpha-1}, \quad \alpha\in R, \quad x>0.$

   $(x^n)'=nx^{n-1}, \quad n\in N, \quad x\in R.$
3. $(a^x)'=a^x\ln a, \ a>0, \ a\neq 1, \ x\in R$

   $(e^x)'=e^x, \quad x\in R.$
4. $(\log_a x)'=\frac{1}{x\ln a}, \quad a>0, \quad a\neq 1, \quad x>0.$

   $(\log_a |x|)'=\frac{1}{x\ln a}, \quad a>0, \quad a\neq 1, \quad x\neq 0.$

   $(\ln x)'=\frac{1}{x}, \quad x>0.$

$$(\ln|x|)' = \frac{1}{x}, \quad x \neq 0.$$

5. $(\sin x)' = \cos x, \quad x \in R.$

6. $(\cos x)' = -\sin x, \quad x \in R.$

7. $(tgx)' = \dfrac{1}{\cos^2 x}, \quad x \neq \dfrac{\pi}{2} + n\pi, \quad n \in Z.$

8. $(ctgx)' = -\dfrac{1}{\sin^2 x}, \quad x \neq n\pi, \quad n \in Z.$

9. $(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}, \quad |x| < 1$

10. $(\arccos x)' = -\dfrac{1}{\sqrt{1-x^2}}, \quad |x| < 1$

11. $(arctgx)' = \dfrac{1}{1+x^2}, \quad x \in R.$

12. $(arcctgx)' = -\dfrac{1}{1+x^2}, \quad x \in R.$

13. $(shx)' = chx, \quad x \in R.$

14. $(chx)' = shx, \quad x \in R.$

15. $(thx)' = \dfrac{1}{ch^2 x}, \quad x \in R.$

16. $(cthx)' = -\dfrac{1}{sh^2 x}, \quad x \neq 0.$

Quydagi funksiyalarni ko'rsatilgan nuqtada hosilasini toping:

1. $y = x^2, x_0 = 0{,}1$

2. $y = 2\sin 3x, x_0 = \dfrac{\pi}{6}$

16. $y = (chx)^{shx}, x_0 = 0$

17. $y = \left(\sqrt{1+3^x}\right)^{\ln x^2}, x_0 = 1$

3. $y = 1 + \ln 2x, x_0 = 1$

4. $y = x + ctgx, x_0 = \dfrac{\pi}{4}$

5. $y = \dfrac{1+x-x^2}{1-x+x^2}, x_0 = 1$

6. $y = (1+x)\sqrt{2+x^2}\sqrt[3]{3+x^3}, x_0 = 0$

7. $y = 2^{tg\left(\frac{1}{x}\right)}, x_0 = \dfrac{1}{\pi}$

8. $y = 3\cos 2x - \sqrt{1-\sin 2x}(\sin x + \cos x), x_0 = \dfrac{\pi}{6}$

9. $y = \log_{\frac{1}{2}}\left(x - \dfrac{1}{2}\right)^{\frac{1}{2}} + \log_2 \sqrt{4x^2 - 4x + 1}, x_0 = 0$

10. $y = \sqrt{\ln x}(\ln x - \log_{ex} x)\sqrt{\ln x + \log_x e + 2}, x_0 = e$

11. $y = \ln(1 + \sin^2 x) - 2\sin x \, arctg \sin x, x_0 = \dfrac{\pi}{2}$

12. $y = \arcsin\left(\dfrac{2x}{1+x^2}\right), x_0 = 2$

13. $y = \arcsin\left(\dfrac{1-x^2}{1+x^2}\right), x_0 = 1$

14. $y = \ln\dfrac{x^4 - x^2 + 1}{x^4 + 2x^2 + 1} + 2\sqrt{3}\, arctg\dfrac{\sqrt{3}}{1-2x^2}, x_0 = 1$

15. $y = \sqrt{arctg\sqrt[5]{\cos\ln^3 x}}, x_0 = 1$

18. $y = \left(\dfrac{\sin x}{x}\right)^x, x_0 = \dfrac{\pi}{2}$

19. $y = 3|x+1|, x_0 = -2$

20. $y = \dfrac{\sqrt{x+1}}{x-1}, x_0 = 2$

21. $y = \dfrac{x^2}{\sqrt{x^2+4}}, x_0 = 0$

22. $y = \sqrt{\dfrac{1-x^2}{1+x^2}}, x_0 = 0$

23. $y = \dfrac{1}{x^2}, x_0 = 1$

24. $y = \sqrt[3]{x^4}, x_0 = 0$

25. $y = \dfrac{(x-2)^3 \ln x}{\sin x}, x_0 = 2$

26. $y = ctgx + 2x, x_0 = \dfrac{\pi}{4}$

27. $y = |\ln x|, x_0 = 1$

28. $y = |\sin x|, x_0 = \pi$

29. $y = e^{-x} + e^x, x_0 = 2$

30. $y = |\pi^2 - x^2|\sin^2 x, x_0 = 0$

5.3. Asosiy teoremalar

1°. Hosilaga ega bo'lgan funksiyalar haqidagi teoremalar. Bu teoremlar funksiyalarni tekshirishda muhim ro'l o'ynaydi.

1-teorema (Ferma teoremasi). $f(x)$ funksiya $X \subset R$ to'plamda berilgan $x_0 \in X$ nuqtaning atrofi uchun $U_\delta(x_0) = (x_0 - \delta,\ x_0 + \delta) \subset X$ $(\delta > 0)$ bo'lib, quyidagi shartlar bajarilsin:

1) $\forall x \in U_\delta(x_0)$ da $f(x) \le f(x_0)$ $\quad (f(x) \ge f(x_0))$,

2) $f'(x_0)$ mavjud va chekli bo'lsin.

U holda $f'(x_0) = 0$ bo'ladi.

Isbot. Aytaylik, $\forall x \in U_\delta(x_0)$ da $f(x) \le f(x_0)$ bo'lsin. Ravshanki, bu holda

$$f(x) - f(x_0) \le 0$$

bo'ladi.

Shartga ko'ra $f(x)$ funksiya x_0 nuqtada chekli $f'(x_0)$ hosilaga ega. Shuning uchun

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0 + 0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0 - 0} \frac{f(x) - f(x_0)}{x - x_0}$$

bo'ladi. Ayni paytda, $x > x_0$ bo'lganda

$$\frac{f(x) - f(x_0)}{x - x_0} \ge 0 \ \Rightarrow \ \lim_{x \to x_0 - 0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \ge 0$$

$x < x_0$ bo'lganda

$$\frac{f(x) - f(x_0)}{x - x_0} \ge 0 \ \Rightarrow \ \lim_{x \to x_0 - 0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \ge 0$$

bo'lishidan $f'(x_0) = 0$ ekani kelib chiqadi.

2-teorema (Roll teoremasi). Faraz qilaylik, $f(x)$ funksiya $[a, b]$ da berilgan bo'lib, quyidagi shartlarni bajarsin:

1) $f(x) \in C[a, b]$,
2) $\forall x \in (a, b)$ da $f'(x)$ mavjud va chekli,
3) $f(a) = f(b)$ bo'lsin.

U holda shunday $x_0 \in (a, b)$ nuqta topiladiki, $f'(x_0) = 0$ bo'ladi.

Isbot. Shartga ko'ra $f(x) \in C[a, b]$. Unda Veyershtrassning ikkinchi teoremasiga ko'ra $f(x)$ funksiya $[a, b]$ da o'zining eng katta va eng kichik qiymatlarga erishadi, ya'ni shunday c_1, c_2 nuqtalar ($c_1, c_2 \in [a, b]$) topiladiki,

$$f(c_1) = \max\{f(x) \mid x \in [a, b]\},$$
$$f(c_2) = \min\{f(x) \mid x \in [a, b]\}$$

bo'ladi.

Agar $f(c_1) = f(c_2)$ bo'lsa,unda $[a, b]$ da $f(x) = const$ bo'lib, $\forall x_0 \in (a, b)$ da $f'(x_0) = 0$ bo'ladi.

Agar $f(c_1) > f(c_2)$ bo'lsa,unda $f(a) = f(b)$ bo'lgani sababli $f(x)$ funksiya $f(c_1)$ hamda $f(c_2)$ qiymatlarning kamida bittasiga $[a, b]$ segmentning ichki x_0 $(a < x_0 < b)$ nuqtasida erishadi. Ferma teoremasiga binoan $f'(x_0) = 0$ bo'ladi.

3-teorema (Lagranj teoremasi). Faraz qilaylik, $f(x)$ funksiya $[a, b]$ da berilgan bo'lib,quydagi shartlar bajarilsin

1) $f(x) \in C[a, b]$,
2) $\forall x \in (a, b)$ da $f'(x)$ hosilasi mavjud va chekli bo'lsin.

U holda shunday $c \in (a, b)$ nuqta topiladiki,

$$f(b) - f(a) = f'(c)(b - a)$$

bo'ladi.

Ushbu

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a) \qquad (1)$$

funksiyani qaraymiz. Bu funksiya Roll teoremasining barcha shartlarini qanoatlantiradi. Ayni paytda, uning hosilasi

$$F'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

bo'ladi.

Roll teoremasiga binoan, shunday c $(c \in (a, b))$ nuqta topiladiki,

$$F'(c) = 0 \qquad (2)$$

bo'ladi.

(1) va (2) munosabatlardan

$$f'(c) - \frac{f(b) - f(a)}{b - a} = 0,$$

ya'ni

$$f(b) - f(a) = f'(c)(b - a)$$

bo'lishi kelib chiqadi.

1-natija. Aytaylik, $f(x)$ funksiya (a, b) da $f'(x)$ hosilaga ega bo'lib, $\forall x \in (a, b)$ da $f'(x) = 0$ bo'lsin.U holda $\forall x \in (a, b)$ da $f(x) = const$ bo'ladi.

x, $x_0 \in (a, b)$ ni olib,chekkalari x va x_0 bo'lgan segmentda $f(x)$ funksiyaga Lagranj teoremasini qo'llab $f(x) = f(x_0) = const$ bo'lishini topamiz.

2-natija. $f(x)$ va $g(x)$ funksiyalari (a, b) da $f'(x)$, $g'(x)$ hosilalarga ega bo'lib, $\forall x \in (a, b)$ da $f'(x) = g'(x)$ bo'lsin.U holda $\forall x \in (a, b)$ da $f(x) = g(x) + const$ bo'ladi.

Bu natijaning isboti $f(x) - g(x)$ funksiyaga nisbatan 1-natijani qo'llash bilan kelib chiqadi.

4-teorema (Koshi teoremasi). Aytaylik, $f(x)$ va $g(x)$ funksiyalar quyidagi shartlarni bajarsin.

1) $f(x) \in C[a, b]$, $g(x) \in C[a, b]$,
2) $\forall x \in (a, b)$ da $f'(x)$ va $g'(x)$ hosilalari mavjud va chekli;

3) $\forall x \in (a,\ b)$ da $g'(x) \neq 0$ bo'lsin.

U holda shunday $c \in (a,\ b)$ nuqta topiladiki,

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

bo'ladi.

Avvalo $g(b) \neq g(a)$ bo'lishini ta'kidlab o'tamiz, chunki $g(b) = g(a)$ bo'ladigan bo'lsa, unda Roll teoremasiga ko'ra shunday $c \in (a,\ b)$ nuqta topilar ediki, $g'(c) = 0$ bo'lar edi. Bu 3)-shartga zid.

Quyidagi

$$\Phi(x) = f(x) - f(a) - \frac{f(b)-f(a)}{g(b)-g(a)}[g(x)-g(a)] \quad (x \in [a,\ b])$$

funksiyani qaraymiz. Bu funksiya Roll teoremasining barcha shartlarini qanoatlantiradi. Unda Roll teoremasiga binoan shunday $c \in (a,\ b)$ nuqta topiladiki,

$$\Phi'(c) = 0 \tag{3}$$

bo'ladi.

Ravshanki,

$$\Phi'(x) = f'(x) - \frac{f(b)-f(a)}{g(b)-g(a)} g'(x) \tag{4}$$

(3) va (4) munosabatlardan

$$f'(c) - \frac{f(b)-f(a)}{g(b)-g(a)} g'(c) = 0$$

ya'ni

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

bo'lishi kelib chiqadi.

1-misol. $\forall x',\ x'' \in R$ uchun $|\sin x' - \sin x''| \leq |x' - x''|$ tengsizlik isbotlansin.

Aytaylik, $x' < x''$ bo'lsin. $f(x) = \sin x$ ga $[x',\ x'']$ da Lagranj teoremasini qo'llaymiz. Unda shunday $c \in (x',\ x'')$ nuqta topiladiki,

$$|\sin x' - \sin x''| = |\cos c| \cdot (x'' - x')$$

bo'ladi.Agar $\forall t \in R$ da $|\cos t| \leq 1$ ekanini e'tiborga olsak, unda yuqoridagi munosabatdan

$$|\sin x' - \sin x''| \leq |x' - x''| \qquad (\forall x',\ x'' \in R)$$

bo'lishi kelib chiqadi.

2-misol. Ushbu

$$e^x \geq 1 + x$$

tengsizlik isbotlansin.

Aytaylik, $x > 0$ bo'lsin.Unda $f(t) = e^t$ funksiyaga $[x,\ 0]$ da Lagranj teoremasini qo'llab topamiz:

$$e^x - e^0 = e^c(x - 0). \qquad c \in (0,\ x)$$

Agar $c > 0$ da $e^c > 1$ bo'lishini e'tiborga olsak, unda keyingi munosabatdan $e^x \geq 1 + x$ bo'lishi kelib chiqadi.

Agar $x < 0$ bo'lsa,unda $f(t) = e^t$ funksiyaga $[x,\ 0]$ da Lagranj teoremasini qo'llab,

$$e^x - e^0 = e^c(0 - x)$$

ni va $-x > 0$, $e^c < 1$ bo'lishini e'tiborga olib, $e^x \geq 1 + x$ ekanligini topamiz.

Ravshanki, $x = 0$ da $e^0 = 1$. Demak, $\forall x \in R$ da $e^x \geq 1 + x$.

3-misol. Ushbu

$$\frac{a-b}{a} < \ln\frac{a}{b} < \frac{a-b}{b} \qquad (0 < b < a)$$

tengsizlik isbotlansin.

$[b,\ a]$ segmentda $f(x) = \ln(x)$ funksiyani qaraymiz. Bu funksiya shu segmentda uzluksiz va $(b,\ a)$ da $f'(x) = \frac{1}{x}$ hosilaga ega. Unda Lagranj teoremasiga ko'ra shunday c $(b < c < a)$ nuqta topiladiki,

$$\frac{\ln a - \ln b}{a - b} = \frac{1}{c} \tag{5}$$

bo'ladi.

Ravshanki,

$$b < c < a \quad \Rightarrow \quad \frac{1}{a} < \frac{1}{c} < \frac{1}{b}. \tag{6}$$

(5) va (6) munosabatlardan

$$\frac{a-b}{a} < \ln\frac{a}{b} < \frac{a-b}{b}$$

bo'lishi kelib chiqadi.

2°. Funksiya hosilasinig uzilishi haqida. Faraz qilaylik, $f(x)$ funksiya $(a,\ b)$ ning x_0 nuqtasidan boshqa barcha nuqtalarida $f'(x)$ hosilaga ega bo'lib, funksiya x_0 nuqtada uzluksiz bo'lsin.

Agar $\lim\limits_{x \to x_0 - 0} f'(x) = b$ limit mavjud bo'lsa, u holda $f(x)$ funksiya x_0 nuqtada chap hosila $f'(x_0 - 0)$ ga ega bo'lib, $f'(x_0 - 0) = b$ bo'ladi.

Agar $\lim\limits_{x \to x_0+0} f'(x) = d$ limit mavjud bo'lsa,u holda $f(x)$ funksiya x_0 nuqtada o'ng hosila $f'(x_0+0)$ ga ega bo'lib, $f'(x_0+0) = d$ bo'ladi.

Aytalik, $\Delta x \neq 0$ va $x_0 + \Delta x \in (a,\, b)$ bo'lsin. Lagranj teoremasidan foydalanib topamiz:

$$\frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} = f'(x_0 + \theta \cdot \Delta x), \quad (0 < \theta < 1).$$

Endi

$$\lim_{x \to x_0-0} f'(x) = b$$

mavjud bo'lsin deylik. Unda

$$\lim_{x \to x_0-0} f'(x) = \lim_{x - x_0 \to -0} f'(x) = \lim_{\Delta x \to -0} f'(x_0 + \Delta x) = b$$

bo'lib,

$$\Delta x \to -0 \text{ da } f'(x_0 + \theta \cdot \Delta x) \to b,$$

ya'ni

$$\Delta x \to -0 \text{ da } \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} \to b$$

bo'ladi.Demak, $f'(x_0-0) = b$.Shunga o'xshash, $f'(x_0+0) = d$ bo'lishi ko'rsatiladi.

Aytaylik, $f(x)$ funksiya x_0 funksiya nuqtada hosilaga ega bo'lsin. Unda, ravshanki,

$$f'(x_0-0) = f'(x_0+0) = f'(x_0)$$

bo'ladi. Ayni paytda,

$$\lim_{x \to x_0-0} f'(x), \quad \lim_{x \to x_0+0} f'(x)$$

limitlarnig mavjud va chekli bo'lishidan

$$\lim_{x \to x_0-0} f'(x) = \lim_{x \to x_0+0} f'(x) = f'(x_0)$$

bo'lishi kelib chiqadi.

Bundan quyidagi xulosa kelib chiqadi: agar $f(x)$ funksiya $(a,\, b)$ da $f'(x)$ hosilaga ega bo'lsa, u holda bu $f'(x)$ hosila birinchi tur uzilishga ega bo'lolmaydi.

Boshqacha aytganda har bir $x_0 \in (a,\, b)$ nuqtada $f'(x)$ funksiya yoki uzluksiz bo'ladi, yoki ikkinchi tur uzilishga ega bo'ladi.

4-misol. Ushbu

$$f(x) = \begin{cases} x^2 \sin\dfrac{1}{x}, & agar\ x \neq 0\ bo'lsa, \\ 0, & agar\ x = 0\ bo'lsa \end{cases}$$

funksiyani qaraylik.

$x \neq 0$ bo'lganda

$$f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$$

bo'ladi.

$x = 0$ bo'lganda, hosila ta'rifiga ko'ra

$$f'(0) = \lim_{x \to 0} \frac{x^2 \sin\frac{1}{x}}{x} = 0$$

bo'ladi.

Demak, $f'(x)$ funksiya R da aniqlangan va $x \neq 0$ da uzluksiz bo'ladi. $f'(x)$ hosila $x = 0$ nuqtada ikkinchi tur uzilishga ega bo'ladi, chunki $x \to 0$ da

$$f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$$

funksiya limitga ega emas.

Berilga f(x) funksiya uchun ko'rsatilgan oraliqda Ferma teoremasi shartlarini bajariladimi?(1-3).

1. $f(x) = 2x^2 - 1 \quad [1:2]$
2. $f(x) = 5\sqrt{2x+1} - x \quad [4:40]$
3. $f(x) = x\ln 5 - x\ln x \quad \left[\frac{5}{3} : 2{,}5\right]$

Berilga f(x) funksiya uchun ko'rsatilgan oraliqda Roll teoremasi shartlarini bajariladimi?(4-7).

4. $f(x) = \ln\sin x \quad \left[\frac{\pi}{6} : \frac{5\pi}{6}\right]$
5. $f(x) = \sqrt[3]{x^2 - 3x + 2} \quad [1:2]$
6. $f(x) = 4^{\sin x} \quad [0:\pi]$
7. $f(x) = \sin x \quad [1:2]$

Quydagi funnksiyalar uchun ko'rsatilgan oraliqda o'rta qiymat haqidagi teorema shartlarini tekshiring va teorema tasdig'ini qanotlantiruvchi barcha c sonlarni toping(8-10).

8. $f(x) = x^2 \quad x \in [1:2]$
9. $f(x) = x^3 \quad x \in [1:3]$
10. $f(x) = \sqrt{1 - x^2} \quad x \in [0:1]$

11. Ushbu $f(x)=3x^2-5$ funksiya $[-2:0]$ kesmada Lagranj teoremasining shartlarini qanoatlantiradimi? Agar qanoatlantirsa, $f(b)-f(a)=f'(c)(b-a)$ Lagranj formulasidagi c nuqtani toping.

12. $f(x)=\ln x$ funksiyaga $[1:e]$ kesmada Lagranj formulasini qo'llang va unda qatnashadigan c nuqtani toping.

13. $f(x)=\sin 3x$ funksiya uchun $[x_1:x_2]$ kesmada Lagranj formulasini yozing.

14. $f(x)=\arcsin 2x$ funksiya uchun $[x_0:x_0+\Delta x]$ kesmada Lagranj formulasini yozing.

15. $f(x)=x^n$ funksiyaning $[0:a]$ kesmada $(n>0,a>0)$ Lagranj teoremasining shartlarini qanoatlantirishini ko'rsating.

16. $y=|x|$ funksiya uchun $[0:a]$ kesmada Roll teoremasi o'rinli emasligini ko'rsating.

17. Agar $a_0x^n+a_1x^{n-1}+a_2x^{n-2}+...+a_nx=0$ tenglama $x=x_0$ musbat ildizga ega bo'lsa, $na_0x^{n-1}+(n-1)a_1x^{n-2}+....+a_{n-1}=0$ tenglama ham, musbat, x_0 dan kichik ildizga ega bo'lishini isbotlang.

18. $x^3-3x+c=0$ tenglamaning $(0:1)$ oraliqda ikkita har xil ildizga ega bo'lmasligini isbotlang.

19. $y=x^3$ chiziqda shunday nuqtani topingki, unga o'tkazilgan urinma, $A(-1:1)$ va $B(2:8)$ nuqtalarni birlashtiruvchi vatarga parallel bo'lsin.

Lagranj teoremasidan foydalanib quyidagi tengsizliklarni isbotlang:

20. $e^x>ex,\ x>1$

21. $\dfrac{a-b}{a}\le\ln\dfrac{a}{b}\le\dfrac{a-b}{b},\ 0<b\le a$

22. $e^x>1+x,\ x\in R$

23. $|\sin x-\sin y|\le|x-y|$

24. $nb^{n-1}(a-b)<a^n-b^n<na^{n-1}(a-b),\ n>1,a>b$

25. $\dfrac{\alpha-\beta}{\cos^2\beta}\le tg\alpha-tg\beta\le\dfrac{\alpha-\beta}{\cos^2\alpha},\ 0<\beta\le\alpha<\dfrac{\pi}{2}$

26. Ushbu $f(x)=e^x,\ g(x)=\dfrac{x^2}{x^2+1}$ funksiyalar uchun $[-3:3]$ kesmada Koshi teoremasi o'rinlimi?

Funksiyaning o'zgarmaslik alomatidan foydalanib, elementar matematikadan ma'lum bo'lgan quyidagi formulalarni isbot qiling:

27. $\arcsin x+\arccos x=\dfrac{\pi}{2}$

28. $\sin^2 x=\dfrac{1-\cos 2x}{2}$

29. $\arccos\dfrac{1-x^2}{1+x^2}=2arctgx$

5.4. Funksiyaning differensiali

1⁰. Funksiya differensiali tushunchasi. Faraz qilaylik, $f(x)$ funksiya $(a,\ b)$ da berilgan bo'lib, $x_0\in(a,b)$, $x_0+\Delta x\in(a,b)$ bo'lsin.

Ma'lumki, $\Delta f(x_0)=f(x_0+\Delta x)-f(x_0)$ ayirma $f(x)$ funksiyaning x_0 nuqtadagi orttirmasi deyiladi.

1-ta'rif. Agar $\Delta f(x_0)$ ni ushbu

$$\Delta f(x_0)=A\cdot\Delta x+\alpha\Delta x$$

ko'rinishda ifodalash mumkin bo'lsa, $f(x)$ funksiya x_0 nuqtada differensiallanuvchi deyiladi, bunda $A=const,\ \Delta x\to 0$, da $\alpha\to 0$.

Teorema. $f(x)$ funksiya $x\in(a,\ b)$ nuqtada differensial-lanuvchi bo'lishi uchun uning shu nuqtada chekli $f'(x)$ hosilaga ega bo'lishi zarur va yetarli.

Zarurligi. $f(x)$ funksiya $x\in(a,\ b)$ nuqtada differensiallanuvchi bo'lsin. Ta'rifga binoan,

$$\Delta f(x)=A\cdot\Delta x+\alpha\Delta x$$

bo'ladi, bunda $A=const$, $\Delta x\to 0$, da $\alpha\to 0$.

Bu tenglikdan foydalanib topamiz:

$$\frac{\Delta f(x)}{\Delta x}=A+\alpha,$$

$$\lim_{\Delta x\to 0}\frac{\Delta f(x)}{\Delta x}=\lim_{\Delta x\to 0}(A+\alpha)=A.$$

Demak, $f'(x)$ mavjud va $f'(x)=A$.

Yetarliligi. $f(x)$ funksiya $x\in(a,\ b)$ da chekli $f'(x)$ hosilaga ega bo'lsin. Ta'rifga ko'ra

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta f(x)}{\Delta x}$$

bo'ladi.Agar

$$\frac{\Delta f(x)}{\Delta x}-f'(x)=\alpha$$

deyilsa,undan

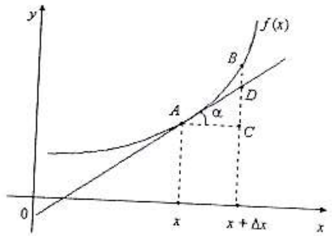
$$\Delta f(x)=f'(x)\cdot\Delta x+\alpha\Delta x$$

bo'lishi kelib chiqadi, bunda $\Delta x\to 0$ da $\alpha\to 0$. Demak, $f(x)$ funksiya differensiallanuvchi.

2-ta'rif. Funksiya orttirmasidagi $f'(x_0)\cdot\Delta x$ ifoda $f(x)$ funksiyaning x_0 nuqtadagi differensiali deyiladi va $df(x_0)$ kabi belgilanadi:

$$df(x_0)=f'(x_0)\cdot\Delta x.$$

Aytaylik, $x\in(a,\,b)$ nuqtada differensiallanuvchi $f(x)$ funksiyaning grafigi 6-chizmada tasvirlangan egri chiziqni ifodalasin:



6-chizma.

Keltirilgan chizmadan ko'rinadiki,

$$\frac{DC}{AC}=tg\alpha$$

bo'lib, $DC=tg\alpha\cdot AC=f'(x)\cdot\Delta x$ bo'ladi.

Demak, $f(x)$ funksiyaning x funksiyaning nuqtadagi differensiali funksiya grafigiga $(x,\,f(x))$ nuqtada o'tkazilgan urinma orttirmasi DC ni ifodalar ekan.

Faraz qilaylik, $f(x)=x,\ x\in R$ bo'lsin. Bu funksiya differensiallanuvchi bo'lib, $df(x)=(x)'\cdot\Delta x=\Delta x$, ya'ni $dx=\Delta x$ bo'ladi. Demak, $(a,\,b)$ da differensiallanuvchi $f(x)$ funksiyaning differensialini

$$df(x)=f'(x)\cdot dx$$

ko'rinishda ifodalash mumkin.

Endi sodda funksiyalarning differensiallarini keltiramiz:

1. $d(x^{\alpha})=\alpha x^{\alpha-1}dx,\quad (x>0);$
2. $d(a^{x})=a^{x}\cdot\ln a\cdot dx,\quad (a>0,\ a\neq 1);$
3. $d(\log_a x)=\frac{1}{x}\log_a e dx,\quad (x>0,\ a>0,\ a\neq 1);$
4. $d(\sin x)=\cos x dx;$
5. $d(\cos x)=-\sin x dx;$
6. $d(tgx)=\frac{1}{\cos^2 x}dx,\quad (x\neq\frac{\pi}{2}+k\pi,\ k=0,\pm1,...);$
7. $d(ctgx)=-\frac{1}{\sin^2 x}dx,\quad (x\neq k\pi,\ k=0,\pm1,...);$
8. $d(\arcsin x)=\frac{1}{\sqrt{1-x^2}}dx,\quad (-1<x<1);$
9. $d(\arccos x)=-\frac{1}{\sqrt{1-x^2}}dx,\quad (-1<x<1);$
10. $d(arctgx)=\frac{1}{1+x^2}dx;$
11. $d(arcctgx)=-\frac{1}{1+x^2}dx;$
12. $d(shx)=chxdx;$
13. $d(chx)=shxdx;$
14. $d(thx)=\frac{1}{ch^2x}dx;$
15. $d(cthx)=-\frac{1}{sh^2x}dx\quad (x\neq 0)$

2^0. Funksiya differensialining sodda qoidalari. Faraz qilaylik, $f(x)$ va $g(x)$ funksiyalari $(a,\,b)$ da berilgan bo'lib, $x\in(a,\,b)$ nuqtada differensiallanuvchi bo'lsin $x\in(a,\,b)$ da

1) $d(c\cdot f(x))=cdf(x),\quad c=const;$
2) $d(f(x)+g(x))=df(x)+dg(x);$
3) $d(f(x)g(x))=g(x)df(x)+f(x)dg(x);$
4) $d\left(\frac{f(x)}{g(x)}\right)=\frac{g(x)df(x)-f(x)dg(x)}{g^2(x)},\quad (g(x)\neq 0).$

bo'ladi.

Bu tasdiqlardan birini, masalan 3)-sini isbotlaymiz.

Ma'lumki,

$$d(f(x)g(x)) = (f(x)g(x))'dx.$$

Agar

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

bo'lishini e'tiborga olsak, unda quyidagi tenglikka kelamiz:

$$d(f(x)g(x) = (f'(x)g(x) + f(x)g'(x))dx =$$

$$= g(x)f'(x)dx + f(x)g'(x)dx = g(x)df(x) + f(x)dg(x).$$

Faraz qilaylik, $y = f(x)$ funksiya $X \subset R$ to'plamda, $g(y)$ funksiya $Y \supset \{f(x) : x \in X\}$ to'plamda berilgan bo'lib, $f'(x)$ va $g'(y)$ hosilalarga ega bo'lsin. U holda

$$d(g(f(x))) = g'(f(x)) \cdot df(x)$$

bo'ladi.

Murakkab funksiyaning hosilasini hisoblash qoidasidan foydalanib topamiz:

$$d(g(f(x))) = [g(f(x))]' dx = g'(f(x)) \cdot f'(x)dx = g'(f(x)) \cdot df(x).$$

1-misol. Ta'rifdan foydalanib, ushbu $f(x) = x - 3x^2$ funksiyaning $x_0 = 2$ nuqtadagi differensiali topilsin.

Bu funksiyaning $x_0 = 2$ nuqtadagi orttirmasini topamiz:

$$\Delta f(2) = f(2 + \Delta x) - f(2) = 2 + \Delta x - 3(2 + \Delta x)^2 - 2 + 12 =$$

$$= -11 \cdot \Delta x - 3\Delta x^2 = -11 \cdot \Delta x + (-3\Delta x) \cdot \Delta x.$$

Demak, $d\,f(2) = -11 \cdot dx$.

3^0. Funksiya differensiali va taqribiy formulalar. Funksiya differensiali yordamida taqribiy formulalar yuzaga keladi.

Aytaylik, $f(x)$ funksiya (a, b) da berilgan bo'lib, $x_0 \in (a, b)$ nuqtada chekli $f'(x_0)$ hosilaga $(f'(x_0) \neq 0)$ ega bo'lsin.U holda $\Delta x \to 0$ da

$$\Delta f(x_0) = f'(x_0) \cdot \Delta x + o(\Delta x)$$

bo'ladi.

Ayni paytda, $f(x)$ funksiya x_0 nuqtada differensiallanuvchi bo'lib, uning differensiali

$$d\,f(x_0) = f'(x_0) \cdot \Delta x$$

bo'ladi.

Ravshanki,

$$\Delta f(x_0) - df(x_0) = o(\Delta x)$$

bo'lib, $\Delta x \to 0$ da

$$\frac{\Delta f(x_0) - df(x_0)}{\Delta x} \to 0$$

bo'ladi.Natijada

$$\Delta f(x_0) \approx df(x_0),$$

ya'ni

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0) \cdot \Delta x \qquad (1)$$

taqribiy formula hosil bo'ladi. (1) formula $x_0 \in (a, b)$ nuqtada differensiallanuvchi $f(x)$ funksiyaning x_0 nuqtadagi orttirmasi $\Delta f(x_0)$ ni uning shu nuqtadagi differensiali $df(x_0)$ bilan almashtirish mumkinligini ko'rsatadi. Bu almashtirishning mohiyati funksiya orttirmasi argument orttirmasining, umuman aytganda murakkab funksiyasi bo'lgan holda, funksiya differensiali esa argument orttir-masining chiziqli funksiyasi bo'lishidadir.

(1) formulada $\Delta x = x - x_0$ deyilsa,unda

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) \qquad (2)$$

bo'ladi.

2-misol. Ushbu $\sin 29^0$ miqdor taqribiy hisoblansin.

Agar $f(x) = \sin x$, $x_0 = 30^0$ deyilsa, unda (2) formulaga ko'ra

$$\sin 29^0 \approx \sin 30^0 + \cos 30^0 \cdot (29^0 - 30^0) \cdot \frac{2\pi}{360^0} = 0{,}5 - \frac{\sqrt{3}}{2} \cdot \frac{2\pi}{360^0} \approx 0{,}4848$$

bo'ladi.

Ma'lumki, $x_0 \in (a, b)$ nuqtada differensiallanuvchi $f(x)$ funksiya grafigiga $(x_0, f(x_0))$ nuqtada o'tkazilgan urinma-ning tenglamasi quyidagi ko'rinishda yoziladi:

$$y = f(x_0) + f'(x_0)(x - x_0).$$

Demak, (2) taqribiy formula geometrik nuqtai nazardan, $f(x)$ funksiya ifodalagan egri chiziqni x_0 nuqtaning yetarli kichik atrofida shu funksiya grafigiga $(x_0, f(x_0))$ nuqtada o'tkazilgan urinma bilan almashtirilishi mumkinligini bildiradi.

(2) formulada $x_0 = 0$ deyilsa,u ushbu

$$f(x) \approx f(0) + f'(0)x \qquad (3)$$

ko'rinishga keladi.

$f(x)$ funksiya sifatida $(1+x)^\alpha$, $\sqrt{1+x}$, e^x, $\ln(1+x)$, $\sin x$, tgx funksiyalarni olib, ularga (3) formulani qo'llash natijasida quyidagi taqribiy formulalar hosil bo'ladi:

$$(1+x)^\alpha \approx 1 + \alpha x,$$

$$\sqrt{1+x} \approx 1+\frac{1}{2}x,$$

$$e^x \approx 1+x,$$

$$\ln(1+x) \approx x,$$

$$\sin x \approx x,$$

$$tgx \approx x.$$

Berilgan funksiyalarning differensialini toping:

1. $y=\ln x+x^2$
2. $y=e^{3x}+\sqrt{x}$
3. $y=\cos^2 x+3$
4. $y=tg4x+\dfrac{2}{x}$
5. $y=\log_3 x+\sin 5x$
6. $y=\ln\ln\left(\dfrac{x}{2}\right)$
7. $y=\cos\dfrac{1}{\log_2 x}$
8. $y=e^{\frac{\sqrt{1-x}}{1+x}}$
9. $y=x^{x^2}$
10. $y=arctg\dfrac{\ln x}{x}, x=e$
11. $y=\arcsin^2 x+arctg^3 x$
12. $y=x^{\sin x}$
13. $y=\sqrt{x}+\ln x-\dfrac{1}{\sqrt{x}}$
14. $y=\dfrac{x^4}{4}\left[(\ln x)^2-\dfrac{1}{2}\ln x+\dfrac{1}{8}\right]$
15. $y=ctg\pi x+\dfrac{\cos\pi x}{2\sin^3 \pi x}$
16. $y=e^{\sqrt[3]{x}}\left(\sqrt[3]{x^2}-2\sqrt[3]{x}+2\right)$
17. $y=\arcsin(\sin x)$
18. $y=\cos(2\arccos x)$
19. $y=\log_2(\log_3(\log_5 x))$
20. $y=\arcsin(\sin x)$
21. $y=\dfrac{1}{2a}\ln\dfrac{x-a}{x+a},(a<0)$
22. $y=\dfrac{\arcsin x}{\sqrt{1-x^2}}$
23. $y=(\sin x)^{\cos x}$
24. $y=x-\ln\sqrt{1+e^{2x}}+e^{-x}\cdot arcctge^x$
25. $y=2^{x^2}$
26. $y=thx+\dfrac{\sqrt{2}}{4}\ln\dfrac{1+\sqrt{2}thx}{1-\sqrt{2}thx}$
27. $y=\sin\ln|x|$
28. $y=\sin\left[\cos^2\left(tg^3 x\right)\right]$
29. $y=\log_2^3(2x+3)^2$
30. $y=A\cdot e^{-k^2x}\sin(\omega x+\alpha)$

5.5. Funksiyaning yuqori tartibli hosila va differensiallari

1°. Funksiyaning yuqori tartibli hosilalari. Faraz qilaylik, $f(x)$ funksiya (a, b) da berilgan bo'lib, $\forall x\in(a, b)$ da $f'(x)$ hosilaga ega bo'lsin. Bu $f'(x)$ funksiyani $g(x)$ orqali belgilaymiz:

$$g(x)=f'(x) \quad (x\in(a, b)).$$

1-ta'rif. Agar $x_0\in(a, b)$ nuqtada $g(x)$ funksiya $g'(x_0)$ hosilaga ega bo'lsa, bu hosila $f(x)$ funksiyaning x_0 nuqtadagi ikkinchi tartibli hosilasi deyiladi va $f''(x_0)$ yoki $\dfrac{d^2 f(x_0)}{dx^2}$ kabi belgilanadi.

Xuddi shunga o'xshash, $f(x)$ ning 3-tartibli $f'''(x)$, 4-tartibli $f^{IV}(x)$ va h.k. tartibli hosilalari ta'riflanadi.

Umuman, $f(x)$ funksiyaning $n-$tartibli hosilasi $f^{(n)}(x)$ ning hosilasi $f(x)$ funksiyaning $(n+1)-$ tartibli hosilasi deyiladi:

$$f^{(n+1)}(x)=\left(f^{(n)}(x)\right)'.$$

Odatda, $f(x)$ funksiyaning $f''(x)$, $f'''(x)$, ... hosilalari uning yuqori tartibli hosilalari deyiladi. Shuni ta'kidlash lozimki, $f(x)$ funksiyaning $x\in(a, b)$ da $n-$tartibli hosilasining mavjudligi bu funksiyaning shu nuqta atrofida $1-$, $2-$, ..., $(n-1)-$ tartibli hosilalari mavjudligini taqoza etadi. Ammo bu hosilalarning mavjudligidan $n-$ tartibli hosila mavjudligi, umuman aytganda, kelib chiqavermaydi.

Masalan,

$$f(x)=\frac{x|x|}{2}$$

funksiyaning hosilasi $f'(x)=|x|$ bo'lib, bu funksiya $x=0$ nuqtada hosilaga ega emas, ya'ni berilgan funksiyaning $x=0$ da birinchi tartibli hosilasi mavjud, ikkinchi tartibli hosilasi esa mavjud emas.

1-misol. $f(x)=a^x$ bo'lsin, $a>0$, $x\in R$. Bu funksiya uchun

$$(a^x)'=a^x\ln a,$$

$$(a^x)''=(a^x\ln a)'=a^x(\ln a)^2,$$

umuman

$$(a^x)^{(n)}=a^x(\ln a)^n \tag{1}$$

bo'ladi. (1) munosabatning o'rinli bo'lishi matematik induksiya usuli bilan isbotlanadi.

2-misol. $f(x) = \sin x$ bo'lsin. Bu funksiya uchun

$$(\sin x)' = \cos x = \sin\left(x + \frac{\pi}{2}\right),$$

$$(\sin x)'' = (\cos x)' = -\sin x = \sin\left(x + 2\frac{\pi}{2}\right),$$

Umuman,

$$(\sin x)^{(n)} = \sin\left(x + n\frac{\pi}{2}\right)$$

bo'ladi.

Shunga o'xshash,

$$(\cos x)^{(n)} = \cos\left(x + n\frac{\pi}{2}\right)$$

bo'ladi.

3-misol. $f(x) = x^{\alpha}$ bo'lsin, $x > 0$, $\alpha \in R$. Bu funksiya uchun

$$(x^{\alpha})' = \alpha x^{\alpha-1},$$

$$(x^{\alpha})'' = (\alpha x^{\alpha-1})' = \alpha(\alpha-1)x^{\alpha-2},$$

umuman,

$$(x^{\alpha})^{(n)} = \alpha(\alpha-1)(\alpha-2)...(\alpha-n+1)x^{\alpha-n}$$

bo'ladi.

Xususan, $f(x) = \frac{1}{x}$, $(x > 0)$ funksiya uchun

$$\left(\frac{1}{x}\right)^{(n)} = \frac{(-1)^n n!}{x^{n+1}}$$

bo'lib,unda

$$(\ln x)^{(n)} = \frac{(-1)^{n-1}(n-1)!}{x^n}$$

bo'lishini topamiz.

Faraz qilaylik, $f(x)$ va $g(x)$ funksiyalar (a, b) da berilgan bo'lib, $\forall x \in (a, b)$ da $f^{(n)}(x)$ va $g^{(n)}(x)$ hosilalarga ega bo'lsin. U holda:

1) $(c \cdot f(x))^{(n)} = c \cdot f^{(n)}(x)$, $c = const$;

2) $(f(x) \pm g(x))^{(n)} = f^{(n)}(x) \pm g^{(n)}(x)$;

3) $$(f(x) \cdot g(x))^{(n)} = \sum_{k=0}^{n} C_n^k f^{(k)}(x) \cdot g^{(n-k)}(x) \qquad (2)$$

$$\left(C_n^k = \frac{n(n-1)...(n-k+1)}{k!}\right), \quad f^{(0)}(x) = f(x)$$

bo'ladi.

Bu tasdiqlardan 3)-sining isbotini keltiramiz. Ravshanki, $n=1$ da (2) munosabat o'rinli bo'ladi. Aytaylik, (2) munosabat $n-1$ da o'rinli bo'lsin:

$$(f(x) \cdot g(x))^{(n-1)} = \sum_{k=0}^{n-1} C_{n-1}^k f^{(k)}(x) \cdot g^{(n-1-k)}(x).$$

Keyingi tenglikni hamda

$$C_{n-1}^k + C_{n-1}^{k-1} = C_n^k$$

bo'lishini e'tiborga olib, topamiz:

$$(f(x) \cdot g(x))^{(n)} = \left((f(x) \cdot g(x))^{(n-1)}\right)' = \left(\sum_{k=0}^{n-1} C_{n-1}^k f^{(k)}(x) \cdot g^{(n-1-k)}(x)\right)' =$$

$$= \sum_{k=0}^{n-1} C_{n-1}^k (f^{(k+1)}(x) g^{(n-1-k)}(x) + f^{(k)}(x) g^{(n-k)}(x)) = C_{n-1}^0 f(x) g^{(n)}(x) +$$

$$+ \sum_{k=0}^{n-1} (C_{n-1}^k + C_{n-1}^k) f^{(k)}(x) g^{(n-k)}(x) + C_{n-1}^{k-1} f^{(n)}(x) g(x) =$$

$$= \sum_{k=0}^{n} C_n^k f^{(k)}(x) g^{(n-k)}(x)$$

Odatda, (2) **Leybnits formulasi** deyiladi.

4-misol. Ushbu

$$y = x^2 \cos 2x$$

funksiyaning $n-$ tartibli hosilasi topilsin.

Leybnits formulasida $f(x) = \cos 2x$, $g(x) = x^2$ deb olamiz. Unda bu formulaga ko'ra, ayni paytda $g(x) = x^2$ funksiya uchun $k > 2$ bo'lganda

$$g^{(k)}(x) = (x^2)^{(k)} = 0, \qquad (k > 2)$$

bo'lishini e'tiborga olib topamiz:

$$(x^2 \cos 2x)^{(n)} = C_n^0 x^2 (\cos 2x)^{(n)} + C_n^1 (x^2)' \cdot (\cos 2x)^{n-1} + C_n^2 (x^2)''(\cos 2x)^{n-2}$$

Ravshanki,

$$(\cos 2x)^{(n)} = 2^n \cos\left(2x + n \cdot \frac{\pi}{2}\right),$$

$$(\cos 2x)^{(n-1)} = 2^{n-1}\cos\left(2x+(n-1)\frac{\pi}{2}\right) = 2^{n-1}\sin\left(2x+n\frac{\pi}{2}\right),$$

$$(\cos 2x)^{(n-2)} = 2^{n-2}\cos\left(2x+(n-2)\frac{\pi}{2}\right) = -2^{n-1}\cos\left(2x+n\frac{\pi}{2}\right).$$

Demak,

$$(x^2\cos 2x)^{(n)} = 2^n\left(x^2-\frac{n(n-1)}{4}\right)\cos\left(2x+n\frac{\pi}{2}\right)+2^n nx\sin\left(2x+n\frac{\pi}{2}\right).$$

2⁰. Funksiyaning yuqori tartibli differensiallari. Faraz qilaylik, $f(x)$ funksiya (a, b) da berilgan bo'lib, $\forall x\in(a, b)$ nuqtada $f''(x)$ hosilaga ega bo'lsin. Ravshanki, $f(x)$ funksiyaning differensiali

$$df(x) = f'(x)dx \qquad (3)$$

bo'lib,bunda $dx=\Delta x$ funksiya argumentning ixtiyoriy orttirmasi.

2-ta'rif. $f(x)$ funksiyaning $x\in(a, b)$ nuqtadagi differensiali $df(x)$ ning differensiali $f(x)$ funksiyaning $x\in(a, b)$ nuqtadagi ikkinchi tartibli differensiali deyiladi va $d^2f(x)$ kabi belgilanadi:

$$d^2f(x) = d(df(x)).$$

Xuddi shunga o'xshash, $f(x)$ funksiyaning uchinchi $d^3f(x)$, to'rtinchi $d^4f(x)$ va h.k. tartibdagi differensiallari ta'riflanadi.

Umuman, $f(x)$ funksiyaning $n-$ tartibli differensiali $d^nf(x)$ ning differensiali $f(x)$ funksiyaning $(n+1)-$ tartibli differensiali deyiladi:

$$d^{n+1}f(x) = d(d^nf(x)).$$

5-misol. Ushbu

$$f(x) = xe^{-x}$$

funksiyaning ikkinchi tartibli differensiali topilsin.

Berilgan funksiyaning ikkinchi tartibli differen-sialini ta'rifiga ko'ra topamiz:

$$d^2f(x) = d(df(x)) = d(d(xe^{-x})) = d(xde^{-x}+e^{-x}dx) = d(-xe^{-x}dx+e^{-x}dx) =$$

$$= -d(xe^{-x})dx+(de^{-x})dx = -(xde^{-x}+e^{-x}dx)dx - e^{-x}(dx)^2 = xe^{-x}(dx)^2 -$$

$$= x\cdot e^{-x}(dx)^2 - e^{-x}(dx)^2 - e^{-x}(dx)^2 = (x-2)e^{-x}(dx)^2.$$

Differensiallash qoidasidan foydalanib topamiz:

$$d^2f(x) = d(df(x)) = d(f'(x)dx) = dx\cdot d(f'(x)) = dx\cdot f''(x)dx = f''(x)(dx)^2, \qquad (4)$$

$$d^3f(x) = d(d^2f(x)) = f'''(x)(dx)^3,$$

....

$$d^nf(x) = f^{(n)}(x)(dx)^n$$

Masalan, yuqorida keltirilgan misol uchun

$$d^2(xe^{-x}) = (xe^{-x})''(dx)^2 = (e^{-x}-xe^{-x})'(dx)^2 =$$

$$= (e^{-x}-e^{-x}-xe^{-x})(dx)^2 = (x-2)e^{-x}(dx)^2$$

bo'ladi.

Aytaylik, $f(x)$ va $g(x)$ funksiyalar (a, b) da berilgan bo'lib, $\forall x\in(a, b)$ nuqtada $n-$ tartibli differensiallarga ega bo'lsin. U holda:

1) $d^n(c\cdot f(x)) = c\cdot d^nf(x)$, $\quad c=const$;

2) $d^n(f(x)\pm g(x)) = d^nf(x)\pm d^ng(x)$;

3) $d^n(f(x)\cdot g(x)) = d^nf(x)\cdot g(x)+C_n^1d^{n-1}f(x)\cdot dg(x)+$

$+\ldots+C_n^kd^{n-k}f(x)\cdot d^kg(x)+\ldots+f(x)\cdot d^ng(x)$

bo'ladi.

1. Ushbu $f(x)=x^3$ funksiyaning $x=x_0$ nuqtadagi hosilasini ta'rif yordamida toping.

◄ Funksiyaning hosilasining ta'rifiga ko'ra quydagi limitni topish kerak, ya'ni

$$\lim_{\Delta x\to 0}\frac{\Delta f(x_0)}{\Delta x}$$

limitni topish lozim bo'ladi.

$$\Delta f(x_0) = f(x_0+\Delta x) - f(x_0)$$

ekanligini e'tiborga olsak,

$$\Delta f(x_0) = (x_0+\Delta x)^3 - x_0^3 = x_0^3+3x_0^3\Delta x+3x_0\Delta x^2+\Delta x^3 - x_0^3 =$$

$$= 3x_0^2\Delta x+3x_0\Delta x^2+\Delta x^3$$

bo'ladi.

Unda

$$\lim_{\Delta x\to 0}\frac{\Delta f(x_0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{3x_0^2\Delta x+3x_0\Delta x^2+\Delta x^3}{\Delta x}=\lim_{\Delta x\to 0}\left(3x_0^2+3x_0\Delta x+\Delta x^2\right)=3x_0^2$$

munosabatdan

$$f'(x_0)=3x_0^2$$

2. Ushbu

$$f(x)=\sin 2x$$

funksiyaning $x=x_0$ nuqtadagi hosilasini ta'rif yordamida toping.

$$\Delta f(x_0)=f(x_0+\Delta x)-f(x_0)=\sin 2(x_0+\Delta x)-\sin 2x_0=$$

$$=\sin(2x_0+2\Delta x)-\sin 2x_0=2\sin\frac{2x_0+2\Delta x-2x_0}{2}\cos\frac{2x_0+2\Delta x+2x_0}{2}=$$

$$=2\sin\Delta x\cos\frac{4x_0+2\Delta x}{2}=2\sin\Delta x\cos(2x_0+\Delta x)$$

Endi

$$\lim_{\Delta x\to 0}\frac{\Delta f(x_0)}{\Delta x}$$

limitni hisoblaymiz. Natijada

$$\lim_{\Delta x\to 0}\frac{\Delta f(x_0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{2\sin\Delta x}{\Delta x}\cdot\lim_{\Delta x\to 0}\cos(2x_0+\Delta x)=2\cos 2x_0$$

bo'ladi.

Demak, qaralayotgan funksiyaning $x=x_0$ hosilasi

$$f'(x_0)=2\cos 2x_0$$

ga teng ekan.

3. Ushbu

$$f(x)=e^x$$

funksiyaning $x_0=1$ nuqtada hosilasini toping.

$$f'(1)=\lim_{\Delta x\to 0}\frac{e^{1+\Delta x}-e}{\Delta x}=\lim_{\Delta x\to 0}\frac{e(e^{\Delta x}-1)}{\Delta x}$$

$$\lim_{x\to 0}\frac{e^x-1}{x}=1$$

ajoyib limitdan foydalansak,

$$f(1)=\lim_{\Delta x\to 0}\frac{e(e^{\Delta x}-1)}{\Delta x}=e$$

bo'ladi.

4-misol: Ushbu

$$f(x)=e^{\frac{x}{7}}$$

funksiyaning $x=x_0$ nuqtadagi hosilasini ta'rif yordamida toping.

Qaralayotgan funksiya uchun

$$\Delta f(x_0)=f(x_0+\Delta x)-f(x_0)=e^{\frac{1}{7}(x_0+\Delta x)}-e^{\frac{x_0}{7}}=e^{\frac{x_0}{7}}(e^{\frac{\Delta x}{7}}-1)$$

ekanligini e'tibor olib topamiz:

Bu munosabatlarning 1), 2) – larning isboti ravshan. 3) – munosabatni isbotlashda (2) formuladan foydalaniladi.

3°. Differensial shaklining invariantligi. Aytaylik, $y=f(x)$ funksiya (a, b) da differensiallanuvchi bo'lib, x o'zgaruvchi o'z navbatida biror t o'zgaruvchining $[\alpha, \beta]$ da differensiallanuvchi funksiyasi bo'lsin:

$$x=\varphi(t)\quad (t\in[\alpha,\beta],\ x=\varphi(t)\in[a,b]).$$

Natijada

$$y=f(x)=f(\varphi(t))$$

bo'ladi.Bu funksiyaning differensiali

$$dy=(f(\varphi(t)))'dt=f'(\varphi(t))\cdot\varphi'(t)dt=f'(\varphi(t))\cdot d\varphi(t)=f'(x)dx$$

bo'lib, u (3) ko'rinishga ega bo'ladi. Shunday qilib, $y=f(x)$ funksiyada x o'zgaruvchi erkli bo'lgan holda ham, u biror t o'zgaruvchiga bog'liq bo'lgan holda ham $y = f(x)$ funksiya differensialining ko'rinishi bir xil bo'ladi. Odatda bu xususiyat differensial shaklining **invariantligi** deyiladi.

$y = f(\varphi(t))$ funksiyaning ikkinchi tartibli differen-siali quyidagicha bo'ladi:

$$d^2y = d(df) = d(f'(x)dx) = df'(x)\cdot dx + f'(x)\cdot d(dx) =$$
$$= f''(x)\cdot(dx)^2 + f'(x)d^2x.$$

Bu munosabatni (4) munosabat bilan solishtirib ikkinchi tartibli differensiallarda differensial shaklining invariantligi xossasi o'rinli emasligini topamiz.

Hisoblang:

1. $y = x(2x-1)^2(x+3)^3,\ y^{(7)}$
2. $y = \frac{a}{x^m},\ y^{(3)}$
3. $y = \sqrt{x},\ y^{(10)}$
4. $y = \frac{x^2}{1-x},\ y^{(8)}$
5. $y = \frac{1+x}{\sqrt{1-x}},\ y^{(100)}$
6. $y = x^2e^{2x},\ y^{(20)}$
7. $y = \frac{e^x}{x},\ y^{(10)}$
8. $y = x\ln x,\ y^{(5)}$
9. $y = \frac{\ln x}{x},\ y^{(5)}$
10. $y = x^2\sin 2x,\ y^{(50)}$
11. $y = \frac{\cos 3x}{\sqrt[3]{1-3x}},\ y^{(3)}$
12. $y = \sin^2 x\ln x,\ y^{(6)}$
13. $y = e^x\cos x,\ y^{(4)}$
14. $y = xshx,\ y^{(100)}$
15. $y = \sin x\sin 2x\sin 3x,\ y^{(10)}$
16. $y = \frac{x}{\sqrt{1-5x}},\ y^{(n)}$
17. $y = \frac{x^2}{\sqrt{1-2x}},\ y^{(n)}$
18. $y = (3-2x)^2e^{2-3x},\ y^{(n)}$
19. $y = x\log_2(1-3x),\ y^{(n)}$
20. $y = x\cos x,\ y^{(n)}$
21. $y = arctgx,\ y^{(n)}$
22. $y = x^{n-1}\ln x,\ y^{(n)}$
23. $y = x^5,\ d^5y$
24. $y = \frac{1}{\sqrt{x}},\ d^3y$
25. $y = x\cos 2x,\ d^{10}y$
26. $y = e^x\ln x,\ d^4y$
27. $y = \cos x chx,\ d^6y$
28. $y = u^2,\ d^{10}y$
29. $y = e^u,\ d^4y$
30. $y = \ln u,\ d^3y$

5.6. Teylor formulasi

1°. Ko'phad uchun Teylor formulasi. Ushbu

$$P(x) = b_0 + b_1(x-x_0) + b_2(x-x_0)^2 + \ldots + b_n(x-x_0)^n \qquad (1)$$

funksiyani (n - darajali ko'phadni) qaraylik, bunda $x_0 \in R$ va $b_0, b_1, \ldots, b_n$ - haqiqiy sonlar. Bu $b_0, b_1, \ldots, b_n$ lar quyidagicha ham aniqlanishi mumkin:

(1) tenglikda $x = x_0$ deyilsa,

$$b_0 = P(x_0)$$

bo'ladi;

$P(x)$ funksiyani differensiallab,

$$P'(x) = 1\cdot b_1 + 2\cdot b_2\cdot(x-x_0) + \ldots + n\cdot b_n(x-x_0)^{n-1}$$

va bu tenglikda $x = x_0$ deb

$$b_1 = \frac{P'(x_0)}{1!}$$

bo'lishini topamiz.

$P(x)$ funksiyani ikki marta differensiallab

$$P''(x) = 2\cdot 1\cdot b_2 + \ldots + n(n-1)\cdot b_n(x-x_0)^{n-2}$$

va bu tenglikda $x = x_0$ deb topamiz:

$$b_2 = \frac{P''(x_0)}{2!}.$$

Bu jarayonni davom ettira borib, $\forall k \ge 0$ da

$$b_k = \frac{P^{(k)}(x_0)}{k!}$$

bo'lishini topamiz.

Natijada $P(x)$ ko'phad quyidagi ko'rinishga keladi:

$$P(x) = P(x_0) + \frac{P'(x_0)}{1!}(x-x_0) + \frac{P''(x_0)}{2!}(x-x_0)^2 + \ldots + \frac{P^{(n)}(x_0)}{n!}(x-x_0)^n \qquad (2)$$

Demak, $P(x)$, ko'phad o'zining hamda hosilalarining biror nuqtasidagi qiymati bilan to'liq aniqlanar ekan.(2) formula $P(x)$ ko'phad uchun Teylor formulasi deyiladi.

2°. Ixtiyoriy funksiyaning Teylor formulasi va uning qoldiq hadlari. Faraz qilaylik, $f(x)$ funksiya (a,b) da berilgan bo'lib, $x_0 \in (a,b)$ bo'lsin. Bu funksiya x_0 nuqtaning

$$U_\delta(x_0) = (x_0-\delta, x_0+\delta) \subset (a,b) \quad \delta > 0$$

atrofida $f'(x), f''(x),..., f^{(n)}(x), f^{(n+1)}(x)$ hosilalarga ega bo‘lsin. Funksiya hosilalaridan foydalanib, ushbu

$$P_n(f;x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + ... + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

ko‘phadni tuzamiz.

Agar $f(x)$ funksiya $n-$ darajali ko‘phad bo‘lsa, ravshanki,

$$f(x) = P_n(f;x)$$

bo‘ladi.

Agar $f(x)$ funksiya ko‘phad bo‘lmasa,

$$f(x) \neq P_n(f;x)$$

bo‘lib, ular orasidagi farq yuzaga keladi. Uni $R_n(x)$ orqali belgilaymiz:

$$R_n(x) = f(x) - P_n(f;x).$$

Natijada ushbu

$$f(x) = P_n(f;x) + R_n(x)$$

ya’ni,

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + ... + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + R_n(x) \qquad (3)$$

formulaga kelamiz. Bu (3) formula $f(x)$ funksiyaning Teylor formulasi deyiladi. (3) formuladagi $R_n(x)$ esa Teylor formulasining qoldiq hadi deyiladi.

Endi qoldiq had $R_n(x)$ ni aniqlaymiz x_0 nuqtaning $U_\delta(x_0)$ atrofidagi x ni tayinlab, ushbu

$$F(t) = f(x) - f(t) - \frac{f'(t)}{1!}(x-t) - \frac{f''(t)}{2!}(x-t)^2 + ... + \frac{f^{(n)}(t)}{n!}(x-t)^n$$

funksiyani $[x_0, x] \subset U_\delta(x_0)$ (yoki $[x, x_0] \subset U_\delta(x_0)$) da qaraymiz.

Bu funksiya $[x_0, x]$ segmentda uzluksiz bo‘lib, (x_0, x) da hosilaga ega bo’ladi.

$$F'(t) = -f'(t) - \left[\frac{f'(t)}{1!}(x-t) - f'(t)\right] - \left[\frac{f''(t)}{2!}(x-t)^2 - \frac{f'(t)}{1!}(x-t)\right] - ... -$$

$$-\left[\frac{f^{(n+1)}(t)}{n!}(x-t)^n - \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1}\right] = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n.$$

Demak,

$$F'(t) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n.$$

Endi $[x_0, x]$ da uzluksiz, (x_0, x) da chekli (nolga teng bo‘lmagan) hosilaga ega $\phi(x)$ funksiyani olib, $F(x)$ va $\phi(x)$ funksiyalarga $[x_0, x]$ da Koshi teoremasini qo’llaymiz. Natijada quyidagi

$$\frac{F(x) - F(x_0)}{\phi(x) - \phi(x_0)} = \frac{F'(c)}{\phi'(c)} \qquad (4)$$

tenglikka kelamiz, bunda $c = x_0 + \theta(x - x_0) \quad (0 < \theta < 1)$.

Ravshanki,

$$F(x) = 0, \quad F(x_0) = R_n(x), \quad F'(c) = -\frac{f^{(n+1)}(c)}{n!}(x-c)^n.$$

Unda (4) tenglikdan

$$R_n(x) = \frac{\phi(x) - \phi(x_0)}{\phi'(c)} \cdot \frac{f^{(n+1)}(c)}{n!}(x-c)^n \qquad (5)$$

bo’lishini topamiz.

a) Koshi ko‘rinishidagi qoldiq hadli Teylor formulasi.

Aytaylik, $\phi(t) = x - t$ bo’lsin.Unda

$$\phi(x) = 0, \quad \phi(x_0) = x - x_0, \quad \phi'(c) = -1$$

bo’lib, (5) tenglik quyidagi

$$R_n(x) = \frac{f^{(n+1)}(c)}{n!} \cdot \frac{-(x-x_0)}{-(n+1)(x-c)^n}(x-c)^n = \frac{f^{(n+1)}(c)}{n!}[x - x_0 - \theta(x-x_0)]^n \cdot (x-x_0) =$$

$$= \frac{f^{(n+1)}(c)}{n!}(x-x_0)^{n+1}(1-\theta)^n$$

ko‘rinishga keladi. Bu holda

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + ... +$$

$$+ \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{f^{(n+1)}(c)}{n!}(x-x_0)^{n+1}(1-\theta)^n$$

formula hosil bo‘lib, uni funksiyaning Koshi ko‘rinishidagi qoldiq hadli Teylor formulasi deyiladi.

b) Lagranj ko‘rinishidagi qoldiq hadli Teylor formulasi.

Aytaylik, $\phi(t) = (x-t)^{n+1}$ bo’lsin.Unda

$$\phi(x) = 0, \quad \phi(x_0) = (x-x_0)^{n+1},$$

$$\phi'(c) = -(n+1)(x-c)^n \quad (c = x_0 + \theta(x-x_0))$$

bo’lib, (5) tenglik quyidagi

$$R_n(x)=\frac{f^{(n+1)}(c)}{n!}\cdot\frac{-(x-x_0)^{n+1}}{-(n+1)(x-c)^n}\cdot(x-c)^n=\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$$

ko'rinishga keladi. Bu holda

$$f(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+...+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\frac{f^{(n+1)}(c)}{n!}(x-x_0)^{n+1}, \quad (6)$$

$$(c=x_0+\theta(x-x_0), \quad 0<\theta<1)$$

formula hosil bo'lib, uni $f(x)$ funksiyaning Lagranj ko'rinishidagi qoldiq hadli Teylor formulasi deyiladi.

c) Peano ko'rinishidagi qoldiq hadli Teylor formulasi.

Yuqoridagi (6) formuladan foydalanib topamiz:

$$f(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+...$$

$$\cdots+\frac{f^{(n-1)}(x_0)}{(n-1)!}(x-x_0)^{n-1}++\frac{f^{(n)}(c)}{n!}(x-x_0)^n,$$

$$(c=x_0+\theta(x-x_0), \quad 0<\theta<1).$$

$f^{(n)}(x)$ **funksiya** x_0 **nuqtada uzluksiz.Demak,** $x\to x_0$ **da** $c\to x_0$ **bo'lib,**

$$f^{(n)}(c)\to f^{(n)}(x_0).$$

Shuni e'tiborga olib, $x\to x_0$ da

$$\frac{f^{(n)}(c)}{n!}(x-x_0)^n=\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+o((x-x_0)^n)$$

bo'lishini topamiz.

Natijada ushbu

$$f(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+...+$$

$$+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+o((x-x_0)^n), \qquad (x\to x_0)$$

formula hosil bo'ladi. Bu formula $f(x)$ funksiyaning Peano ko'rinishidagi qoldiq hadli Teylor formulasi deyiladi.

3^0.Ba'zi funksiyalarning Teylor formulalari. $f(x)$ funksiyaning Peano ko'rinishidagi qoldiq hadli Teylor formulasini olamiz:

$$f(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+...+$$

$$+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+o((x-x_0)^n), \qquad (x\to x_0)$$

Bu tenglikda $x_0=0$ deb,ushbu

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+...+\frac{f^{(n)}(0)}{n!}x^n+o(x^n), \quad (x\to 0) \qquad (7)$$

formulaga kelamiz. (7) formula $f(x)$ funksiyaning Makloren formulasi deyiladi.

1) $f(x)=e^x$ bo'lsin. Bu funksiya uchun $f(0)=1$, $f^{(n)}(0)=1$ bo'lib,

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+..+\frac{x^n}{n!}+o(x^n), \quad x\to 0$$

bo'ladi.

2) $f(x)=(1+x)^\alpha$, $\alpha\in R$ bo'lsin. Bu funksiya uchun

$$f(0)=1, \quad f^{(n)}(0)=\alpha(\alpha-1)...(\alpha-n+1)$$

bo'lib,

$$(1+x)^\alpha=\sum_{k=0}^{n}\frac{\alpha(\alpha-1)...(\alpha-k+1)}{k!}x^k+o(x^n), \quad x\to 0$$

bo'ladi.

Xususan,

$$\frac{1}{1-x}=\sum_{k=0}^{n}x^k+o(x^n), \quad x\to 0$$

$$\frac{1}{1+x}=\sum_{k=0}^{n}(-1)^k x^k+o(x^n), \quad x\to 0$$

bo'ladi.

3) $f(x)=\ln(1+x)$ bo'lsin.Bu funksiya uchun

$$f(0)=0, \quad f^{(k)}(0)=(-1)^{k-1}(k-1)!$$

bo'lib,

$$\ln(1+x)=\sum_{k=1}^{n}\frac{(-1)^k x^k}{k}+o(x^n), \quad x\to 0$$

bo'ladi.

Shuningdek,

$$\ln(1-x)=-\sum_{k=1}^{n}\frac{x^k}{k}+o(x^n), \quad x\to 0$$

bo'ladi.

4) $f(x)=\sin x$ bo‘lsin. Bu funksiya uchun $f(0)=0$, $f^{(2k+1)}(0)=(-1)^k$ bo’lib,

$$\sin x=\sum_{k=0}^{n}(-1)^k\frac{x^{2k+1}}{(2k+1)!}+o(x^{2n+2}),\quad x\to 0$$

bo’ladi.

5) $f(x)=\cos x$ bo‘lsin. Bu funksiya uchun $f(0)=1$, $f^{(2k)}(0)=(-1)^k$ bo’lib,

$$\cos x=\sum_{k=0}^{n}(-1)^k\frac{x^{2k}}{(2k)!}+o(x^{2n+1}),\quad x\to 0$$

bo‘ladi.

Misol. Ushbu $f(x)=\dfrac{1}{3x+2}$

funksiyaning Teylor (Makloren) formulasi yozilsin.

Bu funksiyani quyidagicha

$$f(x)=\frac{1}{3x+2}=\frac{1}{2\left(1+\frac{3}{2}x\right)}$$

yozib,so’ng

$$\frac{1}{1+x}=\sum_{k=0}^{n}(-1)^k x^k+o(x^n),\quad x\to 0$$

bo‘lishidan foydalanib topamiz:

$$\frac{1}{3x+2}=\sum_{k=0}^{n}(-1)^k\frac{3^k}{2^{k+1}}x^k+o(x^n),\quad x\to 0.$$

y=f(x) funksiyalarni ko`rsatilgan nuqtada Teylor formulasini yozing.

1. $f(x)=\frac{1}{x}, x_0=2$

2. $f(x)=\sqrt{x}, x_0=1$

3. $f(x)=\sin(2x-3), x_0=1$

4. $f(x)=xe^{2x}, x_0=-1$

5. $f(x)=x^2e^{-2x}, x_0=-1$

6. $f(x)=(x^2-1)e^{2x}, x_0=-1$

7. $f(x)=\sin(x+1)\sin(x+2), x_0=-1$

16. $f(x)=\ln\frac{(x-1)^{x-2}}{3-x}, x_0=2$

17. $f(x)=\frac{2x+1}{x-1}\ln x, x_0=1$

18. $f(x)=\frac{2x-1}{x-1}, x_0=2$

19. $f(x)=\frac{(x-2)^2}{3-x}, x_0=2$

20. $f(x)=\frac{x}{4+x}, x_0=10$

21. $f(x)=\frac{x+5}{2x-4}, x_0=-\frac{1}{10}$

22. $f(x)=\frac{x^2+3x}{x+1}, x_0=1$

8. $f(x)=(x^2+4x+2)e^{-3x}, x_0=-2$

9. $f(x)=\ln(2x+1), x_0=\frac{1}{2}$

10. $f(x)=\log_3\sqrt[3]{3x-\frac{1}{3}}, x_0=3$

11. $f(x)=\ln(2+x+x^2), x_0=1$

12. $f(x)=\ln(x^2-7x+12), x_0=1$

13. $f(x)=\ln\sqrt[3]{7x-2}, x_0=1$

14. $f(x)=\ln\sqrt[4]{\frac{x-2}{5-x}}, x_0=3$

15. $f(x)=x\ln(2-3x+x^2), x_0=-2$

23. $f(x)=\frac{x^2-3x+3}{x-2}, x_0=3$

24. $f(x)=\frac{x+7}{x(2x+7)}, x_0=-2$

25. $f(x)=\frac{x-2}{(x-3)^2}, x_0=2$

26. $f(x)=\frac{2x}{1-x^2}, x_0=2$

27. $f(x)=\frac{x^2+4x+4}{x^2+10x+25}, x_0=-2$

28. $f(x)=\frac{x^2-4x+5}{x^2-5x+6}, x_0=1$

29. $f(x)=\frac{x^2-5x+7}{x^2-9x+20}, x_0=-2$

30. $f(x)=x\left(\frac{1}{(x-1)^3}-1\right), x_0=2$

VI BOB
FUNKSIYA HOSILALARINING BA'ZI BIR TATBIQLARI

6.1. Funksiya monotonligi. Funksiyaning ekstremumlari

1^0. **Funksiyaning monotonligi.** Faraz qilaylik $f(x)$ funksiya $(a,\ b)$ da $(-\infty \le a < b \le +\infty)$ berilgan bo'lsin.

Ma'lumki,

$\forall x_1, x_2 \in (a,b)$, uchun $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$ $(f(x_1) < f(x_2))$ bo'lsa, $f(x)$ funksiya $(a,\ b)$ da o'suvchi(qat'iy o'suvchi), $\forall x_1, x_2 \in (a,b)$ uchun $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$ $(f(x_1) > f(x_2))$ bo'lsa, $f(x)$ funksiya $(a,\ b)$ da kamayuvchi(qat'iy kamayuvchi) deyiladi.

1-teorema. Aytaylik, $f(x)$ funksiya $(a,\ b)$ da berilgan bo'lib, $\forall x \in (a,\ b)$ da $f'(x)$ hosilaga ega bo'lsin.

$f(x)$ funksiyaning $(a,\ b)$ da o'suvchi bo'lishi uchun $\forall x \in (a,\ b)$ da

$$f'(x) \ge 0$$

bo'lishi zarur va yetarli.

Zarurligi. $f(x)$ funksiya $(a,\ b)$ da o'suvchi bo'lsin. Unda $\Delta x > 0$ bo'lganda

$$f(x+\Delta x) - f(x) \ge 0$$

bo'ladi. Hosila ta'rifidan foydalanib topamiz:

$$f'(x) = f'(x+0) = \lim_{\Delta x \to +0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \ge 0.$$

Yetarliligi. Aytaylik, $\forall x \in (a,b)$ da $f'(x)$ mavjud bo'lib, $f'(x) \ge 0$ bo'lsin. $[x_1, x_2]$ da $(x_1, x_2 \in (a,b),\ x_1 < x_2)$ $f(x)$ funksiyaga Lagranj teoremasini qo'llab topamiz:

$$f(x_2) - f(x_1) = f'(c) \cdot (x_2 - x_1) \ge 0.$$

Demak, $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$, $f(x)$ o'suvchi.

Xuddi shunga o'xshash, quydagi teorema isbotlanadi.

2-teorema. Faraz qilaylik, $f(x)$ funksiya $(a,\ b)$ da berilgan bo'lib, $\forall x \in (a,b)$ da $f'(x)$ hosilaga ega bo'lsin. $f(x)$ funksiya $(a,\ b)$ da kamayuvchi bo'lishi uchun $\forall x \in (a,b)$ da

$$f'(x) \le 0$$

bo'lishi zarur va yetarli.

3-teorema. $f(x)$ funksiya $(a,\ b)$ da berilgan bo'lib, $\forall x \in (a,b)$ da $f'(x)$ hosilaga ega bo'lsin. $f(x)$ funksiyaning $(a,\ b)$ da qat'iy o'suvchi bo'lishi uchun

1) $\forall x \in (a,b)$ da $f'(x) \ge 0$.

2) $\forall x \in (\alpha,\beta)$ da $f'(x) = 0$ tenglik bajariladigan $(\alpha,\beta) \subset (a,b)$ intervalning mavjud bo'lmaslik shartlarining bajarilishi zarur va yetarli.

4-teorema. $f(x)$ funksiya $(a,\ b)$ da berilgan bo'lib, $\forall x \in (a,b)$ da $f'(x)$ hosilaga ega bo'lsin. $f(x)$ funksiyaning $(a,\ b)$ da qat'iy kamayuvchi bo'lishi uchun

1) $\forall x \in (a,b)$ da $f'(x) \le 0$,

2) $\forall x \in (\alpha,\beta)$ da $f'(x) = 0$

tenglik bajariladigan $(\alpha,\beta) \subset (a,b)$ intervalning mavjud bo'lmasligi shartlarning bajarilishi zarur va yetarli

Demak, $(a,\ b)$ da

$$f'(x) \ge 0 \Rightarrow f(x) \text{ o'suvchi} \Rightarrow f'(x) \ge 0,$$
$$f'(x) \le 0 \Rightarrow f(x) \text{ kamayuvchi} \Rightarrow f'(x) \le 0,$$
$$f'(x) > 0 \Rightarrow f(x) \text{ qat'iy o'suvchi} \Rightarrow f'(x) \ge 0,$$
$$f'(x) < 0 \Rightarrow f(x) \text{ qat'iy kamayuvhi} \Rightarrow f'(x) \le 0$$

bo'ladi.

1-misol. Ushbu $f(x) = \dfrac{x^2}{2^x}$ funksiyaning o'suvchi, kamayuvchi bo'lish oraliqlari topilsin.

Ravshanki,

$$f'(x) = x \cdot 2^{-x}(2 - x\ln 2)$$

bo'ladi.

Ushbu $f'(x) > 0$, $x \cdot 2^{-x}(2 - x\ln 2) > 0$ tengsizlik $x \in \left(0,\ \dfrac{2}{\ln 2}\right)$ da o'rinli bo'ladi.Demak, $f(x)$ funksiya $x \in \left(0,\ \dfrac{2}{\ln 2}\right)$ da o'suvchi, $(-\infty,\ 0) \cup (\dfrac{2}{\ln 2},\ +\infty)$ da kamayuvchi bo'ladi.

2^0. **Funksiya ekstremumlari.** Faraz qilaylik, $f(x)$ funksiya $X \subset R$ to'plamda berilgan bo'lib, $x_0 \in X$ bo'lsin.

1-ta'rif. Agar shunday $\delta > 0$ son topilsaki, $\forall x \in U_\delta(x_0) = (x_0 - \delta,\ x_0 + \delta) \subset X$ nuqtalarda

$$f(x) \le f(x_0) \qquad (f(x) \ge f(x_0))$$

tengsizlik bajarilsa, $f(x)$ funksiya x_0 nuqtada maksimumga (minimumga) erishadi deyiladi, x_0 nuqtaga esa $f(x)$ funksiyaning maksimum(minimum) nuqtasi deyiladi.

2-ta'rif. Agar shunday $\delta > 0$ son topilsaki, $\forall x \in U_\delta(x_0) \setminus \{x_0\}$ $(U_\delta(x_0) \subset X)$ nuqtalarda

$$f(x) < f(x_0) \qquad (f(x) > f(x_0))$$

tengsizlik bajarilsa, $f(x)$ funksiya x_0 nuqtada qat'iy maksimumga (qat'iy minumumga) erishadi deyiladi,

Funksiya maksimum hamda minimum umumiy nom bilan uning ekstremumlari,maksimum hamda minimum nuqtalari esa uning ekstremum nuqtalari deyiladi.

5-teorema. Faraz qilaylik, $f(x)$ funksiya $X \subset R$ to'plamda berilgan bo'lib, $x_0 \in X$ nuqtada ekstremumga erishsin.

Agar $f(x)$ funksiya x_0 nuqtada $f'(x_0)$ hosilaga ega bo'lsa, u holda

$$f'(x_0) = 0$$

bo'ladi.

Aytaylik, $f(x)$ funksiya x_0 nuqtada maksimumga erishib, shu nuqtada hosilaga ega bo'lsin. U holda

$$\exists \delta > 0: \quad \forall x \in U_\delta(x_0) \subset X \text{ da } f(x) \le f(x_0)$$

bo'ladi.

$(x_0 - \delta,\ x_0 + \delta)$ intervalda $f(x)$ funksiyaga Ferma teoremasini qo'llab topamiz.

$$f'(x_0) = 0.$$

3-ta'rif. Funksiya hosilasini nolga aylantiradigan nuqta uning stanstionar (kritik) nuqtasi deyiladi.

Eslatma. Agar $f(x)$ funksiya biror nuqtada ekstirumga erishsa, ushu nuqtada hosilaga ega bo'lishi shart emas.

Masalan, $f(x) = |x|$ funksiya $x_0 = 0$ nuqtada minimumga erishadi, biroq u shu nuqtada hosilaga ega emas.

Demak, $f(x)$ funksiyaning ekstremum nuqtalari uning stastionar hamda hosilasi mavjud bo'lmagan nuqtalari bo'lishi mumkin.

4-ta'rif. Agar shunday $\delta > 0$ son topilsaki,

$$\forall x \in (x_0 - \delta, x_0) \ da \ g(x) > 0 \ yoki$$
$$\forall x \in (x_0 - \delta, x_0) \ da \ g(x) < 0$$

bo'lsa, $g(x)$ funksiya x_0 nuqtaning chap tomonida ishora saqlaydi deyiladi.

Agar shunday $\delta > 0$ son topilsaki,

$$\forall x \in (x_0, x_0 + \delta) \ da \ g(x) > 0 \ yoki$$
$$\forall x \in (x_0, x_0 + \delta) \ da \ g(x) < 0$$

bo'lsa, $g(x)$ funksiya x_0 nuqtaning o'ng tomonida ishora saqlaydi deyiladi.

6-teorema. Aytaylik, $f(x)$ funksiya $X \subset R$ to'plamda berilgan bo'lib, quyidagi shartlarni bajarsin:

1) $\exists \delta > 0, \forall x \in U_\delta(x_0) \subset X$ da $f'(x)$ hosilasi mavjud;

2) $f'(x_0) = 0$;

3) $f'(x)$ hosila x_0 nuqtaning o'ng va chap tomonlarida ishora saqlasin.

Agar $f'(x)$ hosila x_0 nuqtani o'tishda ishorasini o'zgartirsa, $f(x)$ funksiya x_0 nuqtada ekstrumga erishsin.

Agar $f'(x)$ hosila x_0 nuqtani o'tishda ishorasini o'zgartirmasa, $f(x)$ funksiya x_0 nuqtada ekstremumga erish-maydi.

Aytaylik,

$$\forall x \in (x_0 - \delta, x_0) \ da \ f'(x) > 0,$$
$$\forall x \in (x_0, x_0 + \delta) \ da \ f'(x) < 0$$

bo'lsin. U holda $\forall x \in (x_0 - \delta,\ x_0)$, $f'(x) > 0 \Rightarrow f(x)$ o'suvchi,ya'ni $f(x) < f(x_0)$, $\forall x \in (x_0,\ x_0 + \delta)$, $f'(x) < 0 \Rightarrow f(x)$ kamayuvchi,ya'ni $f(x) < f(x_0)$ bo'lib, $\forall x \in (x_0 - \delta,\ x_0 + \delta)$ da $f(x) < f(x_0)$ bo'ladi. Demak, bu holda $f(x)$ funksiya x_0 nuqtada maksimumga erishadi.

Aytaylik,

$$\forall x \in (x_0 - \delta, x_0) \ da \ f'(x) > 0,$$
$$\forall x \in (x_0, x_0 + \delta) \ da \ f'(x) < 0$$

bo'lsin.U holda $\forall x \in (x_0 - \delta,\ x_0)$, $f'(x) < 0 \Rightarrow f(x)$ kamayuvchi,ya'ni $f(x) > f(x_0)$, $\forall x \in (x_0,\ x_0 + \delta)$, $f'(x) > 0 \Rightarrow f(x)$ o'suvchi,ya'ni $f(x) > f(x_0)$ bo'lib, $\forall x \in (x_0 - \delta,\ x_0 + \delta)$ da $f(x) > f(x_0)$ bo'ladi.

Demak,bu holda $f(x)$ funksiya x_0 nuqtada minimumga erishadi.

Agar $\forall x \in (x_0 - \delta, x_0)$ da $f'(x) > 0$ $\forall x \in (x_0, x_0 + \delta)$ da $f'(x) > 0$ yoki $\forall x \in (x_0 - \delta,\ x_0)$ da $f'(x) < 0$, $\forall x \in (x_0,\ x_0 + \delta)$ da $f'(x) < 0$ bo'lsa,unda $f(x)$ funksiya $(x_0 - \delta, x_0 + \delta)$ da o'suvchi yoki $(x_0 - \delta,\ x_0 + \delta)$ da kamayuvchi bo'lib $f(x)$ funksiya x_0 nuqtada ekstrumga erishmaydi.

7-teorema. $f(x)$ funksiya $X \subset R$ to'plamda berilgan bo'lib, quyidagi shartlarni bajarsin:

1) $f(x) \in C(X)$;

2) $\exists \delta > 0,\ \forall x \in U_\delta(x_0) \setminus \{x_0\}$ да $f'(x)$ hosilasi mavjud va chekli;

3) $f'(x)$ hosila x_0 nuqtaning o'ng va chap tomonlarida ishora saqlansin.

Agar $f'(x)$ hosila x_0 nuqtani o'tishda ishorasini o'zgartirsa, $f(x)$ funksiya x_0 nuqtada ekstremumga erishadi.

Agar $f'(x)$ hosila x_0 nuqtani o'tishda ishorasini o'zgartirmasa, $f(x)$ funksiya x_0 ekstremumga erishmaydi.

Bu teorema yuqoridagi 6-teorema kabi isbotlanadi.

8-teorema. Faraz qilaylik $f(x)$ funksiya $X \subset R$ to'plamda berilgan va $m \in N,\ \ m \geq 2,\ \ x_0 \in X$ bo'lib, quyidagi shartlarni bajarsin:

1) $\exists \delta > 0,\ \forall x \in U_\delta(x_0) \subset X$ da $f^{(m-1)}(x)$ hosila mavjud;

2) $f^{(m)}(x_0)$ hosila mavjud;

3) $f'(x_0) = f''(x_0) = \ldots = f^{(m-1)}(x_0) = 0,\ \ f^{(m)}(x_0) \neq 0.$

U holda $m = 2k,\ k \in N$ bo'lganda $f(x)$ funksiya x_0 nuqtada ekstremumga erishib, $f^{(m)}(x_0) < 0$ bo'lganda x_0 nuqtada maksimumga, $f^{(m)}(x_0) > 0$ da minimumga erishadi.

Agar $m = 2k+1,\ k \in N$ bo'lsa, $f(x)$ funksiya x_0 nuqtada ekstremumga erishmaydi.

Isbot. $f(x)$ funksiyaning x_0 nuqtadagi Teylor formulasi

$$f(x) = \sum_{k=0}^{n} \frac{f^{(n)}(x_0)}{k!}(x-x_0)^k + o\big((x-x_0)^n\big), \quad x \to x_0$$

ni olamiz. Bu formula teoremaning shartida ushbu

$$f(x) = f(x_0) + \frac{f^{(m)}(x_0)}{m!}(x-x_0)^m + 0\big((x-x_0)^m\big), \ x \to x_0$$

ko'rinishga keladi. Bundan esa $x \neq x_0$ da

$$f(x) - f(x_0) = (x-x_0)^m \left[\frac{f^{(m)}(x_0)}{m!} + \frac{o\big((x-x_0)^m\big)}{(x-x_0)^m} \right], \quad x \to x_0$$

bo'lishi kelib chiqadi.

«o»ning ta'rifiga ko'ra $\frac{1}{m!}|\,f^{(m)}(x_0)\,| > 0$ son uchun $\exists \delta > 0,\ \forall x \in U_\delta(x_0) \setminus \{x_0\}$ nuqtalarda

$$\left| \frac{o\big((x-x_0)^m\big)}{(x-x_0)^m} \right| < \frac{1}{m!}\left| f^{(m)}(x_0) \right|$$

bo'ladi.Demak, $x \in U_\delta(x_0) \setminus \{x_0\}$ uchun

$$\frac{f^{(m)}(x_0)}{m!} + \frac{o((x-x_0)^m)}{(x-x_0)} \quad \text{va} \quad \frac{f^{(m)}(x_0)}{m!}$$

miqdorlar bir xil ishorali bo'ladi. Bundan esa $x \in U_\delta(x_0) \setminus \{x_0\}$ da

$$\frac{f^{(m)}(x_0)}{m!}(x-x_0)^m$$

ning ishorasi $f(x) - f(x_0)$ ayirmaning ishorasi bilan bir xil bo'lishi kelib chiqadi.

Agar $m = 2k,\ k \in N$ bo'lib, $f^{(m)}(x_0) > 0$ bo'lsa,unda $f(x) - f(x_0) > 0$, ya'ni $f(x) > f(x_0)$ bo'ladi. $f(x)$ funksiya x_0 nuqtada minimumga erishadi.

Agar $m = 2k,\ k \in N$ bo'lib, $f^{(m)}(x_0) < 0$ bo'lsa,unda $f(x) - f(x_0) < 0$, ya'ni $f(x) < f(x_0)$ bo'ladi. $f(x)$ funksiya x_0 nuqtada maksimumga erishadi.

Agar $m = 2k+1,\ k \in N$ bo'lsa, $f(x) - f(x_0)$ ayirma ishora saqlamaydi. Bu holda funksiya x_0 nuqtada ekstremumga erishmaydi.

Xususan,agar x_0 nuqta $f(x)$ funksiyaning statsionar nuqtasi bo'lib, $f(x)$ funksiya x_0 nuqtada chekli $f''(x_0) \neq 0$ hosilaga ega bo'lsa, shu nuqtada $f(x)$ funksiya $f''(x_0) < 0$ bo'lganda maksimumga, $f''(x_0) > 0$ minimumga ega bo'ladi.

2-misol. Ushbu

$$f(x) = 2\sqrt[3]{x^5} - 5\sqrt[3]{x^2} + 1$$

funksiya ekstremumga tekshirilsin.

Bu funksiya $R = (-\infty; +\infty)$ aniqlangan bo'lib, u shu to'plamda uzluksiz. Uning hosilasini topamiz:

$$f'(x) = 2 \cdot \frac{5}{3} \cdot x^{\frac{2}{3}} - 5 \cdot \frac{2}{3} \cdot x^{-\frac{1}{3}} = \frac{10(x-1)}{3\sqrt[3]{x}} \qquad (1)$$

Ravshanki, funksiyaning hosilasi $x_1 = 1$ nuqtada nolga alanadi: $f'(1) = 0$; $x_2 = 0$ nuqtada esa funksiyaning hosilasi mavjud emas.

Hosila ifodasi (1) dan ko'rnadiki, $x = 1$ nuqtaning chap tomonidagi nuqtalarda $f'(x) < 0$ o'ng tomonidagi nuqtalarda $f'(x) > 0$ bo'ladi. Demak, berilgan funksiya $x = 1$ nuqtada minimumga erishadi va $\min f(x) = f(1) = -2$ bo'ladi.

Yana hosila ifodasi (1) dan ko'rinadiki, $x = 0$ nuqtaning chap tomonidagi nuqtalarda $f'(x) > 0$, o'ng tomonidagi nuqtalarda $f'(x) < 0$ bo'ladi.

Demak, $f(x)$ funksiya $x=0$ funksiya nuqtada maksimumga erishadi va $\max f(x)=f(0)=1$ bo'ladi.

Funksiyaning ekstremumlarini toping.

1. $y=\frac{1}{1-x^2}$
2. $y=\frac{1}{x}-\frac{1}{x+1}+\frac{1}{x+2}$
3. $y=\frac{x^3}{6x^2-8-x^4}$
4. $y=\frac{x}{1+x^2}$
5. $y=\frac{x^2+2x+1}{x^2+1}$
6. $y=\sqrt{\frac{x-2}{x+2}}$
7. $y=\sqrt{x+1}-\sqrt{x-1}$
8. $y=(2-x)^{2/3}-(2+x)^{2/3}$
9. $y=x+\frac{1}{x^2}$
10. $y=x+\frac{x^2}{x^2+1}$
11. $y=\frac{x^2+8x-6}{x}$
12. $y=x^3-6x^2+9x-4$
13. $y=2x^2-x^4$
14. $y=x(x-1)^2(x-2)^3$
15. $y=x+\frac{1}{x}$
16. $y=\frac{2x}{1+x^2}$
17. $y=\frac{x^2-3x+2}{x^2+2x+1}$
18. $y=\sqrt{2x-x^2}$
19. $y=x\sqrt[3]{x-1}$
20. $y=xe^{-x}$
21. $y=\sqrt{x}\ln x$
22. $y=\frac{\ln^2 x}{x}$
23. $y=\cos x+\frac{1}{2}\cos 2x$
24. $y=\frac{10}{1+\sin^2 x}$
25. $y=arcctgx-\frac{1}{2}\ln(1+x^2)$
26. $y=e^x\sin x$
27. $y=|x|e^{-|x-1|}$
28. $y=\frac{1+x^2}{1+x^4}$
29. $y=|x^2-3x+2|$
30. $y=|x|\left(2+\cos\frac{1}{x}\right)$

6.2. Funksiyaning qavariqligi,egilish nuqtalari va asimtotalari

1⁰. Funksiyaning qavariqligi va botiqligi. Faraz qilaylik, $f(x)$ funksiya $(a,\ b)$ da berilgan bo'lib, $x_1,\ x_2\in(a,\ b)$ uchun $x_1<x_2$ bo'lsin.

$f(x)$ funksiya grafigining $(x_1,\ f(x_1))$, $(x_2,\ f(x_2))$ nuqtalaridan o'tuvchi to'g'ri chiziqni $y=l(x)$ desak,u quydagicha

$$l(x)=\frac{x_2-x}{x_2-x_1}f(x_1)+\frac{x-x_1}{x_2-x_1}f(x_2)$$

bo'ladi.

1-ta'rif. Agar har qanday oraliq $(x_1,\ x_2)\subset(a,\ b)$ da joylashgan $\forall x\in(x_1,\ x_2)$ uchun

$$f(x)\le l(x)\quad (f(x)<l(x))$$

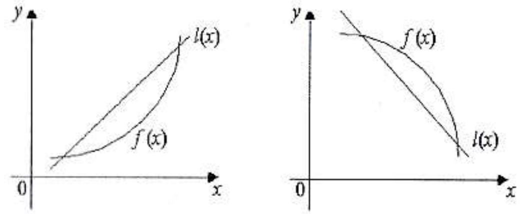
bo'lsa, $f(x)$ funksiya $(a,\ b)$ da botiq (qat'iy botiq) funksiya deyiladi.

2-ta'rif. Agar har qanday oraliq $(x_1,\ x_2)\subset(a,\ b)$ da joylashgan $\forall x\in(x_1,\ x_2)$ uchun

$$f(x)\ge l(x)\qquad (f(x)>l(x))$$

bo'lsa, $f(x)$ funksiya $(a,\ b)$ da qavariq (qat'iy qavariq) funksiya deyiladi.

Botiq hamda qavariq funksiya grafiklari 7-chizmada tasvirlangan:



7-chizma.

Aytaylik, $\alpha_1\ge0,\ \alpha_2\ge0,\ \alpha_1+\alpha_1=1$ bo'lib, $\forall x_1,\ x_2\in(a,\ b)$ bo'lsin. Funksiyaning botiqligi hamda qavariqligini quydagicha ta`riflash mumkin.

3-ta'rif. Agar

$$f(\alpha_1 x_1+\alpha_2 x_2)\le\alpha_1 f_1(x_1)+\alpha_2 f(x_2)$$
$$(f(\alpha_1 x_1+\alpha_2 x_2)<\alpha_1 f(x_1)+\alpha_2 f(x_2))$$

bo'lsa, $f(x)$ funksiya $(a,\ b)$ da botiq (qat'iy botiq) deyiladi.

4-ta'rif. Agar

$$f(\alpha_1 x_1+\alpha_2 x_2)\ge\alpha_1 f_1(x_1)+\alpha_2 f(x_2)$$
$$(f(\alpha_1 x_1+\alpha_2 x_2)>\alpha_1 f(x_1)+\alpha_2 f(x_2))$$

bo'lsa, $f(x)$ funksiya $(a,\ b)$ da qavariq(qat'iy qavariq) deyiladi.

1-misol. Ushbu $f(x)=x^2$ funksiya R da qat`iy botiq funksiya bo`ladi.

3-ta`rifdan foydalanib topamiz:

$$f(\alpha_1 x_1+\alpha_2 x_2)=(\alpha_1 x_1+\alpha_2 x_2)^2=(\alpha_1 x_1)^2+2\alpha_1\alpha_2 x_1 x_2+(\alpha_2 x_2)^2<$$

$$<\alpha_1^2 x_1^2+\alpha_1\alpha_2(x_1+x_2)^2+\alpha_2^2 x_2^2=\alpha_1 x_1^2(\alpha_1+\alpha_2)+\alpha_2 x_2^2(\alpha_1+\alpha_2)=$$

$$=\alpha_1 x_1^2+\alpha_2 x_2^2=\alpha_1 f(x_1)+\alpha_2 f(x_2)$$

1-teorema. Faraz qilaylik, $f(x)$ funksiya (a, b) da berilgan bo'lib,unda $f'(x)$ hosilaga ega bo'lsin. $f(x)$ funksiyaning (a, b) da botiq(qat`iy botiq) bo'lishi uchun $f'(x)$ ning $(a, b)$ da o'suvchi(qat'iy o'suvchi) bo`lishi zarur va yetarli.

Isbot. Zarurligi. $f(x)$ funksiya (a, b) da botiq bo'sin. U holda $\forall x_1, x_2\in(a, b),\ x_1<x_2,\ \forall x\in(x_1, x_2)$ uchun

$$f(x)\le\frac{x_2-x}{x_2-x_1}f(x_1)+\frac{x-x_1}{x_2-x_1}f(x_2)$$

bo`lib undan

$$\frac{f(x)-f(x_1)}{x-x_1}\le\frac{f(x_2)-f(x)}{x_2-x}$$

bo'lishi kelib chiqadi. ($(x_2-x_1)=(x_2-x)+(x-x_1)$ deyildi). Keyingi tengsizlikda $x\to x_1$ so'ng $x\to x_2$ da limitga o'tib,

$$f'(x_1)\le\frac{f(x_2)-f(x_1)}{x_2-x_1},$$

$$f'(x_2)\ge\frac{f(x_2)-f(x_1)}{x_2-x_1}$$

bo'lishini topamiz. Undan $f'(x_1)\le f'(x_2)$ bo'lishi kelib chiqadi.Demak, $f'(x)$ funksiya (a,b) da o`suvchi.

$f(x)$ funksiya (a, b) da qat`iy botiq bo'lsin.U holda

$$\frac{f(x)-f(x_1)}{x-x_1}<\frac{f(x_2)-f(x)}{x_2-x}$$

bo'ladi. Lagranj teoremasiga muvofiq

$$\frac{f(x)-f(x_1)}{x-x_1}=f'(c_1),\quad x_1<c_1<x;$$

$$\frac{f(x_2)-f(x)}{x_2-x}=f'(c_2),\quad x<c_2<x_2$$

bo'lib undan $f'(x_1)<f'(x_2)$ bo'lishi kelib chiqadi.

Yetarliligi. $f'(x)$ funksiya (a, b) da o'suvchi (qat'iy o`suvchi) bo'lsin: $\forall x_1, x_2\in(a, b),\ x_1<x_2$ da

$$f'(x_1)\le f'(x_2)\quad (f'(x_1)<f'(x_2)).$$

Lagranj teoremasidan foydalanib topamiz:

$$\frac{f(x)-f(x_1)}{x-x_1}=f'(c_1),\quad x_1<c_1<x;$$

$$\frac{f(x_2)-f(x)}{x_2-x}=f'(c_2),\quad x<c_2<x_2.$$

Ravshanki, $x_1<c_1<x<c_2<x_2\ \Rightarrow\ c_1<c_2$.Demak, $f'(c_1)\le f'(c_2)$ $(f'(c_1)<f'(c_2))$ bo'lib, yuqoridagi munosabatlardan

$$\frac{f(x)-f(x_1)}{x-x_1}\le\frac{f(x_2)-f(x)}{x_2-x}\qquad\left(\frac{f(x)-f(x_1)}{x-x_1}<\frac{f(x_2)-f(x)}{x_2-x}\right)$$

bo'lishi kelib chiqadi. Bu esa $f(x)$ funksiyaning (a, b) da botiq (qat`iy botiq) ekanligini bildiradi

Xuddi shunga o'xshash, quydagi teorema ham isbotlanadi.

2-teorema. $f(x)$ funksiya (a, b) da berilgan bo'lib, unda $f'(x)$ hosilaga ega bo'lsin.

$f(x)$ funksiyaning (a, b) da qavariq(qat'iy qavariq) bo'lishi uchun $f'(x)$ ning (a, b) da kamayuvchi(qat'iy kamayuvchi) bo'lishi zarur va yetarli.

Aytaylik, $f(x)$ funksiya (a, b) da berilgan bo'lib, u shu intervalda $f''(x)$ hosilaga ega bo'lsin. Bundan tashqari (a, b) intervalning har qanday (α, β) $((\alpha, \beta)\subset(a, b))$ qismida $f''(x)$ aynan nolga teng bo'lmasin.

3-teorema. $f(x)$ funksiya (a, b) intervalda botiq(qavariq) bo'lishi uchun (a, b) da

$$f''(x)\ge 0\quad (f''(x)\le 0)$$

bo'lishi zarur va yetarli.

Bu teoremaning isboti yuqoridagi hamda funksiyaning monotonligi haqidagi teoremadan kelib chiqadi.

2-misol. Ushbu $f(x)=\ln x\quad (x>0)$ funksiya qavariq bo'ladi.

Bu funksiya uchun

$$f''(x)=-\frac{1}{x^2}<0$$

bo'ladi. 2-teoremaga ko'ra berilgan $f(x)=\ln x$ funksiya $(0,+\infty)$ da qat'iy qavariq bo'ladi.

2^0. Funksiyaning egilish nuqtalari. Faraz qilaylik, $f(x)$ funksiya $X\subset R$ to'plamda berilgan bo'lib, $x_0\in X$, $(x_0-\delta,\ x_0+\delta)\subset X$, $\delta>0$ bo'lsin.

5-ta'rif. Agar $f(x)$ funksiya $(x_0-\delta,\ x_0)$ da botiq(qavariq), $(x_0,\ x_0+\delta)$ da qavariq(botiq), x_0 nuqta $f(x)$ funksiyaning **egilish nuqtasi** deyiladi.

Aytaylik, $f(x)$ funksiya $(x_0-\delta,\ x_0+\delta)$ da $f''(x)$ hosilaga ega bo'lsin. Agar $\forall x\in(x_0-\delta, x_0)$ da $f''(x)\ge 0$ $(f''(x)\le 0)$,

$$\forall x\in(x_0, x_0+\delta,)\ \text{da}\ \ f''(x)\le 0\ \ (f''(x)\ge 0),$$

bo'lsa, $f'(x)$ funksiya x_0 nuqtada eksturemumga erishadi va demak, $f''(x_0)=0$ bo'ladi. Demak, $f(x)$ funksiya egilish nuqtasida $f''(x)=0$ bo'ladi.

3-misol. Ushbu $f(x)=x^3$ funksiya $x_0=0$ nuqtada egiladi.

Bu funksiya uchun

$$f''(x)=6x$$

bo'lib,

$$\forall x\in(-\delta,0)\ \text{da}\ f''(x)<0$$

$$\forall x\in(0,\delta,)\ \text{da}\ f''(x)>0\quad(\delta>0)$$

bo'ladi.

3^0. Funksiya grafigining asimptotalari. Faraz qilaylik, $f(x)$ funksiya $X\subset R$ to'plamda berilgan bo'lib, x_0 nuqta X to'plamning limit nuqtasi bo'lsin.

6-ta'rif. Agar ushbu

$$\lim_{x\to x_0+0} f(x),\quad \lim_{x\to x_0-0} f(x)$$

limitlardan biri yoki ikkalasi ham cheksiz bo'lsa, $x=x_0$ to'g'ri chiziq $f(x)$ funksiya grafigining vertikal asimptotasi deyiladi.

Masalan, $f(x)=\dfrac{1}{x}$ funksiya grafigi uchun $x=0$ to'g'ri chizq vertikal asimtota bo'ladi.

Faraz qilaylik, $f(x)$ funksiya $(x_0,+\infty)$ da aniqlangan bo'lsin.

7-ta'rif. Agar shunday k va b sonlari topilsaki,

$$f(x)=kx+b+\alpha(x)\qquad(x\to\infty\ \text{da}\ \alpha(x)\to 0)$$

bo'lsa, $y=kx+b$ to'g'ri chiziq $f(x)$ funksiya grafigining og'ma asimptotasi deyiladi.

4-teorema. $f(x)$ funksiya grafigi $y=kx+b$ og'ma asimptotaga ega bo'lishi uchun

$$\lim_{x\to+\infty}\frac{f(x)}{x}=k,\qquad \lim_{x\to+\infty}(f(x)-kx)=b$$

bo'lishi zarur va yetarli.

Isbot. Zarurligi. $y=kx+b$ to'g'ri chiziq $f(x)$ funksiya grafigining og'ma asimptotasi bo'lsin.Unda

$$f(x)=kx+b+\alpha(x)$$

bo'lib, $x\to+\infty$ da $\alpha(x)\to 0$ bo'ladi. Bu tenglikni e'tiborga olib topamiz:

$$\lim_{x\to+\infty}\frac{f(x)}{x}=\lim_{x\to+\infty}\frac{kx+b+\alpha x}{x}=k;$$

$$\lim_{x\to+\infty}(f(x)-kx)=\lim_{x\to+\infty}(b+\alpha(x))=b.$$

Yetarliligi. Ushbu

$$\lim_{x\to+\infty}\frac{f(x)}{x}=k,\ \ \lim_{x\to+\infty}(f(x)-kx)=b$$

munosabatlar o'rinli bo'lsin. Bu munosabatlardan

$$(f(x)-kx)-b=\alpha(x)\to 0\ \Rightarrow\ f(x)=kx+b+\alpha(x)$$

bo'lishi kelib chiqadi.

4-misol. $f(x)=\dfrac{x^3}{(x-1)^2}$ funksiyaning og'ma asimptotasi topilsin.

Bu funksiya uchun

$$k=\lim_{x\to+\infty}\frac{f(x)}{x}=\lim_{x\to+\infty}\frac{x^2}{(x-1)^2}=1;$$

$$b=\lim_{x\to+\infty}(f(x)-kx)=\lim_{x\to+\infty}\left(\frac{x^3}{(x-1)^2}-x\right)=2$$

bo'ladi. Demak, $y=x+2$ to'g'ri chiziq berilgan funksiya grafigining og'ma asimptotasi bo'ladi.

Funksiyaning asimptotalari,egilish nuqtalarini toping va grafigini chizing.

1. $y=\sqrt{\dfrac{x}{x-2}}$
2. $y=\sqrt[3]{x^3-x^2}$
3. $y=\dfrac{\sqrt{|x^2-3|}}{x}$
4. $y=3x+arctg5x$
5. $y=\dfrac{\ln(x+1)}{x^2}+2x$

16. $y=\dfrac{\ln x}{x^2}$
17. $y=x^2\ln x$
18. $y=\dfrac{1}{x\ln x}$
19. $y=\dfrac{\ln x}{x}$
20. $y=\ln\left(x+\sqrt{x^2+1}\right)$

6. $y=\frac{\sin x}{x}$

7. $y=x\ln\left(e+\frac{1}{x}\right)$

8. $y=x\operatorname{arcsec}x$

9. $y=x^{\frac{1}{x}},\ x>0$

10. $y=x^{x},\ x>0$

11. $y=\ln\left|x^{2}-1\right|$

12. $y=\frac{1}{x^{2}}\ln^{2}|x|$

13. $y=x\ln^{2}|x|$

14. $y=\frac{x^{2}}{\ln|x|}$

15. $y=x^{2}\ln^{2}x$

21. $y=x^{3}e^{-x^{2}/2}$

22. $y=x^{2}e^{2/x}$

23. $y=\left(x^{2}+1\right)e^{-x^{2}/2}$

24. $y=(2x-1)e^{\frac{2}{x}}$

25. $y=(x-2)e^{-\frac{1}{x}}$

26. $y=\frac{1}{x}e^{-\frac{1}{x^{2}}}$

27. $y=xe^{\frac{1}{x}}$

28. $y=\frac{x}{2}+arcctgx$

29. $y=xarctgx$

30. $y=\sin x+\cos x$

6.3. Lopital qoidalari

Ma'lum shartlarda funksiya limitini hisoblash qoidalari o'rganilgan edi. Ko'p hollarda bunday shartlar bajarilmaganda,ya'ni

$$x\to x_{0}\ \text{da}\ f(x)\to 0, g(x)\to 0: \frac{f(x)}{g(x)}\ \text{ning limiti}\ \left(\frac{0}{0}\right)$$

$$x\to x_{0}\ \text{da}\ f(x)\to +\infty, g(x)\to +\infty: \frac{f(x)}{g(x)}\ \text{ning limiti}\ \left(\frac{\infty}{\infty}\right)$$

$$x\to x_{0}\ \text{da}\ f(x)\to \infty, g(x)\to \infty :(f(x)-g(x))\ \text{ning limiti}\ (\infty-\infty)$$

$$x\to x_{0}\ \text{da}\ f(x)\to 0, g(x)\to 0: (f(x))^{g(x)}\ \text{ning limiti}\ (0^{0})$$

$$x\to x_{0}\ \text{da}\ f(x)\to 1, g(x)\to \infty: (f(x))^{g(x)}\ \text{ning limiti}\ (1^{\infty})$$

$x\to x_{0}$ da $f(x)\to\infty, g(x)\to 0$: $f(x)\,g(x)$ ning limiti ∞^{0}ni topishda funksiyaning hosilalariga asoslangan qoidaga ko'ra hisoblash qulay bo'ladi. Bunday usul bilan funksiya limitini topish **Lopital qoidalari deyiladi**.

1^{0}. $\frac{0}{0}$ *va* $\frac{\infty}{\infty}$ Ko'rinishdagi hollar.

1-teorema. Faraz qilaylik, $f(x)$ va $g(x)$ funksiyalar (a, b) da berilgan bo'lib, quydagi shartlarni bajarsin:

1) $\lim\limits_{x\to b-0} f(x)=0,\ \lim\limits_{x\to b-0} g(x)=0;$

2) $\forall x\in(a,b)$ da $f'(x)$ va $g'(x)$ hosilalar mavjud;

3) $\forall\in(a,b)$ da $g'(x)\neq 0;$

4) Ushbu $\lim\limits_{x\to b-0}\frac{f'(x)}{g'(x)}=\ell,\ (\ell\in R)$ mavjud. U holda $\lim\limits_{x\to b-0}\frac{f(x)}{g(x)}=\ell$ bo'ladi.

Isbot. $f(b)=0,\ g(b)=0$ deb olamiz. Unda $f(x)$ va $g(x)$ funksiyalar $(b-\delta,b]$ da $(\delta>0)$ uzluksiz bo'lib qoladi. Teoremaning 4-shartiga ko'ra:

$$\forall\varepsilon>0,\ \exists\delta>0,\ \forall x\in(b-\delta,\ b):\ \left|\frac{f'(x)}{g'(x)}-\ell\right|<\varepsilon$$

bo'ladi.

Endi $(b-\delta,\ b]$ da Koshi teoremasidan foydalanib topamiz:

$$\left|\frac{f(x)}{g(x)}-\ell\right|=\left|\frac{f(b)-f(x)}{g(b)-g(x)}-\ell\right|=\left|\frac{f'(c)}{g'(c)}-\ell\right|<\varepsilon$$

$$(c\in(x,b)\subset[b-\delta,\ b]).$$

Demak,

$$\lim_{x\to b-0}\frac{f(x)}{g(x)}=\ell\ .$$

Shuni isbotlash talab qilingan edi.

1-misol. Ushbu $\lim\limits_{x\to e}\frac{(\ln x)^{\alpha}-\left(\frac{x}{e}\right)^{\beta}}{x-e}=\frac{\alpha-\beta}{e}$ munosabat isbotlansin.

$f(x)=(\ln x)^{\alpha}-\left(\frac{x}{e}\right)^{\beta}$, $g(x)=x-e$ funksiyalar uchun $(1,\ e)$ da 1-teoremaning barcha shartlari bajariladi:

1) $\lim\limits_{x\to e} f(x)=\lim\limits_{x\to e}\left[\ ((\ln x)^{\alpha}-\left(\frac{x}{e}\right)^{\beta}\ )\right]=0,$

$\lim\limits_{x\to e} g(x)=\lim\limits_{x\to e}(x-e)=0;$

2) $f'(x)=\alpha(\ln x)^{\alpha-1}\cdot\frac{1}{x}-\frac{\beta}{e}\left(\frac{x}{e}\right)^{\beta-1},\ \ g'(x)=1;$

3) $g'(x)=1\neq 0;$

$$4)\ \lim_{x\to e}\frac{f'(x)}{g'(x)}=\lim_{x\to e}\frac{\alpha(\ln x)^{\alpha-1}\cdot\frac{1}{x}-\frac{\beta}{e}\cdot\left(\frac{x}{e}\right)^{\beta-1}}{1}=\frac{\alpha-\beta}{e}.$$

Demak,

$$\lim_{x\to e}\frac{f(x)}{g(x)}=\lim_{x\to e}\frac{(\ln x)^{\alpha}-\left(\frac{x}{e}\right)^{\beta}}{x-e}=\frac{\alpha-\beta}{e}.$$

2-teorema. Faraz qilaylik, $f(x)$ va $g(x)$ funksiyalar $(a, +\infty)$ da berilgan bo'lib, quydagi shartlarni bajarilsin:

1) $\lim\limits_{x\to+\infty} f(x)=0,\quad \lim\limits_{x\to+\infty} g(x)=0;$

2) $\forall x\in(a, +\infty)$ da $f'(x),\ g'(x)$ hosilalar mavjud;

3) $\forall x\in(a, +\infty)$ da $g'(x)\neq 0;$

4) Ushbu

$$\lim_{x\to\infty}\frac{f'(x)}{g'(x)}=\ell$$

mavjud $(\ell\in R)$. U holda

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\ell$$

bo'ladi.

$a>0$ deb, $t=\frac{1}{x}$ deymiz.Unda $t\in(0,\ \frac{1}{a})$ bo'lib, $x\to+\infty$ da $t\to+0$. Endi $F(t)$ va $G(t)$ funksiyalarni quydagicha

$$F(t)=f(\tfrac{1}{t}),\qquad G(t)=g(\tfrac{1}{t})$$

aniqlaymiz.Unda

$$t\to+0 \text{ da } F(t)\to 0,\quad G(t)\to 0;$$

$$F'(t)=f'(\tfrac{1}{t})\cdot(-\frac{1}{t^2}),\quad G'(t)=g'(\tfrac{1}{t})\cdot(-\frac{1}{t^2});$$

$$\frac{F'(t)}{G'(t)}=\frac{f'(\frac{1}{t})}{g'(\frac{1}{t})}\to\ell,\quad (t\to+0)$$

bo'lib, 1-teoremaga ko'ra, $t\to+0$ da

$$\frac{F(t)}{G(t)}\to\ell$$

bo'ladi. Keyingi munosabatdan esa

$$\lim_{x\to+\infty}\frac{f(x)}{g(x)}=\ell$$

bo'lishi kelib chiqadi

2-misol. Ushbu $\lim\limits_{x\to+\infty}\dfrac{e^{\frac{1}{x^2}}-1}{2\mathrm{arctg}x^2-\pi}$ limitni hisoblang.

Agar $f(x)=e^{\frac{1}{x^2}}-1,\ g(x)=2\mathrm{arctg}x^2-\pi$ deyilsa,ular uchun 2-teoremaning barcha shartlari bajariladi, jumladan

$$f'(x)=-\frac{2}{x^3}e^{\frac{1}{x^2}},\qquad g'(x)=\frac{4x}{1+x^4}$$

bo'lib,

$$\lim_{x\to+\infty}\frac{f'(x)}{g'(x)}=\lim_{x\to+\infty}\frac{-\frac{2}{x^3}e^{\frac{1}{x^2}}}{\frac{4x}{1+x^4}}=-\lim_{x\to+\infty}\frac{1+x^4}{2x^4}=-\frac{1}{2}$$

bo'ladi. 2-teoremaga ko'ra

$$\lim_{x\to+\infty}\frac{f'(x)}{g'(x)}=\lim_{x\to+\infty}\frac{f(x)}{g(x)}=\lim_{x\to+\infty}\frac{e^{\frac{1}{x^2}}-1}{2\mathrm{arctg}x^2-\pi}=-\frac{1}{2}$$

bo'ladi.

Quydagi teoremalar ham yuqorida keltirilgan teoremalarga o'xshash isbotlanadi.

3-teorema. Faraz qilaylik, $f(x)$ va $g(x)$ funksiyalar (a, b) da berilgan bo'lib quydagi shartlarni bajarsin:

1) $\lim\limits_{x\to b-0} f(x)=\infty,\quad \lim\limits_{x\to b-0} g(x)=\infty;$

2) $\forall x\in(a, b)$ da $f'(x),\ g'(x)$ hosilalar mavjud;

3) $\forall x\in(a, b)$ da $g'(x)\neq 0;$

4) Ushbu $\lim\limits_{x\to b-0}\dfrac{f'(x)}{g'(x)}=\ell,\ (\ell\in R)$ mavjud.U holda

$$\lim_{x\to b-0}\frac{f(x)}{g(x)}=\ell$$

bo'ladi.

4-teorema. Faraz qilaylik, $f(x)$ va $g(x)$ funksiyalar $(a, +\infty)$ da berilgan bo'lib, quydagi shartlarni bajarsin:

1) $\lim\limits_{x\to+\infty} f(x)=\infty, \quad \lim\limits_{x\to+\infty} g(x)=\infty;$

2) $\forall x\in(a,+\infty)$ da $f'(x),\ g'(x)$ hosilalari mavjud;

3) $\forall x\in(a,+\infty)$ da $g'(x)\neq 0;$

4) Ushbu $\lim\limits_{x\to+\infty}\dfrac{f'(x)}{g'(x)}=\ell,\ (\ell\in R)$ mavjud.U holda $\lim\limits_{x\to+\infty}\dfrac{f(x)}{g(x)}=\ell$ bo'ladi.

2^0. $0\cdot\infty,\ \infty-\infty,\ 1^\infty,\ 0^0$ ko'rinishdagi hollar. Bu ko'rinishdagi aniqmasliklar $\dfrac{0}{0},\ \dfrac{\infty}{\infty}$ hollarga keltirib, so'ng yuqoridagi teoremalar qo'llaniladi.

1) $x\to x_0$ da $f(x)\to 0,\ g(x)\to\infty$ bo'lganda $f(x)\cdot g(x)$ funksiyaning limitini topish uchun

$$f(x)\cdot g(x)=\frac{f(x)}{\dfrac{1}{g(x)}}=\frac{g(x)}{\dfrac{1}{f(x)}}$$

deb,so'ng 1- yoki 2-teoremalar qo'llaniladi.

2) $x\to x_0$ da $f(x)\to+\infty,\ g(x)\to+\infty$ bo'lganda $f(x)-g(x)$ funksiyani limitini topish uchun uni

$$f(x)-g(x)=\frac{\dfrac{1}{g(x)}-\dfrac{1}{f(x)}}{\dfrac{1}{f(x)}\cdot\dfrac{1}{g(x)}}$$

deb so'ng 1-teorema qo'llaniladi.

3) $x\to x_0$ da $f(x)\to 0,\ g(x)\to 0$ hamda $x\to x_0$ da $f(x)\to 1,\ g(x)\to+\infty$ bo'lganda $(f(x))^{g(x)}$ funksiyaning limitini topish uchun avvalo

$$y=(f(x))^{g(x)}$$

funksiya logarifmlanadi, so'ng yuqoridagi teoremalar foydalaniladi.

3-misol. Ushbu $\lim\limits_{x\to0}\left(\dfrac{\sin x}{x}\right)^{\frac{1}{x^2}}$

limit hisoblansin.

Avvalo $y=\lim\limits_{x\to0}\left(\dfrac{\sin x}{x}\right)^{\frac{1}{x^2}}$ deb olamiz. Ravshanki, $x\to0$ da

$$f(x)=\frac{\sin x}{x}\to 1,\qquad g(x)=\frac{1}{x^2}\to+\infty.$$

Sodda hisoblashlar yordamida topamiz:

$$\lim_{x\to0}\ln y=\lim_{x\to0}\frac{\ln\dfrac{\sin x}{x}}{x^2}=\lim_{x\to0}\frac{\left(\ln\dfrac{\sin x}{x}\right)'}{\left(x^2\right)'}=$$

$$=\lim_{x\to0}\frac{\dfrac{x}{\sin x}\cdot\dfrac{x\cos x-\sin x}{x^2}}{2x}=$$

$$=\frac{1}{2}\lim_{x\to0}\frac{x\cos x-\sin x}{x^3}=\frac{1}{2}\lim_{x\to0}\frac{(x\cos x-\sin x)'}{\left(x^3\right)'}=$$

$$=-\frac{1}{2}\lim_{x\to0}\frac{x\sin x}{3x^2}=-\frac{1}{6}.$$

Demak, $\lim\limits_{x\to0}\left(\dfrac{\sin x}{x}\right)^{\frac{1}{x^2}}=e^{-\frac{1}{6}}.$

Lopital qoidalaridan foydalanib,limitni hisoblang.

1. $\lim\limits_{x\to0}\dfrac{\sin ax}{\sin bx}$
2. $\lim\limits_{x\to0}\dfrac{chx-\cos x}{x^2}$
3. $\lim\limits_{x\to0}\dfrac{tgx-x}{x-\sin x}$
4. $\lim\limits_{x\to0}\dfrac{3tg4x-12tgx}{3\sin4x-12\sin x}$
5. $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{tg3x}{tgx}$
6. $\lim\limits_{x\to0}\dfrac{xctgx-1}{x^2}$
7. $\lim\limits_{x\to\frac{\pi}{4}}\dfrac{\sqrt[3]{tgx}-1}{2\sin^2x-1}$
8. $\lim\limits_{x\to0}\dfrac{x\left(e^x+1\right)-2\left(e^x-1\right)}{x^3}$
9. $\lim\limits_{x\to0}\dfrac{1-\cos x^2}{x^2\sin x^2}$

16. $\lim\limits_{x\to0}x^{x^x-1}$
17. $\lim\limits_{x\to0}(ctgx)^{\sin x}$
18. $\lim\limits_{x\to+\infty}\dfrac{x^{\ln x}}{(\ln x)^x}$
19. $\lim\limits_{x\to0}\left(\dfrac{1+e^x}{2}\right)^{cthx}$
20. $\lim\limits_{x\to0}\left(\dfrac{\cos x}{chx}\right)^{\frac{1}{x^2}}$
21. $\lim\limits_{x\to0}\left(\dfrac{2}{\pi}\arccos x\right)^{\frac{1}{x}}$
22. $\lim\limits_{x\to+0}x^{\frac{k}{1+\ln x}}$
23. $\lim\limits_{x\to1}x^{\frac{1}{1-x}}$
24. $\lim\limits_{x\to1}(2-x)^{tg\frac{\pi x}{2}}$

10. $\lim\limits_{x\to 0}\dfrac{\arcsin 2x-2\arcsin x}{x^3}$

11. $\lim\limits_{x\to 0}\dfrac{a^x-a^{\sin x}}{x^3}, a>0$

12. $\lim\limits_{x\to 1}\dfrac{x^x-x}{\ln x-x+1}$

13. $\lim\limits_{x\to+\infty}\dfrac{\ln x}{x^{\varepsilon}}, \varepsilon>0$

14. $\lim\limits_{x\to 0}\dfrac{e^{-1/x^2}}{x^{100}}$

15. $\lim\limits_{x\to 1-0}\ln x\cdot\ln(1-x)$

25. $\lim\limits_{x\to\frac{\pi}{4}}(tgx)^{tg2x}$

26. $\lim\limits_{x\to 0}(ctgx)^{\sin x}$

27. $\lim\limits_{x\to+0}\left(\ln\dfrac{1}{x}\right)^x$

28. $\lim\limits_{x\to\infty}\left(tg\dfrac{\pi x}{2x+1}\right)^{\frac{1}{x}}$

29. $\lim\limits_{x\to a}\left(\dfrac{tgx}{tga}\right)^{ctg(x-a)}$

30. $\lim\limits_{x\to 1}\left(\dfrac{1}{\ln x}-\dfrac{1}{x-1}\right)$

VII BOB
ANIQMAS INTEGRAL

7.1. Boshlang'ich funksiya va aniqmas integral tushunchasi

1.1. Boshlang'ich funksiya tushunchasi.

1. Ta'rif. Biror chekli yoki cheksiz (a,b) oraliqdagi har bir x nuqtada differensiallanuvchi va hosilasi

$$F'(x)=f(x)$$

shartni qanotlantiruvchi $F(x)$ berilgan $f(x)$ funksiya uchun **boshlang'ich funksiya** deyiladi.

Berilgan $y=F(x)$ funksiyaning $y=F'(x)=f(x)$ hosilasi bir qiymatli aniqlanadi. Ammo $y=f(x)$ funksiyaning boshlang'ich $F(x)$ funksiyasini topish masalasi bir qiymatli hal qilinmaydi. Haqiqatdan ham, agar $F(x)$ funksiya $f(x)$ uchun boshlang'ich funksiya bo'lsa, u holda ixtiyoriy C o'zgarmas son uchun $F(x)+C$ funksiya ham $f(x)$ uchun boshlang'ich funksiya bo'ladi. Haqiqatan ham, differensiallash qoidalariga asosan,

$$(F(x)+C)'=F'(x)+(C)'=f(x)+0=f(x)$$

va ta'rifga asosan, $F(x)+C$ funksiya $f(x)$ uchun boshlang'ich funksiya bo'ladi. Endi ba'zi bir sodda funksiyalarning boshlang'ichini topib ko'ramiz.

Misollar:

$f(x)=2x \qquad F(x)=\dfrac{2x^2}{2}+C=x^2+C$

$f(x)=3x^2+6x-6 \qquad F(x)=\dfrac{3x^3}{3}+\dfrac{6x^2}{2}-6x+C=x^3+3x^2-6x+C$

$f(x)=3x^2 \qquad F(x)=\dfrac{3x^3}{3}+C=x^3+C$

Bu yerda $F(x)$ funksiya $f(x)$ funksiyaning boshlang'ichidir.

1.2. Aniqmas integral. Aniqmas integralning asosiy xossalari.

Agar $F(x)$ biror (a,b) oraliqda $f(x)$ funksiya boshlang'ichi bo'lsa, unda $F(x)+C$ (C-ixtiyoriy o'zgarmas son) funksiyalar to'plami shu oraliqda $f(x)$ funksiyaning ***aniqmas integrali*** deyiladi.

Berilgan $f(x)$ funksiyaning aniqmas integrali $\int f(x)dx$ kabi belgilanadi va ta'rifga asosan birorta $F(x)$ boshlang'ich funksiya bo'ycha

$$\int f(x)\,dx = F(x) + C$$

tenglik bilan aniqlanadi. Bunda C ixtiyoriy o'zgarmas son ekanligini yana bir marta eslatib o'tamiz.

$\int f(x)\,dx = F(x)+C$ tenglikda $\int$ ***-integral belgisi,*** $f(x)$ ***integral ostidagi funksiya,*** $f(x)\,dx$ ***integral ostidagi ifoda,*** x esa ***integrallash o'zgaruvchisi*** deyiladi.Berilgan $f(x)$ funksiyaning $\int f(x)\,dx$ aniqmas integralini topish bu funksiyani ***integrallash*** deb ataladi.

Endi aniqmas integralning xossalarini keltiramiz. Bundan buyon aniqmas integral haqida gap borganda uni qaralayotgan oraliqda mavjud deb, ya'ni integral ostidagi funksiya qaralayotgan oraliqda boshlang'ich funksiyaga ega deb qaraymiz va oraliqni ko'rsatib o'tirmaymiz.

1)Ushbu

$$d\left(\int f(x)dx\right) = f(x)dx$$

munosabat o'rinli.

Faraz qilaylik, $F(x)$ funksiya $f(x)$ ning boshlang'ich funksiyasi bo'lsin:

$$F'(x) = f(x).$$

U holda,

$$\int f(x)dx = F(x) + C \qquad (C = const)$$

bo'ladi.Bu tenglikka differensial amalini qo'llab topamiz.

$$d\left(\int f(x)dx\right) = d(F(x)+C) = dF(x) = F'(x)dx = f(x)dx$$

Bu xossa avval differensial belgisi d, so'ngra integral belgisi $\int$ kelib, ular yonma-yon turganda o'zaro bir-birini yo'qotishini ifodalaydi.

2)Ushbu

$$\int dF(x) = F(x) + C \qquad (C = const)$$

munosabat o'rinli .

Aytaylik,$F(x)$ funksiya $f(x)$ ning boshlang'ich funksiyasi bo'lsin:

$$F'(x) = f(x).$$

U holda,

$$\int f(x)dx = F(x) + C \qquad (C = const)$$

bo'ladi. Ayni paytda,

$$\int f(x)dx = \int F'(x)dx = \int dF(x) \qquad (1)$$

bo'lib, bu tengliklardan

$$\int dF(x) = F(x) + C$$

bo'lishi kelib chiqadi.

Bu xossa avval integral belgisi $\int$, so'ngra differensial belgisi d kelib, ular yonma-yon turganda o'zaro bir-birini yo'qotishini anglatadi va $F(x)$ ga o'zgarmas C ni qo'shib qo'yish kerkligini ko'rsatadi

3) Ushbu

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx \qquad (2)$$

tenglik o'rinli bo'ladi.

Aytaylik, $F(x)$ va $\Phi(x)$ funksiyalar mos ravishda $f(x)$ va $g(x)$ larning boshlang'ich funksiyasi bo'lsin.

$$F'(x) = f(x), \Phi'(x) = g(x).$$

U holda

$$\int f(x)dx = F(x) + C_1, \int g(x)dx = \Phi(x) + C_2$$

bo'lib,

$$\int f(x)dx + \int g(x)dx = F(x) + \Phi(x) + C_1 + C_2 \qquad (3)$$

bo'ladi.

Ayni paytda,

$$[F(x) + \Phi(x)]' = f(x) + g(x)$$

bo'lganligi sababli

$$\int [f(x) + g(x)]dx = F(x) + \Phi(x) + C_3 \qquad (4)$$

bo'ladi.(3) va (4) munosabatlardan,ulardagi C_1, C_2 va C_3 larning ixtiyoriy ekanligini e'tiborga olib topamiz:

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

Bu xossa aniqmas integralning additivlik xossasi deyiladi.

4) Ushbu

$$\int kf(x)dx = k\int f(x)dx \qquad (5)$$

tenglik o'rinli bo'ladi,bunda k o'zgarmas son va $k \neq 0$.

Misol: Ushbu $J = \int \left(\frac{5}{1+x^2} - 3\sin x\right)dx$ integral topilsin.

Aniq integralning 3)- va 4)- xossalaridan foydalansak,unda

$$\int \left(\frac{5}{1+x^2} - 3\sin x\right)dx = 5\int \frac{1}{1+x^2}dx - 3\int \sin x dx$$

bo'lishi kelib chiqadi.

Endi

$$(-\cos x)' = \sin x, \quad (\mathrm{arctg}x)' = \frac{1}{1+x^2}$$

bo'lishini e`tiborga olib topamiz:

$$5\int \frac{1}{1+x^2}dx - 3\int \sin x dx = 5\mathrm{arctg}x + 3\cos x + C.$$

Demak,

$$J = 5\,\mathrm{arctg}\,x + 3\cos x + C.$$

Aniqmas integralni hisoblang.

1. $\int (2\cdot 3^x - 4shx + 6\cos x + 9)dx$
2. $\int (3x-7)^{10}dx$
3. $\int \frac{x^4}{1+x^2}dx$
4. $\int \frac{dx}{\sqrt{x-3}-\sqrt{x-7}}$
5. $\int \left(\frac{3x^2-2x+5\sqrt{x}}{x\sqrt{x}}\right)dx$
6. $\int \frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1-x^4}}dx$
7. $\int \frac{\cos 2x}{\sin^2 2x}dx$
8. $\int \frac{dx}{\sqrt{3+x+x^2}}$
9. $\int \left(5\cos x - \frac{2}{x^2+1} + x^4\right)dx$
10. $\int \frac{x^2-7}{x+3}dx$
11. $\int \frac{\sqrt[3]{x} - x^2e^x - x}{x^2}dx$
12. $\int \left(\frac{3}{x^2+1} - \frac{2}{\sqrt{1-x^2}}\right)dx$
13. $\int \frac{2\cdot 3^x - 3\cdot 2^x}{3^x}dx$
14. $\int \left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2 dx$

16. $\int a^x\left(1+\frac{a^{-x}}{\sqrt{x^3}}\right)dx$

17. $\int (\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3})dx$

18. $5f(5x)$

19. $3f\left(\frac{x}{2}\right)$

20. $-4f(-4x+3)$

21. $3f(-3x+2)$

22. $\frac{4}{5}f\left(\frac{2}{3}x+7\right)$

23. $cf(ax+b)$

24. $-2f(-2x)$

25. $7f(3x+9)$

26. $3f(x+4)$

27. $2f(\frac{2}{5}x-7)$

28. $5f(2x-9)$

29. $4f(3x+5)$

15. $\int e^x\left(1+\frac{e^{-x}}{\cos^2 x}\right)dx$

30. $5f(\frac{4}{7}x-7)$

1.3.Elementar funksiyalarning aniqmas integral jadvali.

Elementar funksiyalarning hosilalari jadvali hamda aniqmas integral ta'rifidan foydalanib,sodda funksiyalarning aniqmas integrallari topiladi.Ularni jamlab jadval ko'rinishiga keltiramiz:

1) $\int 0\cdot dx = C, \quad C = const.$

2) $\int 1\cdot dx = x + C.$

3) $\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + C, \quad (\alpha \neq -1).$

4) $\int \frac{dx}{x} = \ln|x| + C, \quad (x \neq 0).$

5) $\int a^x dx = \frac{a^x}{\ln a} + C, \quad (a>0,\ a\neq 1).$
$\int e^x dx = e^x + C.$

6) $\int \sin x dx = -\cos x + C.$

7) $\int \cos x dx = \sin x + C.$

8) $\int \frac{dx}{\cos^2 x} = \mathrm{tg}x + C, \quad (x \neq \frac{\pi}{2} + \pi n,\ n\in Z).$

9) $\int \frac{dx}{\sin^2 x} = -\mathrm{ctg}x + C, \quad (x\neq \pi n,\ n\in Z).$

10) $\int \frac{dx}{\sqrt{1-x^2}} = \begin{cases} \arcsin x + C, \\ -\arccos x + C. \end{cases} \quad (-1<x<1).$

11) $\int \frac{dx}{\sqrt{1+x^2}} = \begin{cases} \mathrm{arctg}x + C, \\ -\mathrm{arcctg}x + C. \end{cases}$

12) $\int shxdx = chx + C.$

13) $\int chxdx = shx + C.$

14) $\int \frac{dx}{sh^2x} = -cthx + C, \quad (x\neq 0).$

15) $\int \frac{dx}{ch^2x} = thx + C.$

Integralning formularidan foydalanib hisoblang.

1. $\int \frac{\sqrt{x} - 2\sqrt[3]{x^2} + 1}{\sqrt[4]{x}} dx$

2. $\int \frac{(1-x)^3}{x\sqrt[3]{x}} dx$

3. $\int \frac{(\sqrt{2x} - \sqrt[3]{3x})^2}{x} dx$

4. $\int \frac{e^{3x}+1}{e^x+1} dx$

5. $\int \frac{2^{x+1} - 5^{x-1}}{10^x} dx$

6. $\int (2^x + 3^x)^2 dx$

7. $\int th^2 x\, dx$

8. $\int cth^2 x dx$

9. $\int (2x-3)^{10} dx$

10. $\int (ashx + bchx) dx$

11. $\int (1 + \sin x + \cos x) dx$

12. $\int \sqrt[3]{1-3x}\, dx$

13. $\int \frac{dx}{\sqrt{2-5x}}$

14. $\int \frac{dx}{(5x-2)^{\frac{5}{2}}}$

15. $\int sh(3x-1) + ch(3x-1) dx$

16. $\int (x^2 + 3x^3 + x + 1) dx$

17. $\int (x^4 + \sqrt[5]{x} + 3\sqrt{x} + \frac{1}{x}) dx$

18. $\int (\frac{2}{1+x^2} - \frac{3}{\sqrt{1-x^2}}) dx$

19. $\int \frac{x^2+2}{x^2-1} dx$

20. $\int \frac{x^2}{x^2+1} dx$

21. $\int \frac{3 - 2ctg^2 x}{\cos^2 x} dx$

22. $\int \frac{1 - \sin^3 x}{\sin^2 x} dx$

23. $\int \frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1-x^4}} dx$

24. $\int (\sin\frac{x}{2} + \cos\frac{x}{2})^2 dx$

25. $\int e^x (2 - \frac{e^{-x}}{x^3}) dx$

26. $\int \frac{1}{\sqrt{4-x^2}} + \frac{1}{x^2+3}$

27. $\int \frac{dx}{\sqrt{1+e^x}}$

28. $\int \sin^2(ax+b) dx$

29. $\int 2^{2x} e^x dx$

30. $\int x(x-1)(x+2) dx$

7.2. *Integrallash usullari.*

1.1. O'zgaruvchilarni almashtirish usuli.

Faraz qilaylik, $f(x)$ funksiyaning aniqmas integrali

$$\int f(x)dx \qquad (1)$$

berilgan bo'lib, uni hisoblash talab etilsin.

Ko'pincha, o'zgaruvchi x ni ma'lum qoidaga ko'ra boshqa o'zgaruvchiga almashtirish natijasida berilgan integral sodda integralga keladi va uni hisoblash oson bo'ladi.

Aytaylik, (1) integraldagi o'zgaruvchi x yangi o'zgaruvchi t bilan ushbu

$$t = \varphi(x)$$

munosabatda bo'lib, quydagi shartlar bajarilsin:

1) $\varphi(x)$ funksiya differensiallanuvchi bo'lsin:

2) $g(t)$ funksiya boshlang'ich funksiya $G(t)$ ga ega, ya'ni

$$G'(t) = g(t), \qquad \int g(t)dt = G(t) + C; \qquad (2)$$

3) $f(x)$ funksiya quydagicha

$$f(x) = g(\varphi(x)) \cdot \varphi'((x) \qquad (3)$$

ifodalansin.

U holda

$$\int f(x)dx = \int g(\varphi(x))\varphi'(x)dx = G(\varphi(x)) + C$$

bo'ladi

Murakkab funksiyaning hosilasini hisoblash qoidasidan foydalanib, (2) va (3) munosabatlarni e'tiborga olib topamiz:

$$[G(\varphi(x)) + C]' = G'(\varphi(x)) \cdot \varphi'(x) = g(\varphi(x)) \cdot \varphi'(x) = f(x).$$

Bundan:

$$\int f(x)dx = G(\varphi(x)) + C$$

bo'lishi kelib chiqadi.

Shu yo'l bilan (1) integralni hisoblash o'zgaruvchini almashtirib integrallash usuli deyiladi.

Bu usulda, o'zgaruvchini juda ko'p munosabat bilan almashtirish imkoniyati bo'lgan holda ular orasida qaralayotgan integralni sodda, hisoblash uchun qulay holga keltiradiganini tanlab olish muhimdir.

Misol: $\int x \ln x dx$ integralni hisoblang.

Bu yerda integralni o'zgaruvchisini almashtirib hisoblaymiz:

$$\int x\ln x dx = \left| \begin{matrix} U=\ln x & dv = x \\ dU=\frac{1}{x}dx & v=\frac{x^2}{2} \end{matrix} \right| = \ln x\cdot\frac{x^2}{2} - \int\frac{x^2}{2}\cdot\frac{1}{x}dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$$

2.2.Bo'laklab integrallash usuli.

Faraz qilaylik, $u(x)$ va $v(x)$ funksiyalar uzluksiz $u'(x)$, $v'(x)$ hosilaga ega bo'lsin.

Ravshanki,

$$(u(x)\cdot v(x))' = u'(x)\cdot v(x) + u(x)\cdot v'(x)$$

bo'ladi. Demak,

$$F(x) = u(x)\cdot v(x)$$

funksiya

$$f(x) = u'(x)\cdot v(x) + u(x)\cdot v'(x)$$

funksiyaning boshlang'ich funksiyasi bo'ladi.

Bundan

$$\int [u'(x)\cdot v(x) + u(x)\cdot v'(x)]dx = u(x)\cdot v(x) + C$$

bo'lishi kelib chiqadi.

Aniqmas integralning xossalaridan foydalanib

$$\int u(x)\cdot v'(x)dx = u(x)\cdot v(x) - \int u'(x)\cdot v(x)dx \qquad (5)$$

bo'lishini topamiz.

(5) formulani quydagicha

$$\int u(x)\cdot dv(x) = u(x)\cdot v(x) - \int v(x)du(x) \qquad (5‰)$$

ham yozish mumukin.

Bu (5‰) formula bo'laklab integrallash formulasi deyiladi.Uning yordamida

$$\int u(x)\cdot v'(x)dx$$

integralni hisoblash

$$\int u'(x)\cdot v(x)dx$$

integralni hisoblashga keltiriladi.

Misol: $\int x\cos x dx$ integral hisoblansin.

Bo`laklab integrallash formulasidan foydalanib topamiz:

$$\int x\cos x dx = \left| \begin{matrix} u = x, & du = dx \\ \cos x dx = dv & v = \sin x \end{matrix} \right| = x\sin x - \int \sin x dx = x\sin x + \cos x + C$$

Integralni hisoblang:

1. $\int\frac{\sin x}{x}dx$

16. $\int\frac{dx}{\sqrt{2-3x^2}}$

2. $\int\frac{e^x}{x}dx$
3. $\int\frac{e^x}{x^2}dx$
4. $\int\ln^2 x dx$
5. $\int\frac{dx}{x\sqrt{x^{10}-1}}$
6. $\int\arcsin 2x dx$
7. $\int arctgx dx$
8. $\int\sqrt{x}\log_5 x^4 dx$
9. $\int\frac{x^3}{2^x+1}dx$
10. $\int x arcctgx dx$
11. $\int x arctgx dx$
12. $\int\arcsin x dx$
13. $\int e^{3x}\cos 3x dx$
14. $\int\ln(\sqrt{1-x}+\sqrt{1+x})dx$
15. $\int x\ln x dx$

17. $\int\frac{x}{\sin^2 x}dx$
18. $\int\frac{x}{x^4+4}dx$
19. $\int(x^2+1)d[x]$
20. $\int arctg\sqrt{x}\,dx$
21. $\int\frac{dx}{(x-1)\sqrt{x^2-3x+2}}$
22. $\int\cos x(x^3+1)dx$
23. $\int x^2\arcsin 2x dx$
24. $\int x\sqrt{1-x^2}\arcsin x dx$
25. $\int x^3 arctgx dx$
26. $\int(x^2-x+1)chx dx$
27. $\int x^5\sin 5x dx$
28. $\int\sin x chx dx$
29. $\int 3^x\cos x dx$
30. $\int e^{3x}(x^2-6x+2)dx$

7.3.Ratsional funksiyalarni integrallash.

1.Noma'lum koeffisentlar usuli.

Ushbu

$$\frac{A}{(x-a)^m}\ (x\neq a),\ \frac{Bx+C}{(x^2+px+q)^m}$$

ko'rinishdagi funksiyalar sodda kasrlar deyiladi,bunda $m\in N$; A,B,C,a,p,q haqiqiy sonlar bo'lib, x^2+px+q kvadrat uchhad haqiqiy ildizga ega emas,ya'ni $q-\frac{p^2}{4}>0$.

$m=1$ bo'lganda soda kasrlarning integrallari

$$\int\frac{A}{x-a}dx,\ \int\frac{Bx+C}{x^2+px+q}dx$$

lar quydagicha hisoblanadi:

$$\int \frac{A}{x-a}dx = A\int \frac{d(x-a)}{x-a} = A\ln|x-a| + C;$$

$$\int \frac{Bx+C}{x^2+px+q}dx = \int \frac{Bx+C}{(x+\frac{p}{2})^2 + (q-\frac{p^2}{4})}dx =$$

$$= \left| \begin{matrix} x+\frac{p}{2} = t, & x = t-\frac{p}{2} \\ dx = dt, & q-\frac{p^2}{4} = a^2 \end{matrix} \right| =$$

$$= B\int \frac{tdt}{t^2+a^2} + (C-\frac{Bp}{2})\int \frac{dt}{t^2+a^2} =$$

$$= \frac{B}{2}\ln(t^2+a^2) + (C-\frac{Bp}{2})\frac{1}{a}\text{arctg}\frac{t}{a} + C^* =$$

$$= \frac{B}{2}\ln(x^2+px+q) + \frac{2C-Bp}{2\sqrt{q-\frac{p^2}{4}}}\text{arctg}\frac{x+\frac{p}{2}}{\sqrt{q-\frac{p^2}{4}}} + C^*.$$

Aytaylik, $m \in N,\ m > 1$ bo'lsin.Bu holda soda kasrlarning integrallari

$$\int \frac{A}{(x-a)^m}dx, \quad \int \frac{Bx+C}{(x^2+px+q)^m}dx$$

lar quydagicha hisoblanadi:

$$\int \frac{A}{(x-a)^m}dx = A\int (x-a)^{-m}d(x-a) = -\frac{A}{(m-1)(x-a)^{m-1}} + C,$$

$$\int \frac{Bx+C}{(x^2+px+q)^m}dx = \left| \begin{matrix} x+\frac{p}{2} = t, & x = t-\frac{p}{2} \\ dx = dt, & q-\frac{p^2}{4} = a^2 \end{matrix} \right| =$$

$$= \frac{B}{2}\int \frac{2tdt}{(t^2+a^2)^m} + (C-\frac{p}{2}B)\int \frac{dt}{(t^2+a^2)^m} =$$

$$= -\frac{B}{2}\frac{1}{(m-1)(t^2+a^2)^{m-1}} + (C-\frac{p}{2}B)\int \frac{dt}{(t^2+a^2)^m}.$$

Keyingi munosabatdagi

$$\int \frac{dt}{(t^2+a^2)^m}.$$

integral (6) rekurrent formula yordamida topiladi.

Berilgan ratsional ko`rinishdagi funksiyalarning aniqmas integralini hisoblang.

1. $\int \frac{2x}{(x-2)(x+5)}dx$
2. $\int \frac{x\,dx}{(x+1)(x+2)(x+3)}$
3. $\int \frac{x-4}{(x-2)(x-3)}dx$
4. $\int \frac{2x+3}{(x-2)(x+5)}dx$
5. $\int \frac{3x^2+2x+3}{x(x-1)(x+1)}dx$
6. $\int \frac{x^3+1}{x^3-5x^2+6x}dx$
7. $\int \frac{x^4+3x^3+2x^2+x+1}{x^2+x+1}dx$
8. $\int \frac{5x+2}{x^2+2x+10}dx$
9. $\int \frac{dx}{x^3+1}$
10. $\int \frac{xdx}{x^3+1}$
11. $\int \frac{dx}{x^4-1}$
12. $\int \frac{3x+1}{x(1+x^2)^2}dx$
13. $\int \frac{3x+5}{(x^2+2x+2)^2}dx$
14. $\int \frac{x^{10}}{x^2+x-2}dx$
15. $\int \frac{x^3+1}{x^3-5x^2+6x}dx$
16. $\int \frac{x^4}{x^4+5x^2+4}dx$
17. $\int \frac{xdx}{x^3-3x+2}$
18. $\int \frac{x^2+1}{(x+1)^2(x-1)}dx$
19. $\int \left(\frac{x}{x^2-3x+2}\right)^2 dx$
20. $\int \frac{dx}{(x+1)(x+2)^2(x+3)^3}$
21. $\int \frac{dx}{x^5+x^4-2x^3-2x^2+x+1}$
22. $\int \frac{dx}{(x+1)(x^2+1)}$
23. $\int \frac{dx}{(x^2-4x+4)(x^2-4x+5)}$
24. $\int \frac{xdx}{(x-1)^2(x^2+2x+2)}$
25. $\int \frac{dx}{x^6+1}$
26. $\int \frac{dx}{(1+x)(1+x^2)(1+x^3)}$
27. $\int \frac{dx}{x^4+x^2+1}$
28. $\int \frac{dx}{x(1+x)(1+x+x^2)}$
29. $\int \frac{dx}{x^5-x^4+x^3-x^2+x-1}$
30. $\int \frac{x^2dx}{x^4+3x^3+\frac{9}{2}x^2+3x+1}$

2. Integrallashning Ostrogradskiy usuli.

$\frac{P(x)}{Q(x)}$ to'g'ri kasrning maxraji karrali kompleks ildizlarga ega bo'lganda, uni integrallashda, murakkab hisoblashlar bajarishga to'g'ri keladi.

Bunday hollarda, ushbu

$$\int \frac{P(x)}{Q(x)}dx = \left(\frac{P_1(x)}{Q_1(x)}\right)' + \int \frac{P_2(x)}{Q_2(x)}dx \qquad (1)$$

Ostrogradskiy formulasidan foydalanish qulay bo'ladi,bunda $Q_2(x)$-ildizlari $Q(x)$ ko'phadning hamma sodda (bir karrali) ildizlaridan iborat bo'lgan ko'phad, $Q(x)=Q_1(x)\cdot Q_2(x), P_1(x)$ va $P_2(x)$ lar no'malum koeffisentli ko'phadlar bo'lib $\frac{P_1(x)}{Q_1(x)}$ va $\frac{P_2(x)}{Q_2(x)}$ to'g'ri kasrlardan iborat.

$P_1(x)$ va $P_2(x)$ ko'phadlarni topish uchun ularni no'malum koeffisentlar yordamida yozib olib,so'ngra,(1) ning ikkala tomonini differesiallaymiz,natijada,(1) tenglikka teng kuchli,

$$\frac{P(x)}{Q(x)} = \left\{\frac{P_1(x)}{Q_1(x)}\right\}' + \frac{P_2(x)}{Q_2(x)}$$

tenglikka ega bo'lamiz.Bu tenglikdan no'malum koeffisentlar usulidan foydalanib, $P_1(x)$ va $P_2(x)$ larning tarkibidagi no'malum koeffisentlarni topamiz.Ostrogradskiy formulasi,integralning ratsional qismini (integrallamasdan) ajratishga imkon beradi. $\frac{P(x)}{Q(x)}$ to'g'ri kasrni integrallash masalasi,unga nisbatan integrallanadigan $\frac{P_2(x)}{Q_2(x)}$ to'g'ri kasrni integrallashga keltiriladi.

Misol: $\int \frac{x-1}{(x^2+x+1)^2}dx$ **integralni hisoblang.**

Bu holda, $P(x)=x-1$, $Q(x)=(x^2+x+1)^2$, $Q_1(x)=x^2+x+1$, $Q_2(x)=x^2+x+1$.(1) formulaga asosan,

$$\int \frac{x-1}{(x^2+x+1)^2}dx = \frac{Ax+B}{x^2+x+1} + \int \frac{Cx+D}{x^2+x+1}dx$$

deb yozib olamiz. A, B, C, D noma'lum

Ostrogradskiy usuli yordamida hisoblang:

1. $\int \frac{xdx}{(x-1)^2(x+1)^3}$
2. $\int \frac{dx}{(x^3+1)^2}$
3. $\int \frac{dx}{(x^2+1)^3}$
4. $\int \frac{x^2dx}{(x^2+2x+2)^2}$
5. $\int \frac{dx}{(x^4+1)^2}$
6. $\int \frac{x^3+3x-2}{(x-1)(x^2+x+1)^2}dx$
7. $\int \frac{dx}{(x^4-1)^3}$
8. $\int \frac{2x^3+x^2+5x+1}{(x^2+3)(x^2-x+1)}dx$
9. $\int \frac{5x^2+7x+4}{(x+1)^2(2x^2+3x+2)}dx$
10. $\int \frac{dx}{x^4+1}$
11. $\int \frac{x^3dx}{x^8+3}$
12. $\int \frac{x^3+x}{x^6+1}dx$
13. $\int \frac{x^3-x}{x^8+1}dx$
14. $\int \frac{x^4+1}{x^6+1}dx$
15. $\int \frac{dx}{x(x^{10}+1)^2}$
16. $\int \frac{x^2+1}{x^4+x^2+1}dx$
17. $\int \frac{x^9dx}{(x^{10}+2x^5+2)^2}$
18. $\int \frac{x^{3n-1}}{(x^{2n}+1)}dx$
19. $\int \frac{x^{2n-1}}{x^n+1}dx$
20. $\int \frac{dx}{(x+1)^2(x^2+1)}$
21. $\int \frac{(3x^2-2)xdx}{(x+2)^2(3x^2-2x+4)}$
22. $\int \frac{dx}{x^{2n+1}}$
23. $\int \frac{dx}{x^6+1}$
24. $\int \frac{x^3+x^2+x+3}{(x+3)(x^2+x+1)}dx$
25. $\int \frac{3x^3-x^2+11x-5}{(x+1)^2(x^2-4x+5)}dx$
26. $\int \frac{4x^3-x^2-2x+1}{(2x+1)^2(x^2+x+2)}dx$
27. $\int \frac{dx}{(x^2+4x+5)(x^2-4x+3)}$
28. $\int \frac{5x^2+7x+4}{(x+1)^2(2x^2+3x+2)}dx$
29. $\int \frac{2x^3+x^2+5x+1}{(x^2+3)(x^2-x+1)}dx$
30. $\int \frac{3x^3-8x^2+8x-1}{(x-3)(3x^2+x+2)}dx$

7.4. *Ba'zi irratsional funksiyalarni integrallash.*

1. $\int R\left(x,\sqrt[n]{\frac{ax+b}{cx+d}}\right)dx, \int R(x,(ax+b)^n,(ax+b)^m...)dx, \int R\left(x,\left(\frac{ax+b}{cx+d}\right)^n,\left(\frac{ax+b}{cx+d}\right)^m...\right)dx$ **ko'rinishdagi integrallarni hisoblash.**

Faraz qilaylik, $R(u,v)$ ikki o'zgaruvchining ratsional funksiyasi bo'lib, a,b,c,d lar haqiqiy sonlar, $n\in N$ bo'lsin.

Ushbu

$$\int R\left(x,\sqrt[n]{\frac{ax+b}{cx+d}}\right)dx \ , \quad ad-bc\neq 0,$$

ko'rinishdagi integralniqaraymiz.Bu integral o'zgaruvchini almashtirish yordamida ratsional funksiyaning integragraliga keladi:

$$\int R\left(x,\sqrt[n]{\frac{ax+b}{cx+d}}\right)dx = \left| \begin{array}{l} \sqrt[n]{\dfrac{ax+b}{cx+d}} = t, x = \dfrac{b-t^n d}{ct^n - a} \\ dx = \dfrac{(ad-bc)n}{(a-ct^n)^2}t^{n-1}dt \end{array} \right| =$$

$$= \int R\left(\frac{dt^n - b}{a-ct^n}, t\right)\cdot\frac{(ad-bc)nt^{n-1}}{(a-ct^n)^2}dt$$

Misol: $\int\sqrt{\frac{1+x}{1-x}}\cdot\frac{1}{1-x}dx$ integralni hisoblang.

Bu integralda

$$t=\sqrt{\frac{1+x}{1-x}}$$

almashtirishni bajaramiz.Unda

$$x=\frac{t^2-1}{t^2+1} \quad , \quad dx=\frac{4tdt}{(t^2+1)^2}$$

bo`lib,

$$\int\sqrt{\frac{1+x}{1-x}}\cdot\frac{1}{1-x}dx = 2\int\frac{t^2dt}{t^2+1}$$

bo`ladi.

Ravshanki,

$$\int\frac{t^2dt}{t^2+1}=t-\text{arctg}t+C.$$

Demak,

$$\int\sqrt{\frac{1+x}{1-x}}\cdot\frac{1}{1+x}dx = 2\sqrt{\frac{1+x}{1-x}}-2arctg\sqrt{\frac{1+x}{1-x}}+C$$

Hisoblang:

1. $\int\frac{dx}{1+\sqrt{x}}$
2. $\int\frac{dx}{x(1+2\sqrt{x}+\sqrt[3]{x})}$
3. $\int\frac{x\sqrt[3]{2+x}}{x+\sqrt[3]{2+x}}dx$
4. $\int\frac{1-\sqrt{x+1}}{1+\sqrt[3]{x+1}}dx$
5. $\int\frac{dx}{(1+\sqrt[4]{x})^3\sqrt{x}}dx$
16. $\int\frac{1-x+x^2}{\sqrt{1+x-x^2}}dx$
17. $\int\frac{\sqrt[6]{x}}{1+\sqrt[3]{x}}dx$
18. $\int\frac{dx}{x+2\sqrt{x^3}+\sqrt[3]{x^4}}$
19. $\int\frac{dx}{\sqrt[3]{4x^2+4x+1}-\sqrt{2x+1}}$
20. $\int\frac{x^3-6x^2+11x-6}{\sqrt{x^2+4x+3}}dx$

6. $\int\frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}dx$
7. $\int\frac{dx}{\sqrt[3]{(x+1)^2(x-1)^4}}$
8. $\int\frac{xdx}{\sqrt{x^3(a-x)}} \quad a>0$
9. $\int\frac{dx}{\sqrt[n]{(x-a)^{n+1}(x-b)^{n-1}}}$
10. $\int\frac{dx}{1+\sqrt{x}+\sqrt{1+x}}$
11. $\int\frac{x^2}{\sqrt{1+x+x^2}}dx$
12. $\int\frac{dx}{(x+1)\sqrt{x^2+x+1}}$
13. $\int\frac{dx}{(1-x)^2\sqrt{1-x^2}}$
14. $\int\frac{\sqrt{x^2+2x+2}}{x}dx$
15. $\int\frac{xdx}{(1+x)\sqrt{1-x-x^2}}$
21. $\int\frac{dx}{(\sqrt[6]{x}+\sqrt[12]{x^5})^3}$
22. $\int\frac{dx}{2\sqrt{x}-\sqrt[3]{x}-\sqrt[4]{x}}$
23. $\int\frac{x^3+2x^2+x-1}{\sqrt{x^2+2x-1}}dx$
24. $\frac{\sqrt{x^2+3x+2}-x}{\sqrt{x^2+3x+2}+x}dx$
25. $\int\left(\frac{\sqrt{1+x+x^2}-1}{x}\right)^2\frac{dx}{\sqrt{1+x+x^2}}$
26. $\int\frac{dx}{(1+\sqrt{x+x^2})^2}$
27. $\frac{dx}{1+\sqrt{1-2x-x^2}}$
28. $\int\frac{dx}{1+\sqrt{x^2+2x+2}}$
29. $\int\frac{dx}{x-\sqrt{x^2-x+1}}$
30. $\int\frac{x^2dx}{(4-2x+x^2)\sqrt{2+2x-x^2}}$

2. $\int R\left(x,\sqrt{ax^2+bx+c}\right)dx$ **ko'rinishdagi integrallarni hisoblash.**

Bu integralda a,b,c-haqiqiy sonlar bo'lib, ax^2+bx+c kvadarat uchhad teng ildizlarga ega emas.

Qaralayotgan

$$\int R(x,\sqrt{ax^2+bx+c})dx \qquad (1)$$

integral quydagi uchta almashtirish yordamida ratsional funksiya integraliga keladi.

a) $a>0$ **bo'lsin.**

(1) integralda ushbu

$$t=\sqrt{a}x+\sqrt{ax^2+bx+c} \qquad (\text{yoki } t=-\sqrt{a}x+\sqrt{ax^2+bx+c})$$

almashtirish bajaramiz.U holda

$$ax^2+bx+c=t^2-2\sqrt{a}xt+ax^2,$$

$$x=\frac{t^2-c}{2\sqrt{a}t+b} \quad , \quad dx=\frac{2(\sqrt{a}t^2+bt+c\sqrt{a})}{(2\sqrt{a}t+b)^2}dt,$$

$$\sqrt{ax^2+bx+c}=\frac{\sqrt{a}t^2+bt+c\sqrt{a}}{2\sqrt{a}t+b}$$

bo'ladi.

Natijada: $\int R(x,\sqrt{ax^2+bx+c})dx=$

$$=\int R\left(\frac{t^2-c}{2\sqrt{a}t+b},\ \frac{\sqrt{a}\,t^2+bt+c\sqrt{a}}{2\sqrt{a}\,t+b}\right)\cdot\frac{2(\sqrt{a}\,t^2+bt+c\sqrt{a})}{(2\sqrt{a}\,t+b)^2}dt$$

bo'ladi.

b) $c>0$ **bo'lsin.** Bu holda (1) integralda ushbu

$$t=\frac{1}{x}(\sqrt{ax^2+bx+c}-\sqrt{c})\ \text{yoki}\ \ t=\frac{1}{x}\left(\sqrt{ax^2+bx+c}+\sqrt{c}\right)$$

almashtirish bajaramiz.Unda

$$x=\frac{2\sqrt{c}\,t-b}{a-t^2}\ ,\quad dx=\frac{\sqrt{c}t^2-bt+\sqrt{c}\,a}{(a+t)^2}dt\ ,$$

$$\sqrt{ax^2+bx+c}=\frac{\sqrt{c}t^2-bt+\sqrt{c}a}{(a+t)^2}$$

bo'lib,(1) integral ratsional funksiyaning integraliga keladi:

$\int R(x,\sqrt{ax^2+bx+c})dx=$

$$=\int R\left(\frac{2\sqrt{c}\,t-b}{a-t^2},\ \frac{\sqrt{c}\,t^2-bt+a\sqrt{c}}{a-t^2}\right)\left(\frac{\sqrt{c}t^2-bt+\sqrt{c}\,a}{(a+t)^2}\right)dt$$

c) ax^2+bx+c **kvadrat uchhad turli** x_1 **va** x_2 **haqiqiy ildizlarga ega bo'lsin:**

$$ax^2+bx+c=a(x-x_1)\cdot(x-x_2).$$

Bu holda (1) integralda ushbu

$$t=\frac{1}{x-x_1}\sqrt{ax^2+bx+c}$$

almashtirishni bajaramiz. Natijada

$$x=\frac{-ax_2+x_1t^2}{t^2-a}\ ,\quad \sqrt{ax^2+bx+c}=\frac{a(x_1-x_2)}{t^2-a}t$$

$$dx=\frac{2a(x_1-x_2)t}{(t^2-a)^2}dt$$

bo'lib,

$$\int R(x,\sqrt{ax^2+bx+c})dx=\int R\left(\frac{-ax_2+x_1t^2}{t^2-a}\ ,\ \frac{a(x_1-x_2)}{t^2-a}t\right)\cdot\frac{2a(x_1-x_2)t}{(t^2-a)^2}dt$$

bo'ladi.

1-misol: Ushbu $\int\frac{dx}{x+\sqrt{x^2+x+1}}$ integral hisoblansin.

Bu integralda

$$t=x+\sqrt{x^2+x+1}$$

almashtirishtirish bajaramiz.Natijada

$$x=\frac{t^2-1}{1+2t}\ ,\quad dx=2\frac{t^2+t+1}{(1+2t)^2}dt$$

bo'lib,

$$\int\frac{dx}{x+\sqrt{x^2+x+1}}=2\int\frac{t^2+t+1}{(1+2t)^2t}dt$$

bo'ladi.

Agar

$$\frac{2(t^2+t+1)}{t(1+2t)^2}=\frac{2}{t}-\frac{3}{1+2t}-\frac{3}{(1+2t)^2}$$

bo'lishini e'tiborga olsak,unda

$$\int\frac{dx}{x+\sqrt{x^2+x+1}}=\int\left(\frac{2}{t}-\frac{3}{1+2t}-\frac{3}{(1+2t)^2}\right)dt=$$

$$=2\ln|t|-\frac{3}{2}\ln|1+2t|+\frac{3}{2(1+2t)}+C=$$

$$=2\ln\left|x+\sqrt{x^2+x+1}\right|-\frac{3}{2}\ln\left|1+2x+2\sqrt{x^2+x+1}\right|+$$

$$+\frac{3}{2(1+2x+2\sqrt{x^2+x+1})}+C$$

bo'lishi kelib chiqadi.

Hisoblang:

1. $\int\frac{xdx}{(x-1)^2\sqrt{1+2x-x^2}}$
2. $\int\frac{xdx}{(x^2-1)\sqrt{x^2-x-1}}$
3. $\int\frac{\sqrt{x^2+x+1}}{(x+1)^2}dx$

16. $\int\frac{x^3}{\sqrt{1+2x-x^2}}dx$
17. $\int\frac{dx}{x^4\sqrt{x^2-1}}$
18. $\int\frac{x^{10}dx}{\sqrt{1+x^2}}$

4. $\int \frac{x^3}{(x+1)\sqrt{1+2x-x^2}}dx$

5. $\int \frac{x}{(x^2-3x+2)\sqrt{x^2-4x+3}}dx$

6. $\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

7. $\int \frac{dx}{(x^2+1)\sqrt{x^2-1}}$

8. $\int \frac{dx}{(1-x^4)\sqrt{1+x^2}}$

9. $\int \frac{\sqrt{x^2+2}}{x^2+1}dx$

10. $\int \frac{dx}{(x^2+x+1)\sqrt{x^2+x-1}}$

11. $\int \frac{3x-6}{\sqrt{x^2-4x+5}}dx$

12. $\int \frac{x+3}{\sqrt{4x^2+4x+3}}dx$

13. $\int \frac{x+3}{\sqrt{3+4x-4x^2}}dx$

14. $\int \frac{x^3+x}{\sqrt{1+x^2-x^4}}dx$

15. $\int \frac{x^3}{\sqrt{1+2x-x^2}}dx$

19. $\int \frac{x^3-6x^2+11x-6}{\sqrt{x^2+4x+3}}dx$

20. $\int \frac{dx}{x^3\sqrt{x^2+1}}$

21. $\int \frac{dx}{(x-1)^3\sqrt{x^2+3x+1}}$

22. $\int \frac{dx}{(x+1)^5\sqrt{x^2+2x}}$

23. $\int \frac{dx}{x^4\sqrt{x^2-1}}$

24. $\int \frac{xdx}{(2x^2+1)\sqrt{3x^2+5}}$

25. $\int \frac{dx}{(x^2+1)\sqrt{3x^2+1}}$

26. $\int \frac{dx}{x\sqrt{5x^2-2x+1}}$

27. $\int \frac{dx}{(x-1)^3\sqrt{x^2-2x-1}}$

28. $\int \frac{dx}{(1+x)\sqrt{1+x+x^2}}$

29. $\int \frac{x^3-6x^2+11x-6}{\sqrt{x^2+4x+3}}dx$

30. $\int \frac{x^3+2x^2+x-1}{\sqrt{x^2+2x-1}}dx$

3. $\int R(x,\sqrt{a^2-x^2})dx, \int R(x,\sqrt{a^2+x^2})dx, \int R(x,\sqrt{x^2-a^2})dx$ ko'rinishdagi ifodalarni integrallash.

Quydagi

$$\int R(x,\sqrt{a^2-x^2})dx, \int R(x,\sqrt{a^2+x^2})dx, \int R(x,\sqrt{x^2-a^2})dx$$

ko'rinishdagi integrallar,mos ravishda $x = a\sin t, x = a tg t, x = a\sec t, a \in R, a \neq 0$, almashtirishlar natijasida ratsionallashtirib hisoblanadi.

Misol: $\int x^2\sqrt{4-x^2}dx$ integralni hisoblang.

Berilgan integral $\int R(x,\sqrt{a^2-x^2})dx, \int R(x,\sqrt{a^2+x^2})dx, \int R(x,\sqrt{x^2-a^2})dx$ ko'rinishdagi ifodalarning birinchi integrali ko`rinishida bo`lgani uchun, $x = 2\sin t$ almashtirish bajarib,hisoblaymiz:

$$\int x^2\sqrt{4-x^2}dx = \left| x = 2\sin t, dx 2\cos t dt, \sqrt{4-4\sin^2 t} = 2\cos t \right| = 16\int \sin^2 t\cos^2 t dt = 4\int \sin^2 2t dt =$$

$$= 2\int(1-\cos 4t)dt = 2t - \frac{1}{2}\sin 4t + C$$

$$\sin 4t = 2\sqrt{1-\sin^2 t}\sin t(1-2\sin^2 t), -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}, \sin(\arcsin\frac{x}{2}) = \frac{x}{2}, -2 \leq x \leq 2$$

formulalarni e`tiborga olganda holda eski o`zgaruvchiga qaytib,integralni hisobaymiz:

$$\int x^2\sqrt{4-x^2}dx = 2t - \frac{1}{2}\sin 4t + C = 2\arcsin\frac{x}{2} + \frac{x}{4}(x^2-2)\sqrt{4-x^2} + C.$$

Aniqmas integralni hisoblang:

1. $\int \sqrt{9-x^2}dx$

2. $\int \sqrt{x^2-49}dx$

3. $\int x\sqrt{x^2-1}dx$

4. $\int \sqrt{x^2+16}dx$

5. $\int \sqrt{4-x^2}dx$

6. $\int \sqrt{a^2+x^2}dx$

7. $\int \frac{x^2}{\sqrt{a^2+x^2}}dx$

8. $\int \frac{dx}{\sqrt{(x-a)(b-x)}}$

9. $\int \sqrt{a^2-x^2}dx$

10. $\int \frac{dx}{\sqrt{(16-x^2)^3}}$

11. $\int \frac{x^2dx}{(1+x^2)^3}$

16. $\int x\sqrt{\frac{x}{2x-a}}dx$

17. $\int \sqrt{\frac{a+x}{a-x}}dx$

18. $\int \frac{dx}{(x^2+a^2)^{\frac{3}{2}}}$

19. $\int \frac{dx}{(1-x^2)^{\frac{3}{2}}}$

20. $\int \frac{dx}{\sqrt{a+bx^2}}$

21. $\int \frac{ax}{\sqrt{x+x^2}}dx$

22. $\int \frac{dx}{\sqrt{a+bx^2}}$

23. $\int \frac{x^2}{\sqrt{x^2-2}}dx$

24. $\int \sqrt{x^2+a}dx$

25. $\int \frac{xdx}{\sqrt{a^2+x^2}}$

26. $\int \frac{\sqrt{64+x^2}}{x^2}$

12. $\int \frac{dx}{(x^2+3)^{\frac{5}{2}}}$

13. $\int \frac{dx}{\sqrt{25+4x^2}}$

14. $\int \frac{dx}{\sqrt{81-25x^2}}$

15. $\int \frac{dx}{x^4\sqrt{121-25x^2}}$

27. $\int x\sqrt{(9+4x^2)^3}dx$

28. $\int \frac{dx}{\sqrt{(16-25x^2)^3}}$

29. $\int \frac{dx}{x\sqrt{a^2-b^2x}}$

30. $\int \frac{\sqrt{a^2+b^2x}}{x}dx$

4.Binomial differensialni integrallash.

Ushbu

$$x^m(a+bx^n)^p\,dx \qquad (1)$$

ifoda binomial differensial deyiladi,bunda $a\in R$, $b\in R$, m,n,p-ratsional sonlar.

Binomial differensialning integrali

$$\int x^m(a+bx^n)^p\,dx \qquad (2)$$

ni qaraymiz.Bu integral quydagi hollarda ratsional funksiyaning integraliga keladi:

1) p-butun son. Bu holda m va n ratsional sonlar maxrajining eng kichik umumiy karralisini δ orqali belgilab,(2) integralda

$$x=t^{\delta}$$

almashtirish bajarilsa, (2) integral ratsional funksiyaning integraliga keladi.

2) $\frac{m+1}{n}$-butun son. Bu holda (2) integralda

$$x=t^{\frac{1}{n}}$$

almashtirishni bajarib

$$\int x^m(a+bx^n)^p\,dx=\frac{1}{n}\int(a+bt)^p\cdot t^q\,dt$$

bo'lishini topamiz,bunda

$$q=\frac{m+1}{n}-1.$$

So'ng p ning maxrajini s deb

$$z=(a+bt)^{\frac{1}{s}}$$

almashtirish bajaramiz.Natijada (2) integral ratsional funksiyaning integraliga keladi.

3) $p+q$-butun son. Ma'lumki,(2) integral $x=t^{\frac{1}{n}}$ almashtirish bilan ushbu

$$\frac{1}{n}\int(a+bt)^p\cdot t^{\frac{m+1}{n}-1}dt=\frac{1}{n}\int(a+bt)^p\cdot t^q dt=\frac{1}{n}\int\left(\frac{a+bt}{t}\right)^p\cdot t^{p+q}dt$$

ko'rinishga keladi.

Agar keyingi integralda

$$z=\left(\frac{a+bt}{t}\right)^{\frac{1}{s}}$$

almashtirish bajarilsa (s soni p ning maxraji), u ratsional funksiyaning integraliga keladi.

Misol: $\int\frac{xdx}{\sqrt{1+\sqrt[3]{x^2}}}$ integral hisoblansin.

Bu integralda

$$\int\frac{xdx}{\sqrt{1+\sqrt[3]{x^2}}}=\int x(1+x^{\frac{2}{3}})^{-\frac{1}{2}}dx$$

$$m=1,\quad n=\frac{2}{3},\quad p=-\frac{1}{2}$$

bo'lib,

$$\frac{m+1}{n}=3$$

bo'ladi.

Shuni e`tiborga olib, berilgan integralda,

$$t=(1+x^{\frac{2}{3}})^{\frac{1}{2}}$$

almashtirishni bajaramiz. Unda

$$1+x^{\frac{2}{3}}=t^2\ ,\ x=(t^2-1)^{\frac{3}{2}}\ ,\ dx=\frac{3}{2}(t^2-1)^{\frac{1}{2}}\cdot 2tdt$$

bo'lib,

$$\int x(1+x^{\frac{2}{3}})^{\frac{1}{2}}dx=3\int(t^2-1)^2t^2dt=3\frac{t^7}{7}-6\frac{t^5}{5}+t^3+C,\quad t=\sqrt{1+x^{\frac{2}{3}}}$$

bo'ladi.

Binomial differensialni integrallash usulidan foydalanib hisoblang.

1. $\int\frac{\sqrt{1+\sqrt{x}}}{x\sqrt[4]{x^3}}dx$

2. $\int\frac{\sqrt[3]{(1+\sqrt[3]{x^2})^2}}{x\sqrt[9]{x^5}}dx$

3. $\int\frac{\sqrt[3]{(1+\sqrt{x})^2}}{x\sqrt[6]{x^5}}dx$

4. $\int\frac{\sqrt[4]{(1+\sqrt{x})^3}}{x\sqrt[8]{x^7}}dx$

5. $\int\frac{\sqrt{1+\sqrt[4]{x^3}}}{x^2}dx$

6. $\int\frac{\sqrt[3]{(1+\sqrt[3]{x^2})^2}}{x\sqrt[9]{x^5}}dx$

7. $\int\sqrt[3]{x}(1+\sqrt{x})^2dx$

8. $\int\frac{dx}{x^3\sqrt[5]{1+\frac{1}{x}}}dx$

9. $\int x^4\sqrt[3]{7-3x^2}\,dx$

10. $\int\frac{dx}{\sqrt[4]{1+x^4}}$

11. $x^{-\frac{1}{3}}(1-x^{\frac{1}{6}})^{-1}dx$

12. $\int x^{\frac{1}{2}}(1+x^{\frac{1}{3}})^{-2}dx$

13. $\int x^{-\frac{2}{3}}(1+x^{\frac{1}{3}})^{-3}dx$

14. $\int x^{-\frac{1}{2}}(1+x^{\frac{1}{4}})^{-10}dx$

15. $\int\frac{dx}{x^2\sqrt[3]{(2+x^3)^5}}$

16. $\int x^{-11}(1+x^4)^{-\frac{1}{2}}dx$

17. $\int\frac{dx}{\sqrt[3]{x^2}(\sqrt[3]{x}+1)^3}$

18. $\int\frac{\sqrt{1+\sqrt[3]{x}}}{\sqrt[3]{x^2}}dx$

19. $\int\frac{dx}{\sqrt{x^3}\sqrt[3]{2+\sqrt[4]{x^3}}}$

20. $\int\frac{dx}{x\sqrt{1+x^3}}$

21. $\int\frac{dx}{\sqrt[4]{1+x^4}}$

22. $\int x^3\sqrt[3]{7-3x^2}\,dx$

23. $\int\frac{dx}{\sqrt[4]{x^3}(\sqrt[8]{x}-1)^5}$

24. $\int\frac{\sqrt[4]{(1+\sqrt[5]{x^4})^3}}{x^2\sqrt[5]{x^2}}dx$

25. $\int\frac{\sqrt[3]{(1+\sqrt[4]{x^3})^2}}{x^2\sqrt[4]{x}}dx$

26. $\int\frac{dx}{x\sqrt[6]{x^6+1}}$

27. $\int\frac{dx}{x^3\sqrt[3]{2-x^3}}$

28. $\int\frac{xdx}{\sqrt{1+\sqrt[3]{x^2}}}$

29. $\int\frac{\sqrt[3]{x}dx}{\sqrt{1+\sqrt[3]{x}}}$

30. $\int\frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}dx$

7.5. *Tarkibida trigonometrik funksiyalar qatmnashgan ifodalarni integrallash.*

5.1. $\int R(\sin x,\cos x)dx$, $\int\sin\alpha x\cos\beta x dx$,.... **ko'rinishdagiintegrallarni hisoblash.**

Aytaylik, $R(u,v)$ ikki o'zgaruvchining ratsional funksiyasi bo'lsin.Ushbu

$$\int R(\sin x,\cos x)\,dx \qquad (1)$$

integralini qaraymiz.Bu integralda

$$t=tg\frac{x}{2}$$

almashtirish bajaramiz.

Unda

$$\sin x=\frac{2tg\frac{x}{2}}{1+tg^2\frac{x}{2}}=\frac{2t}{1+t^2},\quad \cos x=\frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}=\frac{1-t^2}{1+t^2},$$

$$x=2arctgt,\quad dx=\frac{2dt}{1+t^2}$$

bo'lib,

$$\int R(\sin x,\cos x)dx=2\int R\left(\frac{2t}{1+t^2},\frac{1-t^2}{1+t^2}\right)\frac{1}{1+t^2}dt$$

bo'ladi.

Ravshanki,

$$R\left(\frac{2t}{1+t^2},\frac{1-t^2}{1+t^2}\right)\frac{1}{1+t^2}$$

ifoda t ning ratsional funksiyasidir.

Demak,(4) integral hisoblash

$$t=tg\frac{x}{2}$$

almashtirish bilan ratsional funksiyani integrallashga keladi.

$$\int\sin mx\cos nxdx,\quad \int\sin mx\sin nxdx,\quad \int\cos mx\cos nxdx \quad (m\neq n).$$

Bu integrallar quydagi trigonometrik formulalar orqali yoyish usulida ikkita oson hisoblanadigan integrallarga keltiriladi:

$$\sin mx\cos nx=\frac{1}{2}\left[\sin(m+n)x+\sin(m-n)x\right],$$

$$\sin mx\sin nx=\frac{1}{2}\left[\cos(m-n)x-\cos(m+n)x\right],$$

$$\cos mx\cos nx=\frac{1}{2}\left[\cos(m+n)x+\cos(m-n)x\right]$$

Misol: $\int\sin 5x\cdot\cos 3x dx$ ni hisoblang.

Yechish. $\int\sin 5x\cdot\cos 3x dx=\frac{1}{2}\int(\sin(5x-3x)+\sin(5x+3x))dx=$

$=\frac{1}{2}\int\sin 2x+\frac{1}{2}\int\sin 8x dx=-\frac{1}{4}\cos 2x-\frac{1}{16}\cos 8x+C.$

1.Hisoblang.

1. $\int\cos\frac{x}{2}\cos\frac{x}{3}dx$
2. $\int\frac{\cos x}{\sqrt{2+\cos 2x}}dx$
3. $\int\frac{dx}{\sin x}$
4. $\int(sh(2x+1)+ch(2x-1))dx$
5. $\int\sin\frac{1}{x}\cdot\frac{dx}{x^3}$
6. $\int\frac{dx}{3+5tgx}$
7. $\int\frac{\sin x+2\cos x-3}{\sin x-2\cos x+3}dx$
8. $\int\frac{\sin x}{\sqrt{2}+\sin x+\cos x}dx$
9. $\int\frac{2\sin x+\cos x}{3\sin x+4\cos x-2}dx$
10. $\int\frac{\sin x-2\cos x}{1+4\sin x\cos x}dx$

2. $\int\sin ax\cos bx\,dx$ **ko'rinishdagi integral berilgan,a va b parametrlarni o'rniga qo'yib aniqmas integralni hisoblang.**

1. $a=1, b=2$
2. $a=2, b=3$
3. $a=3, b=4$
4. $a=3, b=6$
5. $a=2, b=7$
6. $a=4, b=8$
7. $a=6, b=9$
8. $a=8, b=11$
9. $a=12, b=13$
10. $a=5, b=4$

3.Trigonometrik funksiya qatnashgan aniqmas integralni hisoblang.

1. $\int\sin x\sin 2x\sin 3x\,dx$
2. $\int\cos 5x\cos 9x\,dx$
3. $\int sh2x\,sh5x\,dx$
4. $\int\cos px\cos qx\,dx$
5. $\int ch7x\,ch3x\,dx$
6. $\int\cos x\cos 2x\cos 5x\,dx$
7. $\int sh(2x+3)sh(2x+5)dx$
8. $\int\sin x\sin\frac{x}{2}\sin\frac{x}{3}dx$
9. $\int\sin 7x\cos 3x\,dx$
10. $\int\sin px\cos qx\,dx$

4. $\int\frac{dx}{a\cos x+b\sin x}$ **ko'rinishdagi integralni hisoblang.**

1. $a=1, b=3$
2. $a=2, b=5$
3. $a=1, b=2$
4. $a=5, b=3$
5. $a=8, b=6$
6. $a=8, b=3$
7. $a=6, b=5$
8. $a=7, b=9$
9. $a=6, b=10$
10. $a=1, b=2$

5.2. $\int\sin^m x\cos^n x\,dx\quad(m,n\in Z)$ **ko'rinishdagi integrallar.**

$\int R(\sin^2 x,\cos^2 x)dx$ ko'rinishdagi,ya'ni integral ostidagi ifodada $\sin x$ va $\cos x$ funksiyalar faqat juft darajalarda qatnashgan integrallarni qaraymiz.Bu holda $tgx=t$ almashtirmadan foydalanish mumkin.Bunda,

$\cos^2 x=\frac{1}{1+tg^2x}=\frac{1}{1+t^2},\qquad \sin^2 x=\frac{tg^2x}{1+tg^2x}=\frac{t^2}{1+t^2},\qquad dx=\frac{dt}{1+t^2}$

bo'lgani uchun,qaralayotgan integral ostidagi ifoda ratsional kasrga quydagicha almashinadi:

$$\int R(\sin^2 x,\cos^2 x)dx=\int R\left(\frac{t^2}{1+t^2},\frac{1}{1+t^2}\right)\frac{dt}{1+t^2}=\int R_1(t)dt.$$

Shu bilan belgilashni yakunlaymiz.

Misol: $\int\sin^4 x\cdot\cos^2 x dx$ ni hisoblang.

Yechish. $\int\sin^4 x\cdot\cos^2 x dx=\int\sin^2 x(\sin x\cos x)^2 dx=$

$=\int\frac{1}{2}(1-\cos 2x)(\frac{1}{2}\sin 2x)^2 dx=\frac{1}{8}\int\sin^2 2x dx-\frac{1}{8}\int\sin^2 2x\cdot\cos 2x dx=$

$=\frac{1}{16}\int(1-\cos 4x)dx-\frac{1}{16}\int\sin^2 2x d(\sin 2x)=\frac{1}{16}x-\frac{1}{64}\sin 4x-\frac{1}{48}\sin^3 2x+C.$

Mustaqil ishlash uchun misollar:

1.Quydagi berilgan integrallarni hisoblang.

1. $\int\sin^3 x\,dx$
2. $\int\cos^3 x\,dx$
3. $\int\sin^3 3x\sin^3 2x\,dx$
4. $\int tg^3 x\,dx$
6. $\int\cos^5 x\sqrt{\sin x}\,dx$
7. $\int\frac{\sin^2 x}{\cos^6 x}dx$
8. $\int\frac{\sin^2 x}{\cos^6 x}dx$
9. $\int\frac{dx}{1+\cos x}$

5. $\int \cos^2\frac{x}{3}\cos^3\frac{x}{2}dx$

10. $\int \frac{dx}{\sin^2(2x+\frac{\pi}{4})}$

2. $\int \frac{\cos x\,dx}{(a+b\sin x)^n}$ **ko'rinishda berilgan integral berilgan.Bu yerda** $(n \geq 2)$ va $a, b \in const$

1. $a=1, b=1, n=2$
2. $a=4, b=2, n=5$
3. $a=5, b=6, n=3$
4. $a=8, b=3, n=9$
5. $a=7, b=5, n=7$
6. $a=10, b=12, n=14$
7. $a=8, b=16, n=9$
8. $a=11, b=3, n=6$
9. $a=2, b=4, n=3$
10. $a=8, b=5, n=3$

3.Hisoblang.

1. $\int \frac{\sin^2 x}{1+\sin^2 x}dx$
2. $\int \frac{\sin^2 x\cos^2 x}{\sin^8 x+\cos^8 x}dx$
3. $\int \frac{\sin^2 x}{\cos^2 x\sqrt{tgx}}dx$
4. $\int \frac{\sin^2 x-\cos^2 x}{\sin^4 x+\cos^4 x}dx$
5. $\int \frac{dx}{(\sin^2 x+2\cos^2 x)^2}$
6. $\int \frac{\sin x dx}{\sin^3 x+\cos^3 x}$
7. $\int \frac{dx}{\cos^6 x+\sin^6 x}$
8. $\int \frac{\cos^{n-1}\frac{x+a}{2}}{\sin^{n+1}\frac{x-a}{2}}dx$
9. $\int \frac{\cos^2 x dx}{(a^2\sin^2 x+b^2\cos^2 x)^2}$
10. $\int \frac{\sin x\cos x}{\sin x+\cos x}dx$

4. $\int \frac{dx}{a^2\cos^2 x-b^2\sin^2 x}$ **a va b larni o'rniga qo'yib integralni hisoblang.** $(a>0, b>0)$

1. $a=1, b=2$
2. $a=3, b=5$
3. $a=4, b=6$
4. $a=7, b=9$
5. $a=1, b=5$
6. $a=12, b=10$
7. $a=3, b=16$
8. $a=10, b=5$
9. $a=12, b=15$
10. $a=15, b=5$

VIII BOB
ANIQ INTEGRAL

8.1.Aniq integralning ta`rifi.

1.Riman integrali.

$f(x)$ funksiya $[a,b]$ kesmada aniqlangan bo`lsin. $[a,b]$ kesmaning $a=x_0<x_1<....<x_{n-1}<x_n=b$ shartni qanotlantiradiganchekli sondagi $\{x_k\}_{k=1}^n$ nuqtalar sistemasiga $[a,b]$ kesmaning bo`linishi deyiladi va u $P=\{x_k\}_{k=1}^n$ kabi belgilanadi. $x_k (k=\overline{1,n})$ nuqta P bo'linishning bo'luvchi nuqtasi $[x_k, x_{k+1}]$ kesma esa,qism oralig`i deyiladi.Agar $[a,b]$ kesmaning ixtiyoriy $P$ bo`linishdagi qism oralig`ining uzunligi bir xil bo`lsa,u holda,bunday bo`linish, $[a,b]$ kesmaning regulyar bo`linishi deyiladi. $d=d(P)=\max\limits_{0\le k\le n-1}\Delta x_k$ $(\Delta x_k = x_{k+1}-x_k)$, P bo`linishning diametri deb ataladi.Har bir $[x_k, x_{k+1}]$ kesmadan $(k=\overline{0,n-1})$ ξ_k nuqtani olamiz: $x_k \le \xi_k \le x_{k+1}$.

2.2.Darbu yig`indilari.

$f(x)$ funksiya $[a,b]$ da aniqlangan va chegaralangan bo`lsin. Unda $\exists m \in R$, $\exists M \in R$, $\forall x \in [a,b]$ : $m \le f(x) \le M$ bo`ladi.

Aytaylik,

$$P=\{x_0, x_1, x_2, ..., x_{n-1}, x_n\}$$

$[a,b]$ segmentning biror bo`laklash bo`lsin.U holda bu bo`laklashning har bir $[x_k, x_{k+1}]$ $(k=0,1,2,...,n-1)$ oralig`ida

$$m_k = \inf\{f(x)\},\quad x\in[x_k\, x_{k+1}],$$
$$M_k = \sup\{f(x)\},\quad x\in[x_k\, x_{k+1}] \qquad (k=0,1,2,...,n-1)$$

mavjud bo'lib,

$$\inf_{x\in[a,b]}\{f(x)\} \le m_k \le M_k \le \sup_{x\in[a,b]}\{f(x)\} \tag{2}$$

bo`ladi.

Ta'rif: Ushbu

$$s=\sum_{k=0}^{n-1} m_k \cdot \Delta x_k$$

yig`indi $f(x)$ funksiyaning $[a,b]$ segmentning $P$ bo`laklashga nisbatan Darbuning quyi yig`indilari deyiladi.

Ravshanki,bu yig`indi $f(x)$ funksiya hamda $[a,b]$ ning $P$ bo`laklashga bog`liq bo`ladi:

$$s = s(f;P) \ .$$

Ta'rif: Ushbu

$$S = \sum_{k=0}^{n-1} M_k \cdot \Delta x_k$$

yig`indi $f(x)$ funksiyaning $[a,b]$ segmentning $P$ bo`laklashiga nisbatan Darbuning yuqori yig`indisi deyiladi.

Bu yig`indi $f(x)$ funksiyaga hamda $[a,b]$ ning $P$ bo`laklashiga bog`liq bo`ladi:

$$S = S(f;P).$$

Endi har bir $k \in \{0,1,2,...,n-1\}$ ning qiymatida $[x_k, x_{k+1}]$ segmentda ixtiyoriy ξ_k nuqtani tanlaymiz: $\xi_k \in [x_k, x_{k+1}]$ $(k = 0,1,2,...,n-1)$. Natijada $[a,b]$ ning P bo`laklashiga nisbatan,

$$\{\xi_0, \xi_1, \xi_2, ..., \xi_{n-1}\}$$

nuqtalar to`plami hosil bo`ladi. Bu nuqtalardagi $f(x)$ funksiyaning

$$f(\xi_k) \qquad (k = 0,1,2,...,n-1)$$

qiymatlari yordamida ushbu

$$\sum_{k=0}^{n-1} f(\xi_k) \cdot \Delta x_k$$

yig`indini tuzamiz.

Ta'rif: Quydagi

$$\sigma = \sum_{k=0}^{n-1} f(\xi_k) \cdot \Delta x_k$$

yig`indi $f(x)$ funksiyaning $[a,b]$ segmentning $P$ bo`laklashiga nisbatan integral yig`indisi deyiladi. Integral yig`indi, $f(x)$ funksiyaga, P bo`laklashga hamda har bir $[x_k, x_{k+1}]$ da olingan $\xi_k$ nuqtaga bog`liq bo`ladi:

$$\sigma = \sigma(f;P;\xi_k)$$

Ravshanki, $\xi_k \in [x_k \ , \ x_{k+1}]$ uchun

$$m_k \le f(\xi_k) \le M_k$$

bo`lib,ayni paytda

$$s(f;P) \le \sigma(f;P;\xi_k) \le S(f;P) \qquad (3)$$

tengsizliklar bajariladi.

Misol: Ushbu

$$f(x) = |x|$$

funksiyaning [-1, 1] segmentda quydagi

$$P = \left\{-1, -\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$$

bo`laklashga nisbatan Darbu yig`indilari hamda

$$\xi_k = x_k \qquad (k = 0,1,2,...,7)$$

deb,integral yig`indi topilsin.

Berilgan $f(x) = |x|$ funksiya uchun [-1, 1] segmentning $P = \left\{-1, -\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$ bo`laklashida

$$m_0 = \frac{3}{4}, \quad m_1 = \frac{1}{2}, \quad m_2 = \frac{1}{4}, \quad m_3 = 0 \ ,$$

$$m_4 = 0 \ , \quad m_5 = \frac{1}{4}, \quad m_6 = \frac{1}{2}, \quad m_7 = \frac{3}{4}$$

$$M_0 = 1, \ M_1 = \frac{3}{4}, \ M_2 = \frac{1}{2}, \ M_3 = \frac{1}{4},$$

$$M_4 = \frac{1}{4}, \ M_5 = \frac{1}{2}, \ M_6 = \frac{3}{4}, \ M_7 = 1$$

hamda

$$\xi_0 = -1, \ \xi_1 = -\frac{3}{4}, \ \xi_2 = -\frac{1}{2}, \xi_3 = -\frac{1}{4},$$

$$\xi_4 = 0, \ \xi_5 = \frac{1}{4}, \ \xi_6 = \frac{1}{2}, \ \xi_7 = \frac{3}{4},$$

$$f(\xi_0) = 1, \ f(\xi_1) = \frac{3}{4}, \ f(\xi_2) = \frac{1}{2}, \ f(\xi_3) = \frac{1}{4},$$

$$f(\xi_4) = 0, \ f(\xi_5) = \frac{1}{4}, \ f(\xi_6) = \frac{1}{2}, f(\xi_7) = \frac{3}{4}$$

bo`ladi.

Endi $\Delta x_k = \frac{1}{4}$ $(k = 0,1,2,...,7)$ bo`lishini e'tiborga olib topamiz:

$$s(f;P) = (\frac{3}{4} + \frac{1}{2} + \frac{1}{4} + 0 + 0 + \frac{1}{4} + \frac{1}{2} + \frac{3}{4}) \cdot \frac{1}{4} = \frac{3}{4},$$

$$S(f;P) = \left(1 + \frac{3}{4} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{2} + \frac{3}{4} + 1\right) \cdot \frac{1}{4} = 5$$

$$\sigma(f;P;\xi_k)=(1+\frac{3}{4}+\frac{1}{2}+\frac{1}{4}+0+\frac{1}{4}+\frac{1}{2}+\frac{3}{4})\cdot\frac{1}{4}=1.$$

8.2.Aniq integral yordamida limitni hisoblash.

Masalan quydagi limitni qaraymiz:

$\lim\limits_{n\to\infty}\left(\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+...+\frac{n}{n^2+n^2}\right)$ limitini hisoblang.

Yechish: $S_n=\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+...+\frac{n}{n^2+n^2}$ deb belgilaymiz.Bu yig`indini quydagi ko`rinishda yozib olamiz:

$$\sigma_n=\frac{1}{n}\sum_{i=1}^{n}\frac{1}{1+\left(\frac{i}{n}\right)^2}$$

bunda $\frac{1}{1+(\frac{1}{n})^2},\frac{1}{1+(\frac{2}{n})^2},....,\frac{1}{1+(\frac{n}{n})^2}$ qo'shiluvchilar, $\frac{1}{1+x^2}$ funksiyaning $x_1=\frac{1}{n},x_2=\frac{2}{n},...,x_n=\frac{n}{n}=1$ nuqtalardagi qiymatlarini ifoda qiladi. $[0;1]$ kesmaning regulyar $P=\{x_0=0<x_1<x_2,...,<x_{n-1}<x_n=1\}$ bo'linishni olamiz: $[0;1]=[0;x_1]\cup[x_1;x_2]\cup...\cup[x_{n-1};x_n]$,bunda $\Delta x_1=x_i-x_{i-1}=\frac{1}{n}$.Ma'lumki,ta`rifga asosan,integral yig`indining limiti,qism kesmalardan olingan ξ_i nuqtalarni tanlashga bog`liq emas.Shuning uchun, $\xi_i=x_i$ deb olsak,u holda,

$$\sigma_n=\sum_{i=1}^{n}f(\xi_i)\Delta x_i=\sum_{i=1}^{n}\frac{1}{1+(\frac{i}{n})^2}\cdot\frac{1}{n}$$

yig`indi, $f(x)=\frac{1}{1+x^2}$ funksiyaning, $[0;1]$ kesmadagi Riman integral yig`indisi bo`lib hisoblanadi.

Demak, ta'rifga asosan,

$$\lim_{n\to\infty}\left(\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+...+\frac{n}{n^2+n^2}\right)=\int_0^1\frac{1}{1+x^2}dx=arctgx\Big|_0^1=\frac{\pi}{4}.$$

Aniq integral yordamida limitni hisoblang:

1. $\lim\limits_{n\to\infty}(\frac{1}{n^2}+\frac{2}{n^2}+....+\frac{n-1}{n^2})$

2. $\lim\limits_{n\to\infty}(\frac{1}{n+1}+\frac{1}{n+2}+....+\frac{1}{n+3})$
3. $\lim\limits_{n\to\infty}(\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+....+\frac{n}{n^2+n^2})$
4. $\lim\limits_{n\to\infty}\frac{1}{n}(\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+....+\sin\frac{(n-1)\pi}{n})$
5. $\lim\limits_{n\to\infty}\frac{1^p+2^p+.....+n^p}{n^{p+1}}$ $(p>0)$
6. $\lim\limits_{n\to\infty}\frac{1}{n}\left(\sqrt{1+\frac{1}{n}}+\sqrt{1+\frac{2}{n}}+......+\sqrt{1+\frac{n}{n}}\right)$
7. $\lim\limits_{n\to\infty}\frac{\sqrt[n]{n!}}{n}$
8. $\lim\limits_{n\to\infty}\left[\frac{1}{n}\sum_{k=1}^{n}f(a+k\frac{b-a}{n})\right]$
9. $\lim\limits_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\sin\frac{\pi}{n^2}+\left(1+\frac{2}{n}\right)\sin\frac{2\pi}{n^2}+....+\left(1+\frac{n-1}{n}\right)\sin\frac{(n-1)\pi}{n^2}\right]$
10. $\lim\limits_{n\to\infty}\sin\frac{\pi}{n}\sum_{k=1}^{n}\frac{1}{2+\cos\frac{k\pi}{n}}$
11. $\lim\limits_{n\to\infty}\frac{\sum_{k=1}^{n}\sqrt{(nk+k)(nk+k+1)}}{n^2}$
12. $\lim\limits_{n\to\infty}\left(\frac{2^{\frac{1}{n}}}{n+1}+\frac{2^{\frac{2}{n}}}{n+\frac{1}{2}}+....+\frac{2^{\frac{n}{n}}}{n+\frac{1}{n}}\right)$

8.3.Aniq integralning mavjudligi.

1-teorema: $[a,b]$ kesmada chegaralangan $f(x)$ funksiya, shu kesmada Darbu ma'nosida integrallanuvcgi bo`lishi bo`lishi uchun, $\forall\varepsilon>0$ olinganda ham,shunday $\delta=\delta(\varepsilon)>0$ son topilib, $[a,b]$ kesmaning diametri $d(P)<\delta$ bo`lgan har qanday $P$ bo'linishga nisbatan,Darbu yig`indilarning,

$$S_p(f)-s_p(f)<\varepsilon$$

tengsizliklarni qanotlantirishi zarur va yetarlidir.

Agar $f(x)$ funksiyaning $[x_k,x_{k+1}]$ $(k=\overline{0,n-1})$ kesmadagi tebranishini ω_k deb belgilasak,u holda $S_P(f)-s_P(f)<\varepsilon$ tengsizlik,quydagi,

$$\sum \omega_k \Delta x_k < \varepsilon$$

tengsizlikka teng kuchlu bo`ladi.

2-teorema: $f(x)$ funksiyaning $[a,b]$ kesmada Riman ma'nosida integrallanuvchi bo`lishi uchun,uning Darbu ma'nosida integrallanuvchi bo`lishi,zarur va yetarlidir.

Bu holda, Riman integrali,Darbu ma'nosidagi integraliga teng bo`ladi. Biz bundan keyin, "Darbu ma'nosidagi integral" degan terminni ishlatmasdan,o`sha integralni "Riman ma`nosidagi" integral deb ataymiz.

8.4.Integrallanuvchi funksiyalar sinfi

1^0.Uzluksiz funksiyalarning integrallanuvchiligi.

Aytaylik, $f(x)$ funksiya $[a,b]$ oraliqda aniqlangan bo'lsin.

1-teorema. Agar $f(x)$ funksiya $[a,b]$ da uzluksiz bo'lsa, u shu $[a,b]$ da integrallanuvchi,ya'ni

$$C[a,b] \subset R([a,b])$$

bo`ladi.

Isbot. Modomiki, $f(x) \in C[a,b]$ ekan, u Kantor teoremasiga ko`ra $[a,b]$ oraliqda tekis uzluksiz bo`ladi.Kantor teoremasining natijasiga ko`ra,$\forall \varepsilon > 0$ olinganda ham shunday $\delta > 0$ son topiladiki,$[a,b]$ oraliqni uzunliklari $\delta$ dan kichik bo`lgan bo`laklarga ajratganda har bir bo`lakdagi funksiyaning tebranishi

$$\omega_\kappa < \frac{\varepsilon}{b-a}$$

bo`ladi.Unda $[a,b]$ oraliqni diametri $\lambda_P < \delta$ bo`lgan har qanday P bo`laklashda

$$S(f;P) - s(f;P) = \sum_{k=0}^{n-1} \omega_k \Delta x_k < \frac{\varepsilon}{b-a} \sum_{k=0}^{n-1} \Delta x_k = \varepsilon$$

bo`ladi. Demak, $f(x) \in R([a,b])$.

2^0.Monoton funksiyalarning integrallanuvchiligi.

2-teorema. Agar $f(x)$ funksiya $[a,b]$ segmentda chegaralangan va monoton bo`lsa,u shu segmentda integrallanuvchi bo`ladi.

Isbot. Aytaylik,$f(x)$ funksiya $[a,b]$ segmentda o`suvchi bo`lib,$f(a) < f(b)$ bo`lsin.

$\forall \varepsilon > 0$ sonni olib,unga ko`ra $\delta > 0$ ni

$$\delta = \frac{\varepsilon}{f(b) - f(a)}$$

deymiz.

U holda $[a,b]$ segmentning diamerti $\lambda_P < \delta$ bo'lgan ixtiyoriy P bo'laklash uchun

$$S(f;P) - s(f;P) = \sum_{k=0}^{n-1} (M_k - m_k) \cdot \Delta x_k = \sum_{k=0}^{n-1} [f(x_{k+1}) - f(x_k)] \cdot \Delta x_k \le$$

$$\le \lambda_P \sum_{k=0}^{n-1} [f(x_{k+1}) - f(x_k)] = \lambda_P \cdot [f(b) - f(a)] < \frac{\varepsilon}{f(b - f(a)} \cdot [f(b) - f(a)] = \varepsilon$$

bo'ladi.Demak, $f(x) \in R([a,b])$.

3^0.Uziladigan funksiyalarning integrallanuvchiligi.

3-teorema. Agar $f(x)$ funksiya $[a,b]$ segmentda chegaralangan va shu segmentning chekli sondagi nuqtalarida uzilishga ega bo`lib, qolgan barcha nuqtalarida uzluksiz bo`lib, funksiya $[a,b]$ da integrallanuvchi bo`ladi.

Isbot. $f(x)$ funksiya $[a,b]$ da chegaralangan bo`lsin. Demak,

$$\exists C \in R, \ \ \forall x \in [a,b]: \ |f(x)| \le C \qquad (C > 0)$$

bo'ladi.

Soddalik uchun, $f(x)$ funksiya $[a,b]$ segmentning faqat bitta x^* $(x^* \in [a,b])$ nuqtasida uzilishga ega bo`lib, qolgan barcha nuqtalarda uzluksiz bo`lsin. $\forall \varepsilon > 0$ sonni olib,unga ko`ra $\delta > 0$ sonni

$$\delta = \frac{\varepsilon}{16C}$$

deymiz.

x^* nuqtaning δ atrofi $(x^* - \delta, x^* + \delta)$ ni olib,ushbu

$$[a,b] \setminus (x^* - \delta, x^* + \delta)$$

to'plamni qaraymiz.Bu to`plamda $f(x)$ funksiya uzluksiz bo`lib, Kantor teoremasiga binoan u tekis uzluksiz bo`ladi. U holda shunday $\gamma > 0$ son topiladiki,

$$\forall x', x'' \in [a, x^* - \delta], \qquad (\forall x', x'' \in [x^* + \delta, b])$$

lar uchun $|x' - x''| < \gamma$ bo`lishidan

$$|f(x') - f(x'')| < \frac{\varepsilon}{2(b-a)}$$

bo'lishi kelib chiqadi.

Endi $[a,b]$ segmentni diametri $\lambda_P < \min(\delta, \gamma)$ bo'lgan ixtiyoriy P bo'laklashni olib,unga nisbatan

$$\sum_{k=0}^{n-1} \omega_k \cdot \Delta x_k \qquad (1)$$

yig`indini tuzamiz.

Bu yig`indining har bir hadida $[x_k, x_{k+1}]$ $(k=0,1,2,...,n-1)$ oraliqlarning uzunliklari Δx_k lar qatnashadi.

(1) yig`indining ushbu

$$[x_k, x_{k+1}] \cap (x^* - \delta, x^* + \delta) = \emptyset$$

munosabat bajariladigan $[x_k, x_{k+1}]$ ga mos hadlaridan tuzilgan yig`indini

$$\sum_k{}' \ \omega_k \cdot \Delta x_k$$

bilan,qolgan barcha hadlaridan (bunday hadlar uchun)

$$[x_k, x_{k+1}] \cap (x^* - \delta, x^* + \delta) \neq \emptyset$$

yoki

$$[x_k, x_{k+1}] \cap \{x^* - \delta\} \neq \emptyset$$

yoki

$$[x_k, x_{k+1}] \cap \{x^* + \delta\} \neq \emptyset$$

bo`ladi) tashkil topgan yig`indini

$$\sum_k{}'' \omega_k \cdot \Delta x_k$$

bilan belgilaymiz.

Natijada

$$\sum_{k=0}^{n-1} \omega_k \cdot \Delta x_k = \sum_k{}' \omega_k \cdot \Delta x_k + \sum_k{}'' \omega_k \cdot \Delta x_k$$

bo`lib, tenglikning o`ng tomondagi qo`shiluvchilar uchun

$$\sum_k{}' \omega_k \cdot \Delta x_k \le \frac{\varepsilon}{2(b-a)} \cdot \sum_k{}' \Delta x_k \le \frac{\varepsilon}{2(b-a)} \cdot (b-a) = \frac{\varepsilon}{2},$$

$$\sum_k{}'' \omega_k \cdot \Delta x_k \le 2C \cdot \sum_k{}'' \Delta x_k \le 2 \cdot C \cdot 4\delta = 8C \cdot \frac{\varepsilon}{16} = \frac{\varepsilon}{2}$$

bo'ladi.Demak,

$$\sum_{k=0}^{n-1} \omega_k \cdot \Delta x_k < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Bu esa $f(x)$ funksiyaning $[a,b]$ da integrallanuvchi ekanligini bildiradi.

Aniq integralni hisoblang:

1. $\int_0^3 \arcsin\sqrt{\frac{x}{x+1}}\,dx$

2. $\int_0^{2\pi} \frac{dx}{(2+\cos x)(3+\cos x)}$

16. $\int_0^{\frac{\pi}{2}} \ln\cos x \cdot \cos 2nx\,dx$

17. $\int_0^3 \operatorname{sgn} x\,dx$

3. $\int_0^{\ldots} \frac{dx}{\sin^4 x + \cos^4 x}$

4. $\int_0^{\ldots} \sin x \sin 2x \sin 3x\,dx$

5. $\int_0^{\ldots} (x\sin x)^2\,dx$

6. $\int_0^{\ldots} e^x \cos x\,dx$

7. $\int_0^{\ln 2} sh^4 x\,dx$

8. $\int_0^{\ldots} \sin^{2m} \cos^{2n}\,dx$

9. $\int_0^{\ldots} \frac{\sin nx}{\sin x}\,dx$

10. $\int_0^{\ldots} \cos^n x \cos nx\,dx$

11. $\int_0^{\ldots} \frac{\cos(2n+1)x}{\cos x}\,dx$

12. $\int_0^{\ldots} \sin^n x \sin nx\,dx$

13. $\int_0^{\ldots} \sin^{n-1} x \cos(n+1)x\,dx$

14. $\int_0^{\ldots} \cos^{n-1} x \sin(n+1)x\,dx$

15. $\int_0^{\ldots} e^{-m} \cos^{2n} x\,dx$

18. $\int_0^2 [e^x]\,dx$

19. $\int_0^6 [x]\sin\frac{\pi x}{6}\,dx$

20. $\int_0^{\pi} x\,\mathrm{sgn}(\cos x)\,dx$

21. $\int_1^{n+1} \ln[x]\,dx$

22. $\int_0^1 \mathrm{sgn}(\sin(\ln x))\,dx$

23. $\int_0^{2\pi} x\sin x\,dx$

24. $\int_0^{2\pi} \frac{\sin x}{x}\,dx$

25. $\int_{-2}^{2} x^3 2^x\,dx$

26. $\int_{\frac{1}{2}}^{1} x^2 \ln x\,dx$

27. $\int_0^{2\pi} \frac{dx}{1+0.5\cos x}$

28. $\int_0^1 \frac{x^9}{\sqrt{1+x}}\,dx$

29. $\int_0^{100} \frac{e^{-x}}{x+100}\,dx$

30. $\int_a^b \sin x^2\,dx \quad (0 < a < b)$

8.5.Aniq integralning xossalari.

1^0.Integralning chiziqlilik hamda additivlik xossalari.

1-xossa. Agar $f(x)\in R([a,b])$ va $C\in R$ bo'lsa, u holda$(C\cdot f(x))\in R([a,b])$bo`lib,

$$\int_a^b C\cdot f(x)dx = C\int_a^b f(x)dx$$

bo`ladi.

Isbot. $f(x)\in R([a,b])$ va $C\in R$ bo`lsin. Aniq integral ta'rifiga ko`ra

$$\lim_{\lambda_P\to 0}\sigma(f,P,\xi_k)=\int_a^b f(x)dx$$

bo`ladi.

Ravshanki,

$$\sigma(C\cdot f(x;P;\xi_k))=C\sigma(f;P;\xi_k),$$

$$\lim_{\lambda_P\to 0}\sigma(C\cdot f(x;P;\xi_k))=C\cdot\lim_{\lambda_P\to 0}\sigma(f,P,\xi_k).$$

Demak,

$$(C\cdot f(x))\in R([a,b])$$

va

$$\int_a^b C\cdot f(x)dx = C\int_a^b f(x)dx.$$

2-xossa. Agar

$$f(x)\in R([a,b]),\ g(x)\in R([a,b])$$

bo`lsa,u holda

$$(f(x)+g(x))\in R([a,b])$$

bo`lib,

$$\int_a^b (f(x)+g(x))dx=\int_a^b f(x)dx+\int_a^b g(x)dx$$

bo`ladi (additivlik xossasi)

Isbot. Aniq integral ta'rifiga ko'ra

$$\lim_{\lambda_P\to 0}\sigma(f,P,\xi_k)=\int_a^b f(x)dx,$$

$$\lim_{\lambda_P\to 0}\sigma(f,P,\xi_k)=\int_a^b g(x)dx$$

bo`ladi.

Ravshanki,

$$\sigma(f+g,P,\xi_k)=\sigma(f,P,\xi_k)+\sigma(g,P,\xi_k).$$

Limitga ega bo'lgan funksiyalar haqida teoremadan foydalanib, $(f(x)+g(x))\in R([a,b])$ va

$$\int_a^b (f(x)+g(x))dx=\int_a^b f(x)dx+\int_a^b g(x)dx$$

bo'lishini topamiz.

3-xossa Agar $f(x)\in R([a,c])$, $f(x)\in R([c,b])$ bo'lsa,u holda

$$f(x)\in R([a,b])$$

bo'lib,

$$\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b g(x)dx$$

bo'ladi.

Isbot. Aytaylik, $a<c<b$ bo`lib, $f(x)\in R([a,c])$ va $f(x)\in R([c,b])$ bo`lsin.U holda $\forall\varepsilon>0$ son olinganda ham $[a,c]$ oraliqning $\lambda_{P_1}<\delta_1$ bo`lgan $P_1$ bo`laklashi topiladiki

$$S(f;P_1)-s(f;P_1)<\frac{\varepsilon}{2}, \qquad (1)$$

shuningdek $[c,b]$ oraliqning $\lambda_{P_2}<\delta_2$ bo`lgan $P_2$ bo`laklashi topiladik,

$$S(f;P_2)-s(f;P_2)<\frac{\varepsilon}{2}$$

bo'ladi.

Endi $[a,b]$ oraliqning diametri $\lambda_{P_3}<\delta=\min(\delta_1,\delta_2)$ bo`lgan ixtiyoriy $P_3$ bo`laklashini olamiz.Bu P_3 bo`laklashning bo`luvchi nuqtalari qatoriga c nuqtani qo'shib $[a,b]$ ning yangi P bo`lashni hosil qilamiz.Unga nisbatan $f(x)$ funksiyaning Darbu yig`indilari

$$S(f;P),\qquad s(f;P)$$

bo'lsin.

P bo`laklashning $[a,c]$ va $[c,b]$ dagi bo`luvchi nuqtalari mos ravishda ularning P_1' hamda P_2' bo`laklashlarni yuzaga keltiradi.Ravshanki,bu $P_1'$ va $P_2'$ bo'laklashlarga nisbatan quydagi tengsizliklar o`rinli bo`ladi:

$$S(f;P_1')-s(f;P_1')<\frac{\varepsilon}{2},$$
$$S(f;P_2')-s(f;P_2')<\frac{\varepsilon}{2}.$$

Ayni paytda,

$$S(f;P)=S(f;P_1')+S(f;P_2'),$$
$$s(f;P)=s(f;P_1')+s(f;P_2')$$

bo`ladi. Bu munosabatlardan

$$S(f;P)-s(f;P)=\left(S(f;P_1')-s(f;P_1')\right)+$$
$$+\left(S(f;P_2')-s(f;P_2')\right)<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

bo`lishi kelib chiqadi. Demak, $f(x)\in R([a,b])$.

$f(x)$ funksiyaning $[a,b]$, $[a,c]$, $[c,b]$ oraliqlar bo`ycha $P$ bo`laklashga nisbatan integral yig`indilari

$$\sum_{[a,b]} f(\xi_k)\cdot\Delta x_k,\quad \sum_{[a,c]} f(\xi_k)\cdot\Delta x_k,\quad \sum_{[c,b]} f(\xi_k)\cdot\Delta x_k$$

bo`lib,

$$\sum_{[a,b]} f(\xi_k)\cdot\Delta x_k=\sum_{[a,c]} f(\xi_k)\cdot\Delta x_k+\sum_{[c,b]} f(\xi_k)\cdot\Delta x_k$$

bo`ladi. Integral ta'rifidan foydalanib topamiz:

$$\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx.$$

Shunga o`xshash $c<a<b$, $a<b<c$ bo`lgan hollarda ham xossaning o`rinli bo`lishi isbotlandi.

4-xossa. Agar $f(x)\in R([a,b])$, $g(x)\in R([a,b])$ bo`lsa,u holda $f(x)\cdot g(x)\in R([a,b])$ bo`ladi.

Isbot. Modomiki, $f(x)$ va $g(x)$ funksiyalar $[a,b]$ da integrallanuvchi ekan,unda

$$S(f;P)-s(f;P)<\frac{\varepsilon}{2M'}\quad (M'=\sup f(x), x\in[a,b])$$
$$S(g;P)-s(g;P)<\frac{\varepsilon}{2M}\quad (M=\sup g(x),\ x\in[a,b])$$

bo`ladi.

Aytaylik, $\forall x\in[a,b]$ da $f(x)\geq 0$, $q(x)\geq 0$ bo`lsin.U holda $\forall x\in[x_k;x_{k+1}]$ uchun

$$0\leq m_k\leq f(x)\leq M_k',\ m_k=\inf f(x),\ M_k'=\sup f(x);$$
$$0\leq m_k'\leq g(x)\leq M_k,\ m_k'=\inf g(x),\ M_k=\sup g(x)$$

bo`lib, ulardan

$$0\leq m_k\cdot m_k'\leq f(x)\cdot g(x)\leq M_k\cdot M_k',$$

bo`lishi kelib chiqadi.Ayni paytda,

$$m_k^0=\inf\{f(x)\cdot g(x)\},\ M_k^0=\sup\{f(x)\cdot g(x)\}$$

lar uchun

$$m_k\cdot m_k'\leq m_k^0\leq M_k^0\leq M_k\cdot M_k'$$

bo`lib,

$$M_k^0-m_k^0\leq M_k\cdot M_k'-m_k\cdot m_k'=$$
$$=M_k'(M_k-m_k)+m_k(M_k'-m_k')$$

bo`ladi.

Endi $M\geq M_k$, $M'\geq M_k'$ ekanligini e'tiborga olib topamiz:

$$S(f\cdot g;P)-s(f\cdot g;P)=\sum_{k=0}^{n-1}\left(M_k^0-m_k^0\right)\leq$$
$$\leq M'\cdot\sum_{k=0}^{n-1}\left(M_k-m_k\right)\cdot\Delta x_k+M\cdot\sum_{k=0}^{n-1}\left(M_k'-m_k'\right)=$$
$$=M'\left(S(f;P)-s(f;P)\right)+M'\left(S(g;P)-s(g;P)\right)<$$
$$<M'\frac{\varepsilon}{2M'}+M\cdot\frac{\varepsilon}{2M}<\varepsilon.$$

Demak,bu holda $f(x)\cdot g(x)\in R([a,b])$.

Aytaylik, $f(x)$ va $g(x)$ funksiyalar $[a,b]$ da ixtiyoriy integrallanuvchi funksiyalar bo`lsin.

Ravshanki, $\forall x\in[a,b]$ da

$$f(x)-\inf f(x)=f(x)-m\geq 0,$$
$$g(x)-\inf g(x)=g(x)-m'\geq 0$$

bo`ladi.

Endi $f(x)\cdot g(x)$ funksiyani quydagicha yozib olamiz:

$$f(x)\cdot g(x)=\left(f(x)-m\right)\left(g(x)-m'\right)+mg(x)+m'f(x)-mm'.$$

Bu tenglikning o`ng tomonidagi har bir qo`shiluvchi $[a,b]$ da integrallanuvchi bo`lgani sababli $f(x)\cdot g(x)$ ham $[a,b]$ da integrallanuvchi bo`ladi

Natija. Agar $f(x)\in R([a,b])$ bo`lsa, u holda $[f(x)]^n\in R([a,b])$ bo`ladi, bunda $n\in N$.

3^0. Integralning tengsizliklar bilan bog`langan xossalari.

1-xossa. Agar $f(x)\in R([a,b])$ bo`lib, $\forall x\in[a,b]$ da $f(x)\ge 0$ bo`lsa,u holda

$$\int_a^b f(x)dx \ge 0$$

bo`ladi.

Isbot. Integral ta'rifiga ko`ra

$$\lambda_P \to 0 \text{ da } \sum_{k=0}^{n-1} f(\xi_k)\cdot\Delta x_k \to \int_a^b f(x)dx$$

bo`ladi. U holda, $\forall x\in[a,b]$ da $f(x)\ge 0$ bo`lishidan

$$\sum_{k=0}^{n-1} f(\xi_k)\cdot\Delta x_k \ge 0$$

bo`lib,unda

$$\int_a^b f(x)dx \ge 0$$

bo`lishi kelib chiqadi.

1-natija. Agar $f(x)\in R([a,b]), g(x)\in R([a,b])$ bo`lib, $\forall x\in[a,b]$ da $f(x)\le g(x)$ bo`lsa,u holda

$$\int_a^b f(x)dx \le \int_a^b g(x)dx$$

bo`ladi.

Isbot. Ravshanki,

$$f(x)\in R([a,b])\,,\, g(x)\in R([a,b]) \Rightarrow (g(x)-f(x))\in R([a,b])$$

bo`lib,

$$g(x)-f(x)\ge 0 \Rightarrow \int_a^b (g(x)-f(x))dx \ge 0 \Rightarrow \int_a^b g(x)dx - \int_a^b f(x)dx \ge$$

$$\ge 0 \Rightarrow \int_a^b f(x)dx \le \int_a^b g(x)dx$$

bo`ladi.

2-natija. Agar $f(x)\in R([a,b])$, $g(x)\in R([a,b])$ bo`lsa,u holda

$$\left|\int_a^b f(x)g(x)dx\right| \le \sqrt{\int_a^b f^2(x)dx}\cdot\sqrt{\int_a^b g^2(x)dx} \qquad (2)$$

bo`ladi.

Isbot. Ixtiyoriy $\alpha\in R$ uchun

$$\int_a^b (f(x)-\alpha\cdot g(x))^2 dx \ge 0$$

bo`lib,

$$\alpha^2\int_a^b g^2(x)dx - 2\alpha\int_a^b f(x)g(x)dx + \int_a^b f^2(x)dx \ge 0$$

bo`ladi.Kvadrat uchhadning diskriminanti musbat bo`lmagani sababli

$$\left(\int_a^b f(x)g(x)dx\right)^2 - \int_a^b f^2(x)dx\cdot\int_a^b g^2(x)dx \le 0,$$

ya'ni,

$$\left|\int_a^b f(x)g(x)dx\right| \le \sqrt{\int_a^b f^2(x)dx}\cdot\sqrt{\int_a^b g^2(x)dx}$$

bo`ladi.

(2) tengsizlik Koshi-Bunyakovskiy tengsizligi deyiladi.

2-xossa. Agar $f(x)\in R([a,b])$ bo`lsa, $|f(x)|\in R([a,b])$ bo`lib,

$$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$$

bo`ladi.

Isbot. $f(x)\in R([a,b])$ bo`lsin.Integrallanuvchilik me`zoniga ko`ra, $\forall\varepsilon>0$ olinganda ham $[a,b]$ segmentning shunday $P$ bo`laklashi topiladiki, unga nisbatan

$$S(f;P)-s(f;P)=\sum_{k=0}^{n-1}\omega_k\Delta x_k < \varepsilon$$

bo`ladi,bunda $\omega_k - f(x)$ funksiyaning $[x_k,x_{k+1}]$ dagi tebranishi.

Ravshanki, $\forall x', x''\in[a,b]$ uchun

$$\big||f(x')|-|f(x'')|\big| \le |f(x')-f(x'')|$$

bo`lib,undan

$$\sup\big||f(x')|-|f(x'')|\big| \le \sup|f(x')-f(x'')|$$

bo`lishi kelib chiqadi.

Demak,

$$\omega_k' \le \omega_k$$

bo`ladi,bunda $\omega_k' - |f(x)|$ funksiyaning $[x_k,x_{k+1}]$ dagi tebranishi.Shularni e'tiborga olib,

$$S(|f|;P)-s(|f|;P)=\sum_{k=0}^{n-1}\omega_k'\cdot\Delta x_k \le \sum_{k=0}^{n-1}\omega_k\cdot\Delta x_k < \varepsilon$$

bo`lishini topamiz.Demak, $|f(x)| \in R([a,b])$.

$f(x)$ va $|f(x)|$ funksiyalarning integral yig`indilari uchun

$$\left|\sum_{k=0}^{n-1} f(\xi_k)\Delta x_k\right| \le \sum_{k=0}^{n-1} |f(\xi_k)| \cdot \Delta x_k,$$

bo`lib, $\lambda_p \to 0$ da limitga o`tish natijasida

$$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$$

bo`lishi kelib chiqadi.

**3^0.O`rta qiymat haqidagi teoremalar.**Aytaylik, $f(x)$ funksiya $[a,b]$ da berilgan va chegaralangan bo`lsin.U holda $m=\inf\{f(x)\}$, $M=\sup\{f(x)\}$ $(x\in[a,b])$ mavjud va $\forall x\in[a,b]$ uchun

$$m \le f(x) \le M$$

tengsizliklar o`rinli bo`ladi.

1-teorema. Agar $f(x)\in R([a,b])$ bo`lsa,u holda shunday o`zgarmas $\mu(m\le\mu\le M)$ son mavjudki,

$$\int_a^b f(x)dx = \mu\cdot(b-a)$$

bo`ladi.

Isbot. Ravshanki,

$$m \le f(x) \le M \Rightarrow \int_a^b mdx \le \int_a^b f(x)dx \le \int_a^b Mdx \Rightarrow$$

$$\Rightarrow m(b-a) \le \int_a^b f(x)dx \le M(b-a).$$

Keyingi tengsizliklardan

$$m \le \frac{\int_a^b f(x)dx}{b-a} \le M$$

bo`lishi kelib chiqadi.

Agar

$$\mu = \frac{\int_a^b f(x)dx}{b-a}$$

deyilsa,undan

$$\int_a^b f(x)dx = \mu\cdot(b-a)$$

bo`lishini topamiz

3-natija. Agar $f(x)\in C[a,b]$ bo`lsa, u holda shunday $\theta\in[a,b]$ topiladiki,

$$\int_a^b f(x)dx = f(\theta)\cdot(b-a)$$

bo`ladi

Bu tasdiq yuqoridagi teorema va uzluksiz funksiyaning xossalaridan kelib chiqadi.

2-teorema. Agar $f(x)\in R([a,b])$, $g(x)\in R([a,b])$ bo`lib,$[a,b]$ da $g(x)$ funksiya o`z ishorasini o`zgartirmasa,u holda shunday o`zgarmas $\mu(m\le\mu\le M)$ son mavjudki,

$$\int_a^b f(x)g(x)dx = \mu\int_a^b g(x)dx \qquad (3)$$

bo`ladi.

Isbot. Aytaylik, $\forall x\in[a,b]$ da $g(x)\ge 0$ bo`lsin.Unda ravshanki,

$$m\le f(x)\le M \Rightarrow mg(x)\le f(x)\cdot g(x)\le Mg(x)$$

bo`ladi.

Bu munosabatdan hamda aniq integral xossalaridan foydalanib topamiz:

$$m\int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le M\int_a^b g(x)dx.$$

a) $\int_a^b g(x)dx = 0$ bo`lsin.U holda

$$\int_a^b f(x)g(x)dx = 0$$

bo`lib,ixtiyoriy $\mu(m\le\mu\le M)$ da (3) o`rinli bo`ladi.

b) $\int_a^b g(x)dx > 0$ bo'lsin.U holda

$$m \le \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \le M$$

bo`lib,

$$\mu = \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx}$$

deyilsa,unda

$$\int_a^b f(x)g(x)dx = \mu\int_a^b g(x)dx$$

bo`lishi kelib chiqadi.

4-natija. Agar $f(x)\in C[a,b]$ bo`lib, $g(x)\in R([a,b])$ va $g(x)$ funksiya $[a,b]$ da o`z ishorasini o`zgartirmasa,u holda shunday $\theta\in[a,b]$ topiladiki,

$$\int_a^b f(x)g(x)dx = f(\theta)\int_a^b g(x)dx$$

bo`ladi.

Berilgan integrallarni xossalardan foydalanib taqqoslang.(1-7).Tengsizliklarni isbotlang.(8-20).Aniq integralni hisoblang.(21-30).

1. $\int_0^{\frac{\pi}{2}}\frac{\sin x}{x}dx$ va $\int_0^{\pi}\frac{\sin x}{x}dx$

2. $\int_{\frac{1}{2}}^{1}\frac{dx}{\sqrt{x}}$ va $\int_{\frac{1}{2}}^{1}\frac{dx}{\sqrt[3]{x}}$

3. $\int_0^1 e^{-x}\sin x\,dx$ va $\int_0^1 e^{-x^2}\sin x\,dx$

4. $\int_1^2\frac{dx}{\sqrt{1+x^2}}$ va $\int_1^2\frac{dx}{x}$

5. $\int_0^{\frac{\pi}{2}}\sin^{10}x\,dx$ va $\int_0^{\frac{\pi}{2}}\sin^2 x\,dx$

6. $\int_0^1 e^{-x}dx$ va $\int_0^1 e^{-x^2}dx$

7. $\int_0^{\pi}e^{-x^2}\cos^2 x\,dx$ va $\int_{\pi}^{2\pi}e^{-x^2}\cos^2 x\,dx$

16. $0<\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\frac{\sin x}{x}dx<\ln 3$

17. $\sin 1<\int_{-1}^{1}\frac{\cos x}{1+x^2}dx<2\sin 1$

18. $\frac{2}{\pi}\ln\frac{\pi+2}{2}<\int_0^1\frac{\sin x}{x(x+1)}dx<\ln\frac{\pi+2}{2}$

19. $\frac{1}{8}<\frac{\pi}{6}\int_0^1\frac{\sin\left(\frac{\pi}{6}\right)(x+1)}{(x+1)(3-x)}dx<\frac{1}{6}$

20. $0.03<\int_0^1\frac{x^7}{(e^x+e^{-x})\sqrt{1+x^2}}dx<0.05$

21. $\int_0^1\frac{dx}{e^x+e^{-x}}$

22. $\int_0^{\frac{\pi}{4}}\cos^3 x\,dx$

8. $0<\int_0^{\pi}\frac{\sin x}{\sqrt[5]{x^2+2}}dx<\frac{\pi}{\sqrt[5]{2}}$

9. $\frac{1}{\sqrt[3]{9}}<\frac{1}{\pi}\int_{-1}^{1}\frac{\pi+arctgx}{\sqrt[3]{x^2+8}}dx<\frac{3}{2}$

10. $\frac{\sqrt{2}}{3}<\int_0^{\pi}\frac{\cos x}{x^2+2}dx<1$

11. $\frac{1}{10\sqrt{2}}<\int_0^1\frac{x^9}{\sqrt{1+x}}dx<\frac{1}{10}$

12. $\frac{1}{20\sqrt[3]{2}}<\int_0^1\frac{x^{19}}{\sqrt[3]{x^6+1}}dx<\frac{1}{20}$

13. $0<\int_0^{200}\frac{e^{-5x}dx}{x+20}<0.01$

14. $1<\int_0^1\frac{1+x^{20}}{1+x^{10}}dx<1+\frac{1}{42}$

15. $1-\frac{1}{n}<\int_0^1 e^{-x^n}dx<1,\ n>1$

23. $\int_0^{\frac{1}{3}}ch^2 3x\,dx$

24. $\int_1^e\frac{\cos(\ln x)}{x}dx$

25. $\int_1^e\frac{dx}{x(1+\ln^2 x)}$

26. $\int_0^1\sqrt{4-x^2}dx$

27. $\int_2^3\frac{dx}{x^2-2x-8}$

28. $\int_0^4\frac{dx}{\sqrt{x}+1}$

29. $\int_1^2\frac{e^{\frac{1}{x^2}}}{x^3}dx$

30. $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}tg^4 x\,dx$

8.6.Aniq integralni hisoblash usullari.

1⁰.Aniq integrallar ta'rifiga ko`ra hisoblash.

Aytaylik, $f(x)\in R([a,b])$ bo`lsin.Unda integral ta'rifiga ko`ra

$$\lim_{\lambda_P\to 0}\sum_{k=0}^{n-1}f(\xi_k)\Delta x_k = \int_a^b f(x)dx$$

bo`ladi.

Misol. Ushbu

$$\int_a^b \sin x\,dx$$

integral hisoblansin.

Ravshanki, $f(x)=\sin x\in C[a,b]$.Demak, $f(x)\in R([a,b])$.$[a,b]$ oraliqni ushbu

$$a, a+\alpha_n, a+2\alpha_n,\ldots,a+k\alpha_n,\ldots,a+n\alpha_n=b$$

nuqtalar yordamida,bunda $\alpha_n=\frac{b-a}{n}$, n ta teng bo`lakka bo`lib,har bir

$$[a+k\alpha_n, a+(k+1)\alpha_n]\ (k=0,1,2,\ldots,n-1)$$

bo`lakda ξ_k nuqtalarni quydagicha

$$\xi_k = a + (k+1)\alpha_n \quad (k = 0,1,2,...,n-1)$$

tanlaymiz.U holda $f(x) = \sin x$ funksiyaning integral yig`indisi quydagicha

$$\sigma = \sum_{k=0}^{n-1} \sin(a + (k+1)\alpha_n) \cdot \alpha_n = \alpha_n \sum_{k=0}^{n-1} \sin(a + (k+1)\alpha_n)$$

ko`rinishga ega bo`ladi

Ma'lumki,

$$\sin(a + (k+1)\alpha_n) = \frac{1}{2\sin\frac{\alpha_n}{2}} 2\sin\frac{\alpha_n}{2} \sin(a + (k+1)\alpha_n) =$$

$$= \frac{1}{2\sin\frac{\alpha_n}{2}} \left[\cos(a + (k+\frac{1}{2})\alpha_n) - \cos(a + (k+\frac{3}{2})\alpha_n)\right]$$

bo`ladi.

Natijada integral yig`indi uchun ushbu

$$\sigma = \frac{\alpha_n}{2\sin\frac{\alpha_n}{2}} \sum_{k=0}^{n-1} \left[\cos(a + (k+\frac{1}{2})\alpha_n) - \cos(a + (k+\frac{3}{2})\alpha_n)\right] =$$

$$= \frac{\frac{\alpha_n}{2}}{\sin\frac{\alpha_n}{2}} (\cos(a + \frac{1}{2}\alpha_n) - \cos(b + \frac{1}{2}\alpha_n))$$

tenglikka kelamiz.

Keyingi tenglikda $\lambda_P = \Delta x_k = \alpha_n \to 0$ da limitga o`tib topamiz:

$$\int_a^b \sin x dx = \cos a - \cos b.$$

2^0. Nuyuton-Leybnist formulasi.

Aytaylik, $f(x)$ funksiya $[a,b]$ segmentda berilgan va shu segmentda uzluksiz bo`lsin.U holda $f(x)$ boshlang`ich funksiya

$$F(x) = \int_a^x f(t)dt$$

ga teng bo`ladi.

Ravshanki, $F(x)$ funksiya $f(x)$ ning ixtiyoriy boshlang`ich funksiyasi bo`lsa,u holda

$$F(x) = F(x) + C \qquad (C = const)$$

bo`ladi.

Bu tenglikda avval $x = a$ deb

$$F(a) = C,$$

so`ngra $x = b$ deb

$$F(b) = \int_a^b f(x)dx + C$$

bo`lishini topamiz.Demak,

$$\int_a^b f(x)dx = F(b) - F(a). \qquad (1)$$

(1) formula Nyuton-Leybnist formulasi deyiladi.

Odatda, $F(b) - F(a)$ ayirma $F(x)\Big|_a^b$ kabi yoziladi. Demak,

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a).$$

Masalan, $\int_a^b \frac{1}{x} dx = \ln x\Big|_a^b = \ln b - \ln a = \ln\frac{b}{a}. \qquad (a > 0, b > 0)$

3^0.O`zgaruvchialrni almashtirish formulasi.

Faraz qilaylik, $f(x) \in C[a,b]$ bo`lsin.Ravshanki,bu holda

$$\int_a^b f(x)dx$$

integral mavjud bo`ladi.

Ayni paytda,bu funksiya $[a,b]$ da boshlang`ich $F(x)$ funksiyaga ega bo`lib,

$$\int_a^b f(x)dx = F(b) - F(a)$$

bo`ladi.

Aytaylik,aniq integralda x o`zgaruvchi ushbu

$$x = \varphi(t)$$

formula bilan almashtirilgan bo`lib, bunda $\varphi(t)$ funksiya quydagi shartlarni bajarsin:

1) $\varphi(t) \in C[\alpha,\beta]$ bo`lib, $\varphi(t)$ funksiyaning barcha qiymatlari $[a,b]$ ga tegishli;

2) $\varphi(\alpha) = a, \varphi(\beta) = b$;

3) $\varphi(t)$ funksiya $[\alpha,\beta]$ da uzluksiz $\varphi'(t)$ hosilaga ega bo`lsin.

U holda

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\cdot\varphi'(t)dt \qquad (2)$$

bo`ladi.

Ravshanki, $F(\varphi(t))$ murakkab funksiya $[\alpha,\beta]$ segmentda uzluksiz bo`lib,

$$(F(\varphi(t)))' = F'(\varphi(t))\cdot\varphi'(t)$$

bo`lib.

Agar $F'(x) = f(x)$ ekanligini e'tiborga olsak,unda

$$(F(\varphi(t)))' = f(\varphi(t))\cdot\varphi'(t)$$

bo`lishini topamiz.Bu esa $F(\varphi(t))$ funksiya $[\alpha,\beta]$ da $f(\varphi(t))\cdot\varphi'(t)$ funksiyaning boshlang`ich funksiyasi ekanligini bildiradi.Nyuton-Leybnist formulasiga ko`ra

$$\int_\alpha^\beta f(\varphi(t))\cdot\varphi'(t)dt = F(\varphi(\beta)) - F(\varphi(\alpha)) = F(b) - F(a) \qquad (3)$$

bo`ladi.

(2) va (3) munosabatlardan

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\cdot\varphi'(t)dt \qquad (4)$$

bo`lishi kelib chiqadi.

(4) formula aniq integralda o`zgaruvchini almashtirish formulasi deyiladi.

2-misol: Ushbu $\int_0^1 \sqrt{1-x^2}\,dx$ integral hisoblansin.

Berilgan integralda $x = \sin t$ almashtirish bajaramiz.Unda

$$\int_0^1 \sqrt{1-x^2}\,dx = \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2 t}\cos t dt = \int_0^{\frac{\pi}{2}} \cos^2 t dt =$$

$$= \int_0^{\frac{\pi}{2}} (\frac{1}{2} + \frac{1}{2}\cos 2t)dt = (\frac{1}{2}t + \frac{1}{4}\sin 2t)\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

bo`ladi.

**4⁰.Bo`laklab integrallash formulasi.**Aytaylik,$u(x)$ va $v(x)$ funksiyalarning har biri $[a,b]$ segmentda uzluksiz $u'(x)$ ва $v'(x)$ hosilaga ega bo`lsin.U holda

$$\int_a^b u(x)dv(x) = (u(x)\cdot v(x))\Big|_a^b - \int_a^b v(x)du(x) \qquad (5)$$

bo`ladi.

Hosilani hisoblash qoidasiga ko`ra

$$(u(x)\cdot v(x))' = u'(x)\cdot v(x) + u(x)\cdot v'(x)$$

bo`ladi.Demak $u(x)\cdot v(x)$ funksiya $[a,b]$ oraliqda $u'(x)\cdot v(x) + u(x)\cdot v'(x)$ funksiyaning boshlang`ich funksiyasi bo`ladi.Nyuton-Leybnist formulasidan foydalanib topamiz:

$$\int_a^b (u'(x)\cdot v(x) + u(x)\cdot v'(x))'dx = (u(x)\cdot v(x))\Big|_a^b .$$

Keyingi tenglikdan

$$\int_a^b u(x)dv(x) = (u(x)\cdot v(x))\Big|_a^b - \int_a^b v(x)du(x)$$

bo`lishi kelib chiqadi.

(5) formula aniq integralda bo`laklab integrallash formulasi deyiladi.

3-misol. Ushbu $\int_1^2 x\ln x dx$ integral hisoblansin.

Bu intervalda $u(x) = \ln x, dv(x) = x$ deb $du(x) = \frac{1}{x}dx, v(x) = \frac{x^2}{2}$ bo`linishni topamiz.Unda (5) formulaga ko`ra:

$$\int_1^2 x\ln x dx = (\frac{x^2}{2}\ln x)\Big|_1^2 - \int_1^2 \frac{x^2}{2}\cdot\frac{1}{x}dx = 2\ln 2 - \frac{1}{2}\int_1^2 x dx = 2\ln 2 - \frac{3}{4} \text{ bo`ladi.}$$

4-misol. Ushbu $J_n = \int_0^{\frac{\pi}{2}} \sin^n x dx \quad (n = 0,1,2,...)$ integral hisoblansin.

Ravshanki,

$$J_0 = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}, \qquad J_1 = \int_0^{\frac{\pi}{2}} \sin x dx = (-\cos x)\Big|_0^{\frac{\pi}{2}} = 1.$$

$n \geq 2$ bo`lganda berilgan integralni

$$J_n = \int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \sin^{n-1} x\, d(-\cos x)$$

ko`rinishda yozib,unga bo`laklab integrallash formulasini qo`llaymiz.Natijada

$$J_n = (-\sin^{n-1} x \cdot \cos x)\Big|_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x dx =$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x(1-\sin^2 x)dx = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x dx - (n-1)\int_0^{\frac{\pi}{2}} \sin^n x dx =$$

$$= (n-1)J_{n-2} - (n-1)J_n$$

bo`lib,unda ushbu

$$J_n = \frac{n-1}{n} J_{n-2}$$

rekurent formula kelib chiqadi.

Bu formula yordamida berilgan integralni $n = 1,2,3,\ldots\ldots..$ bo`lganda ketma-ket hisoblash mumkin.

Aytaylik, $n = 2m$-juft son bo`lsin.Unda

$$J_{2m} = \frac{2m-1}{2m} \cdot \frac{2m-3}{2m-2} \cdots\cdots \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot J_0 = \frac{(2m-1)!!}{(2m)!!} \cdot \frac{\pi}{2}$$

bo`ladi.

Aytaylik, $n = 2m+1$-toq son bo`lsin.Unda

$$J_{2m+1} = \frac{2m}{2m+1} \cdot \frac{2m-2}{2m-1} \cdots\cdots \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot J_1 = \frac{(2m)!!}{(2m+1)!!}$$

bo`ladi. ($m!!$ simvol $m$ dan katta bo`lmagan va u bilan bir xil juftlikka ega bo`lgan natural sonlarning ko`paytmasini bildiradi.)

5^0.Vallis formulasi. Ma'lumki, $0 < x < \frac{\pi}{2}$ bo`lganda

$$\sin^{2n+1} x < \sin^{2n} x < \sin^{2n-1} x \qquad (n = 1,2,3,\ldots)$$

tengsizliklar o`rinli bo`ladi.Bu tengsizliklarni $[0, \frac{\pi}{2}]$ oraliq bo`ycha integrallab,

$$\int_0^{\frac{\pi}{2}} \sin^{2n+1} x dx < \int_0^{\frac{\pi}{2}} \sin^{2n} x dx < \int_0^{\frac{\pi}{2}} \sin^{2n-1} x dx,$$

so`ngra 4^0 da keltirilgan formuladan foydalanib topamiz:

$$\frac{(2n)!!}{(2n+1)!!} < \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2} < \frac{(2n-2)!!}{(2n-1)!!}.$$

Bu tengsizliklardan

$$\left(\frac{(2n)!!}{(2n-1)!!}\right)^2 \cdot \frac{1}{2n+1} < \frac{\pi}{2} < \left(\frac{(2n)!!}{(2n-1)!!}\right)^2 \cdot \frac{1}{2n}$$

bo`lishi kelib chiqadi.

Keyingi tengsizliklardan topamiz:

$$\frac{\pi}{2} = \lim_{n \to \infty} \frac{1}{2n+1} \left[\frac{(2n)!!}{(2n-1)!!}\right]^2. \qquad (6)$$

(6) formula Vallis formulasi deyiladi.

Integralni hisoblang:

1. $\int \sqrt[3]{x}\, dx$
2. $\int \sin x\, dx$
3. $\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{dx}{1+x^2}$
4. $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}}$
5. $\int \frac{dx}{\sqrt{1+x^2}}$
6. $\int_0 |1-x|\, dx$
7. $\int \frac{dx}{x^2-2x\cos\alpha+1}, \quad (0<\alpha<\pi)$
8. $\int_0 \frac{dx}{1+\varepsilon\cos x}, \quad (0 \le \varepsilon < 1)$
9. $\int \frac{dx}{\sqrt{(1-2ax+a^2)(1-2bx+b^2)}}$
   $(|a|<1, |b|<1, ab>0)$
10. $\int \frac{dx}{a^2\sin^2 x+b^2\cos^2 x}, \quad (ab \ne 0)$
11. $\int_{-1} \frac{dx}{x}$
12. $\int_0 \frac{\sec^2 x\, dx}{2+tg^2 x}$
13. $\int_0 x|x-a|\, dx$
16. $\int_0^{\pi} x\sin x\, dx$
17. $\int_0^{2\pi} x^2 \cos x\, dx$
18. $\int_{\frac{1}{e}}^{e} |\ln x|\, dx$
19. $\int_0^1 \arccos x\, dx$
20. $\int_0^{\sqrt{3}} x\, arctgx\, dx$
21. $\int_{-1}^{1} \frac{xdx}{\sqrt{5-4x}}$
22. $\int_0^a x^2\sqrt{a^2-x^2}\, dx$
23. $\int_0^{0.75} \frac{dx}{(x+1)\sqrt{x^2+1}}$
24. $\int_0^{\ln 2} \sqrt{e^x-1}\, dx$
25. $\int_0^1 \frac{\arcsin\sqrt{x}}{\sqrt{x(1-x)}} dx$
26. $\int_0^1 x(2-x^2)^{12} dx$
27. $\int_{-1}^{1} \frac{xdx}{x^2+x+1}$
28. $\int_1^e (x\ln x)^2 dx$

14. $\int_{0}^{\pi}\frac{\sin^2 x}{1+2\alpha\cos x+\alpha^2}dx$

15. $\int_{0}^{\ln 2} xe^{-x}dx$

29. $\int_{1}^{9} x\sqrt[3]{1-x}\,dx$

30. $\int_{-2}^{-1}\frac{dx}{x\sqrt{x^2-1}}$

8.7.Aniq integralning tadbiqlari. Aniq integral yordamida tekis shakl yuzini hisoblash.

1^0.Tekis shaklning yuzi tushunchasi.

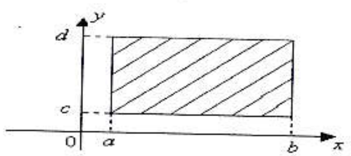
Ma'lumki, (x,y) juftlik, $(x\in R, y\in R)$,tekislikda nuqtalarni ifodalaydi. Koordinatalari ushbu

$$a\le x\le b,\ \ c\le y\le d \qquad (a\in R, b\in R, c\in R, d\in R)$$

tengsizliklarni qanotlantiruvchi tekislik nuqtalaridan hosil bo`lgan $D_0$ to`plam:

$$D_0=\{(x,y); x\in[a,b], y\in[c,d]\}$$

to`g`ri to`rtburchak deyiladi.



Bu to`g`ri to`rtburchak tomonlari (chegaralari) mos ravishda koordinata o`qlariga parallel bo`ladi.

D_0 to`g`ri to`rtburchakning yuzi deb (uning chegarasining,ya'ni

$$x=a\ ,x=b \qquad (c\le y\le d),$$

$$y=c\ ,y=d \qquad (a\le x\le b)$$

to`g`ri chiziq kesmalarining D_0 ga tegishli bo`lishi yoki tegishli bo`lmasligidan qat'iy nazar) ushbu

$$\mu(D_0)=(b-a)\cdot(d-c)$$

miqdorga aytiladi.

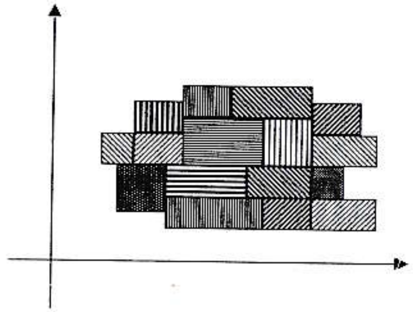
Aytaylik, tekislik nuqtalardan iborat biror Q to`plam berilgan bo`lsin. Agar shunday D_0 to`g`ri to`rtburchak topilsinki,

$$Q\subset D_0$$

bo`lsa, $Q$ chegaralangan to`plam deyiladi.

Har qanday chegaralangan tekislik nuqtalaridan iborat to`plam tekis shakl deyiladi.

Agar tekis shakl chekli sondagi kesishmaydigan to`g`ri to`rtburchak birlashmasi sifatida ifodalansa,uni ko`pburchak deymiz.



Bunday to`g`ri ko`pburchakning yuzi deb, uni tashkil etgan to`g`ri to`rtburchak yuzalari yig`indisiga aytiladi.

To`g`ri ko`pburchak yuzi quydagi xossalarga ega:

1) To`g`ri ko`pburchak yuzi har doim manfiy bo`lmaydi: $\mu(D)\ge 0$;

2)Kesishmaydigan ikki D_1 va D_2 to`g`ri ko`pbuchaklardan tashkil topgan to`g`ri ko`pburchak yuzi

$$\mu(D_1\cup D_2)=\mu(D_1)+\mu(D_2)\ \ ;$$

3) Agar D_1 ва D_2 to`g`ri ko`pburchaklar uchun

$$D_1\subset D_2$$

bo`lsa,u holda

$$\mu(D_1)\le\mu(D_2)$$

bo`ladi.

Tekislikda biror chegaralangan Q shakl berilgan bo`lsin.Bu shaklning ichiga $A$ to`g`ri ko`pburchak $(A\subset Q)$,so`ngra $Q$ shaklni o`z ichiga olgan B to`g`ri ko`pburchak $(Q\subset B)$ lar chizamiz. Ularning yuzalari mos ravishda $\mu(A)$ va $\mu(B)$ bo`lsin.

Ravshanki,bunday to`g`ri ko`pburchaklar ko`p bo`lib,ularning yuzalaridan iborat $\{\mu(A)\}$ va $\{\mu(B)\}$ to`plamlar hosil bo`ladi.

Ayni paytda,bu sonli to`plamlar chegaralangan bo`ladi. Binobarin, ularning aniq chegaralari

$$\sup\{\mu(A)\},\ \ \inf\{\mu(B)\}$$

lar mavjud.

1-ta'rif. Agar

$$\sup\{\mu(A)\} = \inf\{\mu(B)\}$$

bo`lsa,Q shakl yuzaga ega deyiladi.Ularning umumiy qiymati Q shaklning yuzi deyiladi va $\mu(Q)$ kabi belgilanadi:

$$\mu(Q) = \sup\{\mu(A)\} = \inf\{\mu(B)\}$$

1-teorema. Tekis shakl Q yuzaga ega bo`lishi uchun $\forall\varepsilon > 0$ son olinganda ham shunday $A$ $(A \subset Q)$ va $B$ $(Q \subset B)$ to`g`ri ko`pburchaklar topilib,ular uchun

$$\mu(B) - \mu(A) < \varepsilon$$

tengsizlikning bajarilishi zarur va yetarli.

Isbot. Zarurligi. Aytaylik, Q shakl yuzaga ega bo`lsin. Unda tarifga binoan

$$\sup\{\mu(A)\} = \inf\{\mu(B)\} = \mu(Q)$$

bo`ladi.

Modomiki,

$$\sup\{\mu(A)\} = \mu(Q),$$
$$\inf\{\mu(B)\} = \mu(Q)$$

ekan, unda $\forall\varepsilon > 0$ olinganda ham shunday to`g`ri ko`pburchak $A(A \subset Q)$ hamda shunday to`g`ri ko`pburchak $B(Q \subset B)$ topiladiki,

$$\mu(Q) - \mu(A) < \frac{\varepsilon}{2},$$

$$\mu(B) - \mu(Q) < \frac{\varepsilon}{2}$$

bo`ladi.Bu tengsizliklardan

$$\mu(B) - \mu(A) < \varepsilon$$

bo`lishi kelib chiqadi.

Yetarliligi. Aytaylik, A $(A \subset Q)$ ва B $(Q \subset B)$ to`g`ri ko`pburchak uchun $\mu(B) - \mu(A) < \varepsilon$ tengsizligi bajarilsin.

Ravshanki,

$$\mu(A) \le \sup\{\mu(A)\} ,$$
$$\mu(B) \ge \inf\{\mu(B)\} .$$

Bu munosabatlardan

$$\inf\{\mu(B)\} - \sup\{\mu(A)\} \le \mu(B) - \mu(A) < \varepsilon$$

bo`lishini topamiz.

ε-ixtiyoriy musbat son bo`lganda

$$\sup\{\mu(A)\} = \inf\{\mu(B)\}$$

bo`lishi kelib chiqadi. Demak, Q shakl yuziga ega.

Shunga o`xshash quydagi teorema isbotlanadi.

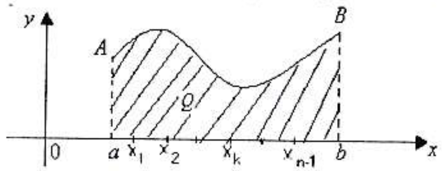
2-teorema. Tekis shakl Q yuzaga ega bo`lishi uchun $\forall\varepsilon > 0$ son olinganda ham shunday yuzaga ega tekis shakllar P va S lar $(P \subset Q$, $Q \subset S)$ topilib, ular uchun

$$\mu(S) - \mu(P) < \varepsilon$$

tengsizlikning bajarilishi zarur va yetarli.

2^0.Egri chiziqli trapetsiya yuzini hisoblash. Faraz qilaylik, $f(x) \in C[a,b]$ bo`lib, $\forall x \in [a,b]$ da $f(x) \ge 0$ bo`lsin.

Yuqoridan $f(x)$ funksiya grafigi,yon tomonlardan $x = a$, $x = b$ vertikal chiziqlar hamda pastdan abisissa o`qi bilan chegaralangan shaklni qaraylik.



Odatda,bu shakl egri chiziqli trapetsiya deyiladi. $[a,b]$ segmentning ixtiyoriy

$$P = \{x_0, x_1, x_2, ..., x_n\} \quad (a = x_0 < x_1 < x_2 < ... < x_n = b)$$

bo`laklashini olamiz.Bu bo`laklashning har bir $[x_k, x_{k+1}]$ oralig`ida

$$\inf\{f(x)\} = m_k, \ \sup\{f(x)\} = M_k \quad (k = 0,1,2,...,n-1)$$

mavjud bo`ladi.

Endi asosi $\Delta x_k = x_{k+1} - x_k$,balandligi m_k bo`lgan $(k = 0,1,2,...,n-1)$ to`g`ri to`rtburchaklarning birlashmasidan tashkil topgan to`g`ri ko`pburchak A deylik.

Shunningdek,asosi $\Delta x_k = x_{k+1} - x_k$, balandligi M_k bo`lgan $(k = 0,1,2,...,n-1)$ to`g`ri to`rtburchaklarning birlashmasidan tashkil topgan to`g`ri ko`pburchakni B deylik. Ravshanki,

$$A \subset Q \ , \ Q \subset B$$

bo`lib,ularning yuzalari

$$\mu(A) = \sum_{k=0}^{n-1} m_k \cdot \Delta x_k, \ \mu(B) = \sum_{k=0}^{n-1} M_k \cdot \Delta x_k$$

bo`ladi.

Bu yig`indilarni $f(x)$ funksiyaning $[a,b]$ segmentining $P$ bo`laklashiga nisbatan Darbuning quyi hamda yuqori yig`indilari ekanligini payqash qiyin emas:

$$\mu(A) = s(f;P) \quad , \quad \mu(B) = S(f;P).$$

$f(x)\in C[a,b]$ bo`lgani uchun $f(x)$ funksiya $[a,b]$ da integrallanuvchi bo`ladi. Unda integrallanuvchilik mezoniga ko`ra, $\forall\varepsilon>0$ olinganda ham $[a,b]$ segmentning shunday $P$ bo`laklashi topiladiki,

$$S(f;P)-s(f;P)<\varepsilon$$

bo`ladi. Binobarin,ushbu

$$\mu(B)-\mu(A)<\varepsilon$$

tengsizlik bajariladi.Bu esa,1-teoremaga muvofiq, qaralayotgan egri chiziqli trapetsiyaning yuziga ega bo`lishini bildiradi.
Unda ta'rifga ko`ra

$$\sup\{\mu(A)\}=\inf\{\mu(B)\}$$

bo`ladi.

Ayni paytda,

$$\sup\{\mu(A)\}=\int_{\bar{a}}^{b}f(x)dx,$$

$$\inf\{\mu(B)\}=\int_{a}^{\bar{b}}f(x)dx$$

bo`lganligi sababli Q egri chiziqli trapetsiya yuzi

$$\mu(Q)=\int_a^b f(x)dx \qquad (1)$$

ga teng bo`ladi.

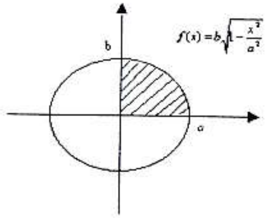
1-misol. Tekislikda ushbu $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

ellips bilan chegaralangan Q shaklning yuzi topilsin.

Ellips bilan chegaralangan Q shaklning yuzi OX va OY koordinata o`qlari hamda

$$f(x)=b\cdot\sqrt{1-\frac{x^2}{a^2}}\quad,\qquad 0\le x\le a$$

chiziqlar bilan chegaralangan egri chiziqli trapetsiya yuzining 4 tasiga teng bo`ladi.



11 – *chizma*

Unda (1) formuladan foydalanib topamiz:

$$\mu(Q)=4\int_0^a b\sqrt{1-\frac{x^2}{a^2}}\,dx=\frac{4b}{a}\int_0^a\sqrt{a^2-x^2}\,dx=$$

$$=\left|\begin{matrix}x=a\sin t\,, & \\ & 0\le t\le\frac{\pi}{2}\\ dx=a\cos t dt\,, & \end{matrix}\right|=$$

$$=\frac{4b}{a}\cdot a^2\int_0^{\frac{\pi}{2}}\cos^2 t dt=4ab\cdot\frac{\pi}{4}=ab\pi.$$

Aytaylik, $f_1(x)\in C[a,b]$, $f_2(x)\in C[a,b]$ bo`lib, $\forall x\in[a,b]$ da

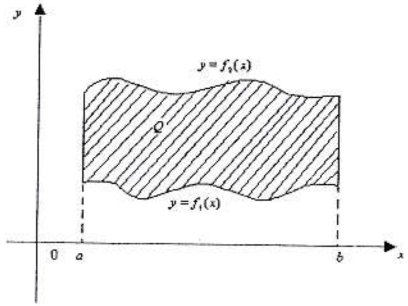
$$0\le f_1(x)\le f_2(x)$$

bo`lsin.

Tekislikdagi Q shakl quydagi

$$y=f_1(x), y=f_2(x),\ x=a, x=b$$

chiziqlar bilan chegaralangan shaklni ifodalasin.



12 – *chizma*

Bu shaklning yuzi

$$\mu(Q)=\int_a^b f_2(x)dx-\int_a^b f_1(x)dx=\int_a^b [f_2(x)-f_1(x)]dx \qquad (2)$$

bo`ladi.

2-misol. Tekislikda ushbu $y=4-x^2, y=x^2-2x$

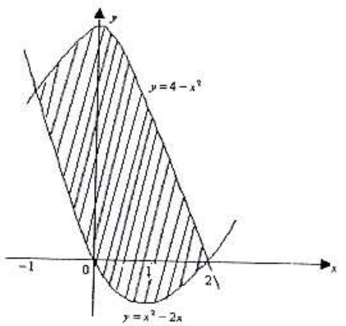
chiziqlar (parabolalar) bilan chegaralangan Q shaklning yuzi topilsin.

Parabolalarning tenglamalari

$$y=4-x^2 \ ,$$
$$y=x^2-2x$$

ni birgalikda yechib, ularning kesishish nuqtalarini topamiz:

$$4-x^2=x^2-2x \ ,$$
$$x_1=-1 \ , \ x_2=2 \ ; \ y_1=3 \ , \ y_2=0: \ A(-1;3) \ , \ B(2;0).$$



13 – *chizma*

Bu shaklnimg yuzini (2) formuladan foydalanib hisoblaymiz:

$$\mu(Q)=\int_{-1}^{2}[(4-x^2)-(x^2-2x)]dx=\int_{-1}^{2}(4+2x-2x^2)dx=(4x+x^2-\frac{2}{3}x^3)\Big|_{-1}^{2}=9.$$

Eslatma. Agar $f(x)\in C[a,b]$ funksiya $[a,b]$ da ishorasini saqlamasa, (1) integral egri chiziqli trapetsiyalar yuzalarning yig`indidan iborat bo`ladi. Bunda OX o`qining yuqorisidagi yuza musbat ishora bilan, $OX$ o`qining pastdagi yuza manfiy ishora bilan olinadi.

Masalan, OX o`qi hamda $f(x)=\sin x$, $0\le x\le 2\pi$ funksiya grafigi bilan chegaralangan shakl yuzi

$$\mu(Q)=\int_0^{\pi}\sin x+(-\int_{\pi}^{2\pi}\sin x dx)=(-\cos x)\Big|_0^{\pi}-(-\cos x)\Big|_{\pi}^{2\pi}=4$$

bo`ladi.

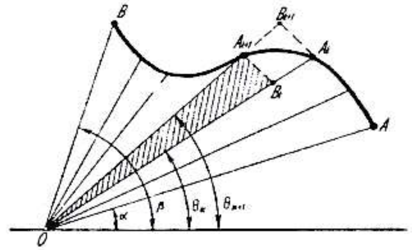
3°. Egri chiziqli sektorning yuzini hisoblash. Aytaylik, $A\breve{B}$ egri chiziq qutb koordinatalar sistemasida ushbu

$$\rho=\rho(\theta) \ , \ \alpha\le\theta\le\beta \quad (\alpha\in R,\beta\in R)$$

tenglama bilan berilgan bo`lsin. Bunda

$$\rho(\theta)\in C[\alpha,\beta] \ , \quad \forall\theta\in[\alpha,\beta] \ \text{ da } \ \rho(\theta)\ge 0 \ .$$

Tekislikda $A\breve{B}$ egri chiziq hamda OA ва OB radius-vektorlar bilan chegaralangan Q shaklni qaraymiz.



$[\alpha,\beta]$ segmentni ixtiyoriy

$$P=\{\theta_0,\theta_1,...,\theta_n\} \qquad (\alpha=\theta_0<\theta_1<...<\theta_n=\beta)$$

bo`laklashni olamiz. $O$ nuqtadan har bir qutb burchagi $\theta_k$ ga mos $OA_k$ radius-vektor o`tkazamiz. Natijada OAB -egri chiziqli sektor

$$OA_kA_{k+1} \ (k=0,1,2,...,n-1 \ ; \ A_0=A \ , \ A_n=B)$$

egri chiziqli sektorchalarga ajratadi.

Ravshanki, $\rho=\rho(\theta)\in C[\alpha,\beta]$ bo`lgani uchun $[\theta_k,\theta_{k+1}]$ da $(k=0,1,2,...,n-1)$

$$m_k=\inf\{\rho(\theta)\} \ , \quad M_k=\sup\{\rho(\theta)\}$$

lar mavjud.

Endi har bir $[\theta_k,\theta_{k+1}]$ segment uchun radius-vektorlari mos ravishda m_k hamda M_k bo`lgan doiraviy sektorlarni hosil qilamiz. Bunday doiraviy sektorlar yuzaga ega bo`lib, ularning yuzi mos ravishda

$$\frac{1}{2}m_k^2\cdot\Delta\theta_k \quad , \ \frac{1}{2}M_k^2\cdot\Delta\theta_k \qquad (\Delta\theta_k=\theta_{k+1}-\theta_k)$$

bo`ladi.

Radius-vektorlari m_k $(k=0,1,2,...,n-1)$ bo`lgan barcha doiraviy sektorlar birlashmasidan hosil bo`lgan shaklni Q_1 desak, unda $Q_1\subset Q$ bo`lib, uning yuzi

$$\mu(Q_1)=\frac{1}{2}\sum_{k=0}^{n-1}m_k^2\cdot\Delta\theta_k \qquad (3)$$

bo`ladi.

Shuningdek,radius-vektorlari $M_k \quad (k=0,1,2,...,n-1)$ bo`lgan barcha doiraviy sektorlar birlashmasidan hosil bo`lgan shaklni Q_2 desak,unda

$Q\subset Q_2$ bo`lib,uning yuzi

$$\mu(Q_2)=\frac{1}{2}\sum_{k=0}^{n-1}M_k^2\cdot\Delta\theta_k \qquad (4)$$

bo`ladi.

(3) va (4) yig`indilar $\frac{1}{2}\rho^2(\theta)$ funksiyaning Darbu yig`indilari bo`ladi. Ayni paytda, $\frac{1}{2}\rho^2(\theta)$ funksiya $[\alpha,\beta]$ da uzluksiz bo`lgani uchun u integrallanuvchidir. Demak, $\forall\varepsilon>0$ olinganda ham $[\alpha,\beta]$ segmentning shunday P bo`laklashi topiladiki,

$$S(\frac{1}{2}\rho^2(\theta);P)-s(\frac{1}{2}\rho^2(\theta);P)<\varepsilon$$

bo`ladi. Binobarin,ushbu

$$\mu(Q_2)-\mu(Q_1)<\varepsilon$$

tengsizlik bajariladi.Bu esa, 2-teoremaga muvofiq,qaralayotgan egri chiziqli sektorning yuzaga ega bo`lishini bildiradi.Unda ta'rifga ko`ra

$$\sup\{\mu(Q_1)\}=\inf\{\mu(Q_2)\}$$

bo`ladi.

Ayni paytda,

$$\sup\{\mu(Q_1)\}=\int_\alpha^\beta\rho^2(\theta)d\theta,$$

$$\inf\{\mu(Q_2)\}=\int_\alpha^\beta\rho^2(\theta)d\theta$$

bo`lgani sababli Q egri chiziqli sektorning yuzi

$$\mu(Q)=\frac{1}{2}\int_\alpha^\beta\rho^2(\theta)d\theta$$

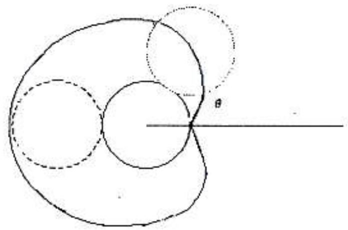
ga teng bo`ladi.

3-misol. Ushbu

$$\rho=\rho(\theta)=a(1-\cos\theta)\quad(a\in R,\ 0\le\theta\le2\pi)$$

funksiya grafigi bilan chegaralangan shaklning yuzi topilsin.

Bu funksiya grafigi kardioidani ifodalaydi.Ma'lumki, kardioida radiusi r ga teng bo`lgan aylananing shu radiusli ikkinchi qo`zg`almas aylana bo`ylab harakati (sirpanmasdan dumalashi) natijasida birinchi aylana ixtiyoriy nuqtasining chizgan chizig`idir.



Kardioida qutb o`qiga nisbatan simmetrik bo`lganligi sababli yuqori yarim tekislikdagi shaklning yuzini topib,so`ngra uni 2 ga ko`paytirsak,izlayotgan yuza kelib chiqadi.

θ o`zgaruvchi $[0,\pi]$ da o`zgarganda ρ radius-vektor kardiodning yuqori yarim tekislikdagi qismini chizadi.Shuning uchun

$$\mu(Q)=2\cdot\frac{1}{2}\int_0^\pi\rho^2(\theta)d\theta=\int_0^\pi a^2(1-\cos\theta)^2d\theta=$$

$$=a^2\int_0^\pi\left[\frac{3}{2}-2\cos\theta+\frac{1}{2}\cos2\theta\right]d\theta=$$

$$=a^2(\frac{3}{2}\theta-2\sin\theta+\frac{1}{2}\cdot\frac{1}{2}\sin2\theta)\Big|_0^\pi=\frac{3}{2}\pi a^2$$

bo`ladi.

Mustaqil ishlash uchun masalalar:

Quydagi chiziqlar bilan chegaralangan shaklning yuzini toping.

1. $y=x-x^2,\ y=x\sqrt{1-x}$
2. $y=|\log_a x|,\ y=0,\ x=\frac{1}{a},\ x=a,\ a>1$
3. $y=\ln(1+x),\ y=-xe^{-x}, x=1$
4. $y=e^x\sin x,\ y=0,\ x=\frac{\pi}{4}$
5. $y=a^x,\ y=a, x=0, a>1$
6. $y=tgx,\ y=\frac{2\cos x}{3},\ x=0$

16. $y=4^{-x}, y=-\log_4 x, y=0,\ x=0;$
17. $y=x^{-\alpha}, y=0, x=a, \alpha>0, a>1$
18. $y=\arcsin x, y=\arccos x, y=0$
19. $y=|x|^3e^{-x^2}, |x|=a, a>0$
20. $2y=x^2, x^2+y^2=4y, 2y\ge x^2$
21. $x^2+y^2=2, y^2=2x-1, x\ge\frac{1}{2}$

7. $y=\sin^2 x,\ y=x\sin x,\ 0\le x\le \pi;$

8. $y=\sin^3 x+\cos^3 x,\ y=0,\ -\frac{\pi}{4}\le x\le \frac{3\pi}{4}.$

9. $y=x(x-a)^2,\ y=0;$

10. $y=\frac{x^2}{2},\ y=\frac{1}{1+x^2}$

11. $y=ax^2e^x,\ y=-x^3e^x;$

12. $y=e^{-x}|\sin x|,\ y=0,\ \pi n\le x\le \pi(n+1)$

13. $x^2+y^2=8,\ 2y=x^2,\ y\ge 0;$

14. $y=\frac{\sqrt{x}}{1+x^3},\ y=0,\ x=1$

15. $y=\frac{(ab)^{\frac{3}{2}}}{a^2\cos^2 x+b^2\sin^2 x},\ y=0, x=0, x=\pi$

22. $y=x, y=\frac{1}{x}, y=\frac{10}{3-x}, x\ge 1;$

23. $y=\sqrt{3}x^2, y=\sqrt{4-x^2}$

24. $y=\sin x, y=0,\ 0\le x\le \frac{\pi}{4}$

25. $y=-x^2, y=x^2-2x-4$

26. $y=\frac{a^2}{\sqrt{a^2-x^2}}, y=2a, a>0$

27. $y=\frac{6}{x+5}, y=|x|,\ x\ge -2$

28. $y=x, y=\frac{\pi}{2}\sin x, x\ge 0$

29. $y=(x^2-2x)e^x, y=0, x\ge 0$

30. $y=\frac{x^2}{2}, y=\frac{1}{1+x^2}$

8.8.Aniq integral yordamida yoy uzunligini hisoblash.

Ma'lumki, tekislikdagi iki $A(x_1,y_1)$ va $B(x_2,y_2)$ nuqtalarni birlashtiruvchi to`g`ri chiziq $\mu(l_0)=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ kesmasi l_0 uzunlikka ega va uning uzunligi

ga teng bo`ladi.

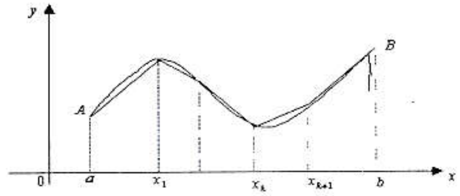
Aytaylik,tekislikdagi l chiziq $A_0(x_0,y_0), A_1(x_1,y_1),...,$ $A_n(x_n,y_n)$ nuqtalarni$(n\in N)$ birin-ketin to`g`ri chiziq kesmalari bilan birlashtirishidan hosil bo`lgan bo`lsin.Odatda,bunday chiziq siniq chiziq deyiladi.Siniq chiziq uzunligi (peremetri) deb,uni tashkil etgan to`g`ri chiziq kesmalari uzunliklari yig`indisiga aytiladi.

$$\mu(l)=\sum_{k=0}^{n-1}\sqrt{(x_{k+1}-x_k)^2+(y_{k+1}-y_k)^2}.$$

Faraz qilaylik,tekislikdagi $A\breve{B}$ egri chizig`I (uni $A\breve{B}$ yoyi deb ham ataymiz) ushbu

$$y=f(x) \qquad (a\le x\le b)$$

tenglama bilan berilgan bo`lsin,bunda $f(x)\in C[a,b]$.



$[a,b]$ segmentning ixtiyoriy

$$P=\{x_0,x_1,...,x_n\} \qquad (a=x_0<x_1<...<x_n=b$$

bo`laklashini olib,bo`luvchi x_k $(k=0,1,2,...,n)$ nuqtalar orqali OY o`qiga parallel to`g`ri chiziqlar o`tkazamiz.Bu to`g`ri chiziqlarning $A\breve{B}$ yoyi bilan kesishgan nuqtalari

$$A_k(x_k,f(x_k)) \qquad (k=0,1,2,...,n\ ;\ \ A_0=A\ ,\ A_n=B)$$

bo`ladi.

$A\breve{B}$ yoydagi bu $A_k(x_k,f(x_k))$ nuqtalarni bir-biri bilan to`g`ri chiziq kesmalari yordamida birlashtirib,l siniq chiziqni hosil qilamiz.Odatda,l siniq chiziq $A\breve{B}$ yoyiga chizilgan siniq chiziq deyiladi.u uzunlikka ega bo`lib,uzunligini (peremetrini) $\mu(l)$ deylik.

Agar P_1 va P_2 lar $[a,b]$ segmentning ikkitabo`laklashi bo`lib,$P_1\subset P_2$ bo`lsa,u holda bub o`laklashlarga mos $A\breve{B}$ yoyiga chizilgan siniq chiziq l_1 , l_2 larning peremetrlari uchun

$$\mu(l_1)\le \mu(l_2)$$

bo`ladi.

$[a,b]$ segmentning P_1 bo`laklashi quydagi

$$P_1 = \{x_0, x_1, ..., x_k, x_{k+1}, ..., x_n\}$$
$$(a = x_0 < x_1 < ... < x_k < x_{k+1} < ... < x_n = b)$$

ko`rinishda bo`lib, P_2 bo`laklash esa $P_1$ bo`laklashning barcha bo`luvchi nuqtalari hamda qo`shimcha bitta $x^* \in [a,b]$ nuqtani qo`shish natijasida hosil bo`lgan bo`llaklash bo`lsin.Bu x^* nuqta x_k hamda x_{k+1} nuqtalar orasida joylashsin: $x_k < x^* < x_{k+1}$ .Demak,

$$P_2 = \{x_0, x_1, ..., x_k, x^*, x_{k+1}, ..., x_n\}$$
$$(a = x_0 < x_1 < ... < x_k < x^* < x_{k+1} < ... < x_n = b\ )$$

Ravshanki, $P_1 \subset P_2$ bo`ladi.

$A\breve{B}$ yoyiga chizilgan P_1 bo`laklashga mos siniq chiziq $l_1$,shu yoyga chizilgan $P_2$ bo`laklashga mos siniq chiziq l_2 dan faqatgina bitta bo`lagi bilangina farq qiladi: $l_1$ da $A_kA_{k+1}$ bo`lak bo`lgan holda $l_2$ da ikkita $A_kA^*$ hamda $A^*A_{k+1}$ bo`laklar bo`ladi.

Ammo A_kA_{k+1} to`g`ri chiziq kesmasining uzunligi $\mu(A_kA_{k+1}), A_kA^*$ hamda A^*A_{k+1} kesmalar uzunliklari $\mu(A_kA^*)$, $\mu(A^*A_{k+1})$ yig`indisidan har doim katta bo`lmaganligi,yani

$$\mu(A_kA_{k+1}) \le \mu(A_kA^*) + \mu(A^*A_{k+1})$$

uchun

$$\mu(l_1) \le \mu(l_2)$$

bo`ladi.

Demak, P bo`laklashning bo`luvchi nuqtalari sonini orttira borilsa, $A\breve{B}$ yoyiga chizilgan ularga mos siniq chiziqlar peremetrlari ham ortib boradi

1-ta'rif. Agar $\lambda_p \to 0$ da $A\breve{B}$ yoyiga chizilgan siniq chiziq peremetri

$$\mu(l) = \sum_{k=0}^{n-1} \sqrt{(x_{k+1} - x_k)^2 + [f(x_{k+1}) - f(x_k)]^2}$$

chekli limitga ege bo`lsa, $A\breve{B}$ yoy uzunlikka ega deyiladi.

Ushbu

$$\lim_{\lambda_p \to 0} \mu(l) = \mu(A\breve{B})$$

limit $A\breve{B}$ yoyning uzunligi deyiladi.

Masalan,agar

$$f(x) = kx + C \qquad (a \le x \le b)$$

bo`lsa,unda $A\breve{B}$ ning uzunligi

$$\mu(A\breve{B}) = \lim_{\lambda_p \to 0} \sum_{k=0}^{n-1} \sqrt{(x_{k+1} - x_k)^2 + k^2(x_{r+1} - x_k)^2} =$$
$$= \lim_{\lambda_p \to 0} \sum_{k=0}^{n-1} \sqrt{1+k^2} \cdot (x_{k+1} - x_k) = \sqrt{1+k^2} \cdot (b-a)$$

bo`ladi.

Aytaylik, $A\breve{B}$ egri chiziq ushbu

$$\begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases}, \qquad (\alpha \le t \le \beta)$$

tenglamalar sistemasi bilan berilgan bo`lsin.

(Bu holda egri chiziq parametric ko`rinishda berilgan deyiladi).Bunda:

1) $\varphi(t) \in C[\alpha, \beta]$, $\psi(t) \in C[\alpha, \beta]$;

2) $\forall t_1, t_2 \in [\alpha, \beta]$, $t_1 \ne t_2$ uchun (1)

$$A_1(x_1, y_1) = A_1(\varphi(t_1), \psi(t_1)) ,$$
$$A_2(x_2, y_2) = A_2(\varphi(t_2), \psi(t_2))$$

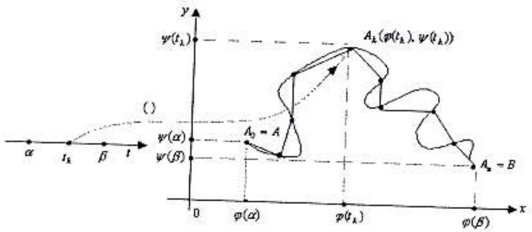
nuqtalar turlicha;

3) $t = \alpha$ ga A nuqta, $t = \beta$ ga B nuqta mos kelsin.

$[\alpha, \beta]$ segmentning ixtiyoriy

$$P = \{t_0, t_1, ..., t_n\} \qquad (\alpha = t_0 < t_1 < ... < t_n = \beta$$

bo`laklashni olib,bub o`laklashni bo`luvchi $t_k$ $(k=0,1,2,...,n)$ nuqtalariga mos kelgan $A\breve{B}$ yoydagi $A_k = A_k(x_k, y_k)$ $(x_k = \varphi(t_k)$ , $y_k = \psi(t_k)$ ; $k = 0,...,n)$ nuqtalarni bir-biri bilan to`g`ri chiziq kesmalari yordamida birlashtirib, $A\breve{B}$ yoyga chizilgan siniq chiziq l ni hosil qilamiz.



Bu siniq chiziq peremetri

$$\mu(l) = \sum_{k=0}^{n-1} \sqrt{[\varphi(t_{k+1}) - \varphi(t_k)]^2 + [\psi(t_{k+1}) - \psi(t_k)]^2}$$

bo`ladi.

2-ta'rif. Agar $\lambda_p \to 0$ da $A\breve{B}$ yoyiga chizilgan siniq chiziq peremetri $\mu(l)$ chekli limitga ega bo`lsa, $A\breve{B}$ yoy uzunlikka ega deyiladi.

Ushbu

$$\lim_{\lambda_p \to 0} \mu(l) = \mu(A\breve{B})$$

limit $A\breve{B}$ yoyining uzunligi deyiladi.

Yuqorida keltirilgan ta'riflardan yoy uzunligining (agar u mavjud bo`lsa ) musbat bo`lishi kelib chiqadi.

Endi yoy uzunligining ikkita xossasini isbotsiz keltiramiz:

1) Agar $A\breve{B}$ yoyi uzunlikka ega bo`lib, u $A\breve{B}$ yoydagi nuqtalar yordamida $n$ ta $A_k\breve{A}_{k+1}$ yoylarga $(k = 0,1,2,...,n$ ; $A_0 = A, B = A_{n+1})$ ajralgan bo`lsa, u holda har bir $A_k\breve{A}_{k+1}$ yoy uzunlikka ega va

$$\mu(A\breve{B}) = \sum_{k=0}^{n} \mu(A_k\breve{A}_{k+1})$$

bo`ladi.

2) Agar $A\breve{B}$ yoyi n ta $A_k\breve{A}_{k+1}$ yoylarga ajratgan bo`lib,har bir $A_k\breve{A}_{k+1}$ yoy uzunlikka ega bo`lsa,u holda $A\breve{B}$ yoyi ham uzunlikka ega bo`ladi.

$y = f(x)$ **tenglama bilan berilgan egri chiziq uzunligini hisoblash.**Faraz qilaylik, $A\breve{B}$ egri chiziq ushbu

$$y = f(x) \quad , \quad a \le x \le b$$

tenglama bilan berilgan bo`lsin.Bunda $f(x)$ funksiya $[a,b]$ segmentda uzluksiz va uzluksiz $f'(x)$ hosilaga ega.

$[a,b]$ segmentning ixtiyoriy

$$P = \{x_0, x_1, ..., x_n\} \quad (a = x_0 < x_1 < ... < x_n = b)$$

bo`laklashini olib,unga mos $A\breve{B}$ yoyiga chizilgan l siniq chiziq hosil qilamiz.

Bu siniq chiziq peremetri

$$\mu(l) = \sum_{k=0}^{n-1} \sqrt{(x_{k+1} - x_k)^2 + [f(x_{k+1}) - f(x_k)]^2}$$

bo`ladi.

Har bir $[x_k, x_{k+1}]$ segmentda $f(x)$ funksiyaga Lagranch teoremasini qo`llab topamiz:

$$\mu(l) = \sum_{k=0}^{n-1} \sqrt{(x_{k+1} - x_k)^2 + [f'(\tau_k)\cdot(x_{k+1} - x_k)]^2} =$$

$$= \sum_{k=0}^{n-1} \sqrt{1 + f'^2(\tau_k)}\cdot(x_{k+1} - x_k) = \sum_{k=0}^{n-1} \sqrt{1 + f'^2(\tau_k)}\,\Delta x_k \quad ,$$

bunda $\tau_k \in [x_k, x_{r+1}]$.

Bu tenglikdagi yig`indining $\sqrt{1 + f'^2(x)}$ funksiyaning integral yig`indisidan farqi shuki, integral yig`indida $\xi_k \in [x_k, x_{k+1}]$ nuqta ixtiyoriy bo`lgan holda yuqoridagi yig`indida esa $\tau_k$ nuqta Lagranj teoremasiga muvofiq olingan tayin nuqta bo`lishidadir.Ammo $\sqrt{1 + f'^2(x)}$ funksiya integrallanuvchi bo`lgani sababli $\xi_k = \tau_k$ deb olishimiz mumkin.Natijada

$$\mu(l)=\sum_{k=0}^{n-1}\sqrt{1+f'^2(\xi_k)}\cdot\Delta x_k$$

bo`lib,undan

$$\lim_{\lambda_p\to 0}\mu(l)=\lim_{\lambda_p\to 0}\sum_{k=0}^{n-1}\sqrt{1+f'^2(\xi_k)}\cdot\Delta x_k=\int_a^b\sqrt{1+f'^2(x)}dx$$

bo`lishi kelib chiqadi.

Demak, $A\breve{B}$ yoyining uzunligi

$$\mu(A\breve{B})=\int_a^b\sqrt{1+f'^2(x)}dx \qquad (2)$$

bo`ladi.Bu formula yordamida yoy uzunligi hisoblanadi.

1-misol. Ushbu

$$f(x)=\frac{a}{2}(e^{\frac{x}{a}}+e^{-\frac{x}{a}}) \qquad (a>0\ ,\quad -a\le x\le a)$$

tenglama bilan berilgan $A\breve{B}$ egri chiziqning uzunligi topilsin.

Bu tenglama bilan aniqlanadigan chiziq zanjir chizig`i deyiladi.

Ravshanki,

$$f'(x)=\frac{1}{2}(e^{\frac{x}{a}}-e^{-\frac{x}{a}})\ ,$$

$$1+f'^2(x)=\frac{1}{4}(e^{\frac{x}{a}}+e^{-\frac{x}{a}})^2\ ,$$

$$\sqrt{1+f'^2(x)}=\frac{1}{2}(e^{\frac{x}{a}}+e^{-\frac{x}{a}})$$

bo`ladi.(2) formuladan foydalanib,zanjir chizig`ining uzunligini topamiz:

$$\mu(A\breve{B})=\int_{-a}^{a}\frac{1}{2}(e^{\frac{x}{a}}+e^{-\frac{x}{a}})dx=\frac{a}{2}(e^{\frac{x}{a}}-e^{-\frac{x}{a}})\Big|_{-a}^{a}=a(e-\frac{1}{e}).$$

Parametrik ko`rinishda berilgan egri chiziqning uzunligini hisoblash.

Faraz qilaylik, $A\breve{B}$ egri chiziq ushbu

$$\begin{cases}x=\varphi(t),\\ y=\psi(t)\end{cases} \qquad (\alpha\le t\le\beta)$$

tenglamalar sistemasi bilan berilgan bo`lib,(1) shartlarning bajarilishi bilan birga $\varphi(t),\psi(t)$ funksiyalari $[\alpha,\beta]$ da uzluksiz $\varphi'(t)$ hamda $\psi'(t)$ hosilaga ega bo`lsin.

$[\alpha,\beta]$ segmentning ixtiyoriy

$$P=\{t_0,t_1,...,t_n\} \qquad (\alpha=t_0<t_1<...<t_n=\beta)$$

bo`laklanishini olib,ularga mos $A\breve{B}$ yoyning $A_k=A_k(x_k,y_k)$ $(x_k=\varphi(t_k)\ ,\ y_k=\varphi(t_k))$ nuqtalarni bir-biri bilan to`g`ri chiziq kesmasi yordamida birlashtirishdan hosil bo`lgan l siniq chiziq peremetri

$$\mu(l)=\sum_{k=0}^{n-1}\sqrt{[\varphi(t_{k+1})-\varphi(t_k)]^2+[\psi(t_{k+1})-\psi(t_k))]^2}$$

ni qaraymiz.

Lagranj teoremasidan foydalanib topamiz:

$$\mu(l)=\sum_{k=0}^{n-1}\sqrt{\varphi'^2(\tau_k)\cdot(t_{k+1}-t_k)^2+\psi'^2(\theta_k)\cdot(t_{k+1}-t_k)^2}=$$

$$=\sum_{k=0}^{n-1}\sqrt{\varphi'^2(\tau_k)+\psi'^2(\theta_k)}\cdot\Delta t_k \qquad (\Delta t_k=t_{k+1}-t_k)$$

bunda

$$\tau_k\in[t_k,t_{k+1}]. \qquad \theta_k\in[t_k,t_{k+1}].$$

Keyingi tenglikni quydagicha yozib olamiz:

$$\mu(l)=\sum_{k=0}^{n-1}\sqrt{\varphi'^2(\xi_k)+\psi'^2(\xi_k)}\cdot\Delta t_k+$$

$$+\sum_{k=0}^{n-1}[\sqrt{\varphi'^2(\tau_k)+\psi'^2(\theta_k)}-\sqrt{\varphi'^2(\xi_k)+\psi'^2(\xi_k)}]\cdot\Delta t_k \qquad (*)$$

bunda,

$$\xi_k\in[t_k,t_{k+1}].$$

Modomiki,

$$\sqrt{\varphi'^2(t)+\psi'^2(t)} \in C[\alpha,\beta]$$

ekan unda

$$\sqrt{\varphi'^2(t)+\psi'^2(t)} \in R[\alpha,\beta]$$

bo`lib,

$$\lim_{\lambda_p\to 0}\sum_{k=0}^{n-1}\sqrt{\varphi'^2(\xi_k)+\psi'^2(\xi_k)}\cdot\Delta t_k=\int_\alpha^\beta\sqrt{\varphi'^2(t)+\psi'^2(t)}dt \qquad (3)$$

bo`ladi.

Ixtiyoriy a, b, c, d haqiqiy sonlar uchun ushbu

$$\left|\sqrt{a^2+b^2}-\sqrt{c^2+d^2}\right|\le|a-c|+|b-d|$$

tengsizlik o`rinli bo`ladi.

Haqiqatdan ham,

$$\left|\sqrt{a^2+b^2}-\sqrt{c^2+d^2}\right|=\left|\frac{(a^2-c^2)+(b^2-d^2)}{\sqrt{a^2+b^2}+\sqrt{c^2+d^2}}\right|\le|a-c|\cdot$$

$$\cdot\frac{|a+c|}{\sqrt{a^2+b^2}+\sqrt{c^2+d^2}}+|b-d|\cdot\frac{|b+d|}{\sqrt{a^2+b^2}+\sqrt{c^2+d^2}}\le$$

$$\le|a-c|+|b-d|.$$

Bu tengsizlikdan foydalanib topamiz:

$$\left|\sum_{k=0}^{n-1}[\sqrt{\varphi'^2(\tau_k)+\psi'^2(\theta_k)}-\sqrt{\varphi'^2(\xi_k)+\psi'^2(\xi_k)}]\Delta t_k\right|\le$$

$$\le\sum_{k=0}^{n-1}|\varphi'(\tau_k)-\varphi'(\xi_k)|\Delta t_k+\sum_{k=0}^{n-1}|\psi'(\theta_k)-\psi'(\xi_k)|\Delta t_k\le$$

$$\le\sum_{k=0}^{n-1}\omega_k(\varphi')\cdot\Delta t+\sum_{k=0}^{n-1}\omega_k(\psi')\cdot\Delta t.$$

$$\varphi'(t)\in R[\alpha,\beta], \qquad \psi'(t)\in R[\alpha,\beta]$$

bo`lgani sababli

$$\lim_{\lambda_p\to 0}\sum_{k=0}^{n-1}[\sqrt{\varphi'^2(\tau_k)+\psi'^2(\theta_k)}-$$
$$-\sqrt{\varphi'^2(\xi_k)+\psi'^2(\xi_k)}]\Delta t_k=0 \qquad (4)$$

bo`ladi.

(3) va (4) munosabatlarni e`tiborga olib, $\lambda_p\to 0$ da (*) tenglikda limitga o`tsak, u holda $A\breve{B}$ yoyning uzunligi uchun

$$\mu(A\breve{B})=\int_\alpha^\beta\sqrt{\varphi'^2(t)+\psi'^2(t)}dt \qquad (5)$$

bo`lishi kelib chiqadi.Bu formula yordamida yoy uzunligi hisoblanadi.

2-misol. Ushbu

$$x=a(t-\sin t)$$
$$y=a(1-\cos t) \qquad (0\le t\le\pi)$$

tenglamalar sistemasi bilan berilgan $A\breve{B}$ egri chiziqning (stikloidaning) uzunligi topilsin.

Ravshanki,

$$x'(t)=a(1-\cos t)\quad,\qquad y'(t)=a\sin t\quad,$$

$$x'^2(t)+y'^2(t)=a^2(1-\cos t)^2+a^2\sin^2 t=a^2 2(1-\cos t)\ ,$$

$$\sqrt{x'^2(t)+y'^2(t)}=a\sqrt{2(1-\cos t)}$$

bo`ladi.(5) formulaga ko`ra izlanayotgan egri chiziq uzunligi

$$\mu(A\breve{B})=\int_0^{2\pi}a\sqrt{2(1-\cos t)}dt=2a\int_0^{2\pi}\sin\frac{t}{2}dt=-4a\cdot(\cos\frac{t}{2})_0^{2\pi}=8a$$

bo`ladi.

Quydagi berilgan egri chiziqlarning uzunligini toping:

1. $y=\frac{x^2}{2},\ 0\le x\le\sqrt{3}$
2. $y=chx,\ 0\le x\le 1$
3. $y^2=x^3,\ 0\le x\le 5$
4. $y=\arccos\sqrt{x}-\sqrt{x-x^2},0\le x\le 1$

16. $x=at, y=abt^2, z=\frac{2}{3}ab^2t^3,$ $0\le t\le t_0$
17. $x=3t-t^3, y=3t^2, z=3t+t^3$ $0\le t\le t_0$
18. $x=2t, y=\ln t, z=t^2,$ $0<t_1\le t\le t_2$
19. $x=a\sin^2\varphi, y=a\sin\varphi\cos\varphi,$ $z=a\ln\cos\varphi, |\varphi|\le\varphi_0<\frac{\pi}{2}$

5. $x=\frac{1}{4}y^2-\frac{1}{2}\ln y,\ 1\le y\le 2$

6. $x=1-\ln(y^2-1),\ 3\le y\le 4$

7. $x=t^2, y=\frac{t^3}{3}-t, koordinata o`qlari bilan$ *kesishgan nuqtalari*

8. $x=t^2, y=t^3, 0\le t\le 1$

9. $x=2(t-\sin t), y=2(1-\cos t), sikloida bitta$ *arkasi*

10. $x=3(2\cos t-\cos 2t), y=3(2\sin t-\sin 2t)$

11. $x=a\cos t, y=a\sin t, z=bt, 0\le t\le t_0$

12. $x=2t, y=\ln t, z=t^2, 0<t_1\le t\le t_2$

13. $x=at\cos t, y=at\sin t, z=bt, 0\le t\le t_0$

14. $x=at\cos t, y=at\sin t, z=\frac{at^2}{2}, 0\le t\le t_0$

15. $x=a\cos t, y=a\sin t, z=be^t, 0\le t\le \ln\frac{a}{b}, a>b$

20. $x=\cos^3 t, y=\sin^3 t, z=\cos 2t,$ $0\le t\le 2\pi$

21. $x=e^t, y=e^{-t}, z=\sqrt{2}t, 0\le t\le t_0$

22. $x=a\,ch t, y=b\,sh t, z=at,$ $0\le t\le t_0$

23. $x=a(t-\sin t), y=a(1-\cos t)$ $z=4a\cos\left(\frac{t}{2}\right), 0\le t\le 2\pi$

24. $x=a(t-\cos t), y=a(1-\sin t)$ $z=4a\sin\left(\frac{t}{2}\right), 0\le t\le t_0$

25. $x=at, y=a\sqrt{t}\sin t, z=a\sqrt{t}\cos t$ $0<t_1\le t\le t_2$

26. $x=ae^{k\varphi}\cos\varphi, y=ae^{k\varphi}\sin\varphi,$ $z=be^{k\varphi}, z_1\le z\le z_2$

27. $x=a\cos t, y=a\sin t, z=bt,$ $0\le t\le t_0$

28. $y=\ln x,\ \frac{3}{4}\le x\le\frac{12}{5}$

29. $y=\sqrt{\frac{x}{3}}(1-x),\ (0\le x_0<x\le 1)$

30. $x=\frac{2}{3}\sqrt{(y-1)^3}, (0\le x\le 2\sqrt{3})$

Qutb koordinatalar sistemasida berilgan egri chiziq uzunligini hisoblash.

Faraz qilaylik, $A\breve{B}$ egri chiziq qutb koordinatalar sistemasida quydagi

$$r=\rho(\theta),\quad (\alpha\le\theta\le\beta)$$

tenglama bilan berilgan bo`lsin.Bunda $\rho(\theta)\in C[\alpha,\beta]$ bo`lib,u uzluksiz $\rho'(\theta)$ hosilaga ega bo`lsin.

Qutb koordinatalari (ρ,θ) dan Dekart koordinatalari (x,y) ga o`tish formulasiga binoan

$$x=\rho(\theta)\cdot\cos\theta,$$
$$y=\rho(\theta)\cdot\sin\theta\qquad(\alpha\le\theta\le\beta)$$

bo`ladi.Natijada $A\breve{B}$ parametrik ko`rinishda

$$\varphi(\theta)=\rho(\theta)\cdot\cos\theta,$$
$$\psi(\theta)=\rho(\theta)\cdot\sin\theta\qquad(\alpha\le\theta\le\beta)$$

berilgan egri chiziq sifatida ifodalanadi,bunda $\varphi(\theta)$, $\psi(\theta)$ funksiyalari 3^0 da keltirilgan shartlarni bajaradigan funksiyalar bo`ladi.

(5) formuladan foydalanib $A\breve{B}$ egri chiziqning uzunligini topamiz:

$$\mu(A\breve{B})=\int_\alpha^\beta\sqrt{(\rho(\theta)\cdot\cos\theta)'^2+(\rho(\theta)\cdot\sin\theta)'^2}\,d\theta=$$
$$=\int_\alpha^\beta\sqrt{\rho'^2(\theta)+\rho^2(\theta)}\,d\theta.$$

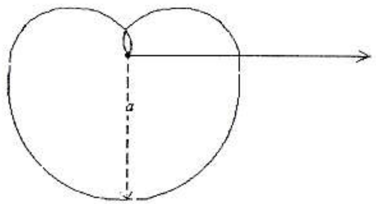
Bu formula yordamida egri chiziqning uzunligi hisoblanadi.

3-misol. Ushbu

$$\rho=a\cdot Sin^3\frac{\theta}{3}$$

tenglama bilan berilgan egri chiziqning uzunligi topilsin.

θ o`zgaruvchi 0 dan $3\pi$ gacha o`zgargandan (ρ,θ) nuqta chizmada tasvirlangan l egri chiziqni chizib chiqadi:



(2) formuladan foydalanib l chiziqning uzunligini topamiz:

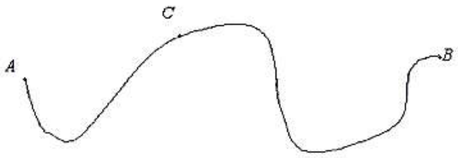
$$\mu(l)=\int_0^{3\pi}\sqrt{(a\sin^3\frac{\theta}{3})'^2+(a\sin^3\frac{\theta}{3})'^2}\,d\theta=$$
$$=\int_0^{3\pi}\sqrt{a^2\cdot\sin^4\frac{\theta}{3}\cos^2\frac{\theta}{3}+a^2\sin^6\frac{\theta}{3}}\,d\theta=$$
$$=a\int_0^{3\pi}\sin^2\frac{\theta}{3}\,d\theta=\frac{3\pi a}{2}.$$

Yoy differensiali.

Aytaylik,tekislikdagi $A\bar{B}$ egri chiziq ushbu

$$\begin{cases} x=\varphi(t)\,, \\ y=\psi(t) \end{cases} \qquad (\alpha \le t \le \beta)$$

tenglamalar sistemasi bilan berilgan bo`lib,bunda $\varphi(t)$ hamda $\psi(t)$ funksiyalari $[\alpha,\beta]$ da uzluksiz $\varphi'(t)$ hamda $\psi'(t)$ hosilaga ega bo`lsin.



Ma`lumki,$t$ o`zgaruvchining $t=\alpha$ qiymatiga $A\breve{B}$ egri chiziqda A nuqta mos keladi.

Endi ixtiyoriy $t\in[\alpha,\beta]$ ni olib unga mos $A\breve{B}$ egri chiziqdagi nuqtani C bilan belgilaylik: $C(\varphi(t),\psi(t))\ \alpha\le t\le \beta$ Ravshanki,$A\breve{C}$ yoyning uzunligi C nuqtaning $A\breve{B}$ egri chiziqdagi holatiga qarab o`zgaradi va ayni paytda $t$ ning har bir tayin qiymatida yagona $A\breve{C}$ yoyning uzunligiga ega bo`lamiz.Binobarin $A\breve{C}$ yoyning uzunligi $\mu(A\breve{C})$ t o`zgaruvchining funksiyasi bo`ladi.

$$\mu_t(A\breve{C}) \qquad (\alpha \le t \le \beta)$$

(5) formuladan foydalanib topamiz:

$$\mu_t(A\breve{C})=\int_\alpha^t\sqrt{\varphi'^2(t)+\psi'^2(t)}dt.$$

Modomiki, $\sqrt{\varphi'^2(t)+\psi'^2(t)}\in C[\alpha,\beta]$ ekan,unda $\mu_t(A\breve{C})$ funksiya hosilaga ega bo`lib,

$$(\mu_t(A\breve{C}))'=\sqrt{\varphi'^2(t)+\psi'^2(t)}$$

bo`ladi.

Keyingi tenglikning kvadratini dt^2 ga ko`paytirib,ushbu

$$(\mu_t(A\breve{C}))'^2\cdot dt^2=\varphi'^2(t)dt^2+\psi'^2(t)dt^2\ ,$$

ya`ni

$$d(\mu_t(A\breve{C}))'^2=dx^2+dy^2$$

munosabatga kelamiz.Bu munosabat yoy differensialining kvadratini ifodalaydi.Demak,yoy differensiali $d\mu_t(A\breve{C})$ yuqoridagi $x=\varphi(t)$,$y=\psi(t)$ funksiyalarning differensiallari dx hamda dy lar orqali ifodalanadi.Binobarin,(5) formula,uzluksiz hosilaga ega bo`lgan $x(t)$ , $y(t)$ funksiyalar yordamida egri chiziq yoyining turli usullarda parametrlashtirishda o`z ko`rinishini saqlaydi.

Quydagi qutb koordinatalar sistemasida berilgan egri chiziqning uzunligini hisoblang.

1. $r=a\sin\varphi$
2. $r=ae^{k\varphi},\varphi_1\le\varphi\le\varphi_2$
3. $r=a(1-\cos\varphi)$
4. $r=2(1+\cos\varphi),r\le1$
5. $r=a(1-\sin\varphi),\ -\frac{\pi}{2}\le\varphi\le-\frac{\pi}{6}$
6. $r=ath(\frac{\varphi}{2}),\ 0\le\varphi\le\varphi_0$
7. $r=a\cos^3\left(\frac{\varphi}{3}\right)$
8. $r=a\sin^4\left(\frac{\varphi}{4}\right)$
9. $r=a\cos^5\left(\frac{\varphi}{5}\right)$
10. $r=\frac{p}{\sin^2\left(\frac{\varphi}{2}\right)},\ \frac{\pi}{2}\le\varphi\le\frac{3\pi}{2}$
11. $r=\frac{a}{\sin^2\frac{\varphi}{3}}$
12. $r=\frac{a}{\cos^4\frac{\varphi}{4}}$
13. $r=a\varphi,\ 0\le\varphi\le\varphi_0$
14. $r=a\varphi^2,\ 0\le\varphi\le4$

16. $r=a\varphi^3,\ 0\le\varphi\le4$
17. $\varphi=\frac{\sqrt{r^2-a^2}}{a}-\arccos\left(\frac{a}{r}\right),\ a<r_1\le r\le r_2$
18. $r=8\cos^3\frac{\varphi}{3},\ 0\le\varphi\le\frac{\pi}{2}$
19. $r=3(1+\sin\varphi),-\frac{\pi}{6}\le\varphi\le0$
20. $r=1-\sin\varphi,\ -\frac{\pi}{2}\le\varphi\le-\frac{\pi}{6}$
21. $r=\cos^3\frac{\varphi}{3},0\le\varphi\le\frac{3\pi}{2}$
22. $r=4\varphi,\ 0\le\varphi\le\frac{3}{4}$
23. $r=2e^{-\frac{4}{3}\varphi}$
24. $2\sin^3\frac{\varphi}{3},\ 0\le\varphi\le\frac{\pi}{2}$
25. $r=3e^{\frac{3}{4}\varphi},\ -\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}$
26. $r=2(1-\cos\varphi),\ -\pi\le\varphi\le-\frac{\pi}{2}$
27. $r=3\varphi,\ 0\le\varphi\le\frac{4}{3}$
28. $r=4(1-\sin\varphi),\ 0\le\varphi\le\frac{\pi}{6}$
29. $r=5(1+\cos\varphi),\ 0\le\varphi\le\frac{\pi}{2}$

15. $r = a\varphi^3,\ 0 \le \varphi \le 3$ 30. $r = \dfrac{1}{\sin^2\dfrac{\varphi}{2}},\ 0 \le \varphi \le \dfrac{\pi}{2}$

8.9.Aniq integral yordamoda aylanma jism hajmini hisoblash.

1.Dekart koordinatalar sistemasida berilgan aylanma jismning hajmini hisoblash.

Faraz qilaylik,bizga biror T jism berilgan bo'lib,uning Oy o'qqa parallel bo'lgan kesimlarning yuzasi ma'lum bo'lsin.Bu yuza,x o'zgaruvchining funksiyasi bo'ladi,uni $S = S(x)$ orqali belgilaymiz.

Agar $S = S(x)$ funksiya $[a,b]$ kesmada uzluksiz bo'lsa,T jismning V hajmi,ushbu

$$V = \int_a^b S(x)dx \qquad (11.1)$$

formula bilan hisoblanadi.

1-misol: Ushbu $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ ellipsoid sirt bilan chegaralangan jismning hajmini toping.

Dastlab berilgan tenglama bo'yicha ellipsoidni yasaymiz.Ellopsoidni Oxz tekislikka parallel bo'lgan, $y\in[-b;b]$ kesmada o'zgaruvchi $y = p$ tekislik bilan kesamiz.Kesimda ellips hosil bo'ladi:

$$\frac{x^2}{a^2}+\frac{z^2}{c^2}=1-\frac{p^2}{b^2},1-\frac{p^2}{b^2}.0,\ \frac{x^2}{a_1^2}+\frac{z^2}{c_1^2}=1$$

Bunda ellipsning yarim o'qlari,

$a_1 = \dfrac{a}{b}\sqrt{b^2-p^2},\ c_1=\dfrac{c}{b}\sqrt{b^2-p^2}$.Bu kesimlarning yuzlari,p ga bog'liq bo'lgan,ellips bilan chegaralangan yuzaga teng bo'ladi:

$$S(p) = \pi a_1 b_1 = \frac{\pi ac}{b^2}(b^2-p^2)$$

$S(p)$ kesimlarning yuzasini (11.1) formulaga keltirib qo'yib,V jismning hajmini topamiz:

$$V = \int_{-b}^{b}\frac{\pi ac}{b^2}(b^2-p^2)dp = 2\frac{\pi ac}{b^2}\int_{-b}^{b}(b^2-p^2)dp = 2\frac{\pi ac}{b^2}\left(b^2p-\frac{p^3}{3}\right)\Big|_0^b=\frac{4}{3}\pi acb$$

Faraz qilaylik, $y = f(x)$ funksiya $[a,b]$ kesmada aniqlangan va uzluksiz bo'lib, $\forall x\in[a,b]$ uchun $f(x) > 0$ bo'lsin.Yuqoridan $y = f(x)$ funksiya grafigi,yon tomonlardan $x=a, x=b$ vertikal to'g'ri chiziqlar,pasdan Ox o'qidagi $[a,b]$ kesma bilan chegaralangan shaklni Ox o'qi atrofida aylantirishdan hosil bo'lgan aylanma T jism hajmi

$$V_x = \pi\int_a^b f^2(x)dx$$

formula bo'ycha topiladi.

Agar D egri chiziqli trapetsiya,yuqoridan $f(x)$,pastdan $g(x)$ uzluksiz egri chiziqlar bilan,yon tomonlaridan esa $x=a, x=b$ to'g'ri chiziqlar bilan chegaralangan bo'lsa,uning Ox o'qi atrofida aylantirishdan hosil bo'lgan aylanma T jismning hajmi

$$V_x = \pi\int_a^b\left[f^2(x)-g^2(x)\right]dx$$

formula bo'yicha topiladi.

$y = y(x)$ funksiya,$[\alpha,\beta]$ kesmada $x = x(t), y = y(t)$ pqarametrik tenglamalari bilan berilgan bo'lsin.Bu funksiyalar $[\alpha,\beta]$ da uzluksiz,$\forall t\in[\alpha,\beta]$ kesmada $y(t)\ge 0$ va $x(t)$ funksiya,uzluksiz,manfiy bo'lmagan $x'(t)$ hosilaga ega,hamda $a = x(\alpha)$, $b = y(\beta)$ bo'lsa,u holda, T aylanma jismning hajmi,

$$V = \pi\int_\alpha^\beta y^2(t)\cdot x'(t)dt$$

formula bilan topiladi.

Agar $x(t)$ funksiya $[\alpha,\beta]$ kesmada kamayuvchi va $a = x(\alpha)$, $b = y(\beta)$ bo'lsa, u holda,yuqoridagi shartlar bajarilganda, T aylanma jismning hajmi,

$$V_x = -\pi\int_a^b\left[f^2(x)-g^2(x)\right]dx$$

formula bo'yicha topiladi.

Oy $(x=0)$ o'q atrofida aylantirishdan hosil bo'lgan T aylanma jism hajmi yuqoridagi formulalarga asosan,quydagicha topiladi.

$$V_y = \pi\int_c^d g^2(y)dy,$$

$$V_y = \pi\int_c^d \left[f^2(y) - g^2(y)\right]dy,$$

$$V = \pi\int x^2(t)\cdot y'(t)dt$$

formulalar bo`ycha topiladi.

Misol: Quydagi, $y^2 = 2px, y = 0, x = a$ chiziqlar bilan chegaralangan shaklni, $Ox\ (y = 0)$ o`q atrofida aylantirishdan hosil bo`lganaylanma jismning hajmini toping.

D soha,yuqoridan $y^2 = 2px$ uzluksiz funksiya bilan,yon tomonlardan, $x = a$ va $x = 0$ to`g`ri chiziqlar,pastdan esa, $Ox\ (y = 0)$ o`q bilan chegaralangan.Endi, $D$ egri chiziqli sohani $Ox\ (y = 0)$ o`q atrofida aylantirishdan hosil bo`lgan aylanma jismning hajmini, $V_x = \pi\int_a^b f^2(x)dx$ formula bo`ycha hisoblaymiz:

$$V_x = \pi\int_0^a y^2\,dx = \pi\int_0^a 2pxdx = 2p\pi\frac{x^2}{2}\Big|_0^a = \pi pa^2$$

Misol: Quydagi, $x = a(t - \sin t), y = a(1 - \cos t), (1 \le t \le 2\pi), y = 0$ chiziqlar bilan chegaralangan shaklni $Ox\ (y = 0)$ o`q atrofida aylantirishdan hosil bo`lgan aylanma jismning hajmini toping.

Aylanma jismning hajmini $V = \pi\int_\alpha^\beta y^2(t)\cdot x'(t)dt$ formula bo`ycha topamiz:

$$V_y = \pi\int a^2(1-\cos t)^2 a(1-\cos t)dt = \pi a^3\int(1-\cos t)^3 dt = 8\pi a^3\int\left(\sin^2\frac{t}{2}\right)^3 dt = 5\pi a^3.$$

Quydagi berilgan egri chiziqlardan hosil bo`lgan shakl hajmini toping.(1-20) Egri chiziqni $Ox$ o`qi atrofida (21-25) Oy o`qi atrofida (26-30) hosil bo`lgan jism hajmini toping.

1. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, |z| = H$
2. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{(z-H)^2}{H^2}, z = 0$
3. $x^2 + y^2 = ax, |x| = |z|$

16. $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \ z = \pm c$
17. $\frac{x^2}{a^2} + \frac{y^2}{z^2} = 1, \ (0 < z < a)$
18. $z^2 = b(a - x), x^2 + y^2 = ax$

4. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{(z-H)^2}{H^2}, \frac{z}{H} = \frac{y}{b}, z = 0 \ (y \ge 0)$
5. $y^2 + z^2 = a^2 ch^2(\frac{x}{a}), |x| = b$
6. $4a^2x^2\left(1 - \frac{y^2}{b^2}\right) = (x^2 + z^2)^2$
7. $\frac{x^2}{(z+a)^2} + \frac{y^2}{(z-a)^2} = 1$
8. $z^2 = a(a - y), x = k_1y, \ x = k_2y, k_2 > k_1$
9. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \ \frac{z}{H} = \frac{x}{a}, \ z = 0$
10. $\left(1 - \frac{x^2}{a^2}\right)\left(1 - \frac{z}{a}\right)^2 = \frac{y^2}{b^2}, \ z = 0$
11. $\frac{y^2}{a^2} + \frac{x+z}{a} = 1, x = 0, \ y = 0, z = 0$
12. $x^2 + y^2 + z^2 + xy + yz + xz = a^2$
13. $a^2(z^2 - y^2) = x^2z^2, z = a$
14. $x^2 = b(a - y), \ y^2 + z^2 = ay$
15. $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1, \ z = c - H$

19. $x^2 + y^2 = a^2, y^2 + z^2 = a^2$
20. $x^2 + y^2 + z^2 = a^2, x^2 + y^2 = ax$
21. $x^4 + y^4 = ay^3$
22. $x^4 + y^4 = 2axy^2$
23. $(x^2 + y^2)^3 = a^4y^2$
24. $(x^2 + y^2)^3 = a^2(x^2 - y^2)^2$
25. $(x^2 + y^2)^3 = a^2x^2y^2$
26. $(x^2 + y^2)^2 = a^2(x^2 - y^2)$
27. $(x^2 + y^2)^3 = a^2x^4$
28. $(x^2 - y^2)x = a(x^2 + y^2), x = 3a$
29. $x^4 + y^4 = 2axy^2$
30. $x^4 + y^4 = a\,y^3$

2.Qutb koordinatalar sistemasida berilgan aylanma jismning hajmini hisoblash: Agar $\breve{AB}$ egri chiziqning tenglamasi qutb koordinatalar sistemasida $r = r(\varphi), \ 0 \le \alpha \le \varphi \le \beta \le 2\pi$, ko`rinishda berilgan bo`lib, $[\alpha, \beta]$ kesmada $r(\varphi)$-uzluksiz bo`lsa,u holda,qutb nuri atrofida $T = \{(r, \varphi) : \alpha \le \varphi \le \beta, 0 \le r \le r(\varphi)\}$ aylanma sektorning hajmi,

$$V = \frac{2\pi}{3}\int_\alpha^\beta r^3(\varphi)\sin\varphi\, d\varphi$$

formula bo`ycha topiladi.

Agar $\breve{AB}$ egri chiziqning tenglamasi qutb koordinatalar sistemasida $r = r(\varphi), \ -\frac{\pi}{2} \le \alpha \le \varphi \le \beta \le \frac{\pi}{2}$, ko`rinishda berilgan bo`lib, $[\alpha, \beta]$ kesmada $r(\varphi)$-uzluksiz bo`lsa,u holda, $\varphi = \frac{\pi}{2}$ qutb nuri atrofida $T = \{(r, \varphi) : \alpha \le \varphi \le \beta, 0 \le r \le r(\varphi)\}$ aylanma sektorning hajmi,

$$V = \frac{2\pi}{3}\int_{\alpha}^{\beta} r^{3}(\varphi)\cos\varphi\, d\varphi$$

formula bo`ycha topiladi.

Misol: Ushbu $r = a(1+\cos\varphi)$ kardiodaning qutb o`qi atrofida aylantirishdan hosil bo`lgan aylanma jismning hajmini toping.

$r = a(1+\cos\varphi)$ kardiodaning qutb o`qi atrofida aylantirishdan hosil bo`lgan aylanma jismning hajmini $V = \frac{2\pi}{3}\int_{\alpha}^{\beta} r^{3}(\varphi)\sin\varphi\, d\varphi$ formulaga ko`ra, topamiz:

$$V = \frac{2\pi}{3}\int r^{3}(\varphi)\sin\varphi\, d\varphi = \frac{2\pi}{3}\int a^{3}(1-\cos\varphi)^{3}\sin\varphi\, d\varphi = \frac{\pi}{6}a^{3}(1-\cos\varphi)^{4} = \frac{8\pi}{3}a^{3}.$$

Qutb koordinatalar sistemasida berilgan aylanma jismning hajmini hisoblang:

1. $0 \le r \le a\varphi, \quad 0 \le \varphi \le \pi$
2. $\pi r^{3} \le \varphi \le \pi$
3. $0 \le r \le a\cos^{2}\varphi$
4. $0 \le r \le a\sqrt[3]{\cos 3\varphi}, \quad \frac{7\pi}{6} \le \varphi \frac{3\pi}{2}$
5. $0 \le r \le ae^{k\varphi}, \ 2\pi n \le \varphi \le 2\pi n + \pi$
6. $0 \le r \le a(1+\cos\varphi)$
7. $0 \le r \le 2a\frac{\sin^{2}\varphi}{\cos\varphi}, 0 \le \varphi \le \frac{\pi}{3}$
8. $0 \le r \le a\sqrt{\sin\varphi}$
9. $0 \le r \le a\cos 2\varphi$
10. $0 \le r \le \frac{p}{1+\cos\varphi}, |\varphi| \le \frac{\pi}{2}$

16. $0 \le r \le \frac{p}{1+e\cos\varphi}, 0 \le \varphi \le \frac{\pi}{2}$
17. $0 \le r \le -\frac{3a\cos 2\varphi}{(2+\cos 2\varphi)\sin\varphi}, \frac{\pi}{4} \le \varphi \le \frac{3\pi}{4}$
18. $r = 4a + \frac{a}{\cos\varphi}$
19. $r = 2a + \frac{a}{\cos\varphi}, \varphi = \frac{\pi}{2}$
20. $r^{2} = a^{2}\sin 2\varphi, \varphi = \frac{\pi}{4}$
21. $r^{2} = a^{2}\sin 2\varphi, \varphi = -\frac{\pi}{4}$
22. $r = 1+\cos t, \varphi = t - tg\frac{t}{2}, (0 \le t \le T < \pi)$
23. $\varphi = \sqrt{r}, (0 \le r \le 5)$
24. $\varphi = \int_{0}^{r}\frac{shp}{p}dp, \ (0 \le r \le R)$
25. $\varphi = \frac{1}{2}\left(r + \frac{1}{r}\right), \ 1 \le r \le 3$

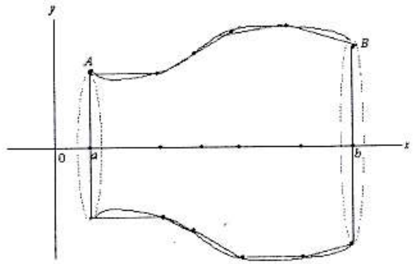
11. $0 \le r \le a\frac{\cos 2\varphi}{\cos\varphi}, |\varphi| \le \frac{\pi}{4}$
12. $0 \le r \le a\sqrt{\cos 2\varphi}$
13. $0 \le r \le a\sqrt[3]{\cos 3\varphi}, \frac{\pi}{2} \le \varphi \le \frac{5\pi}{6}$
14. $0 \le r \le a\sqrt[3]{\cos 3\varphi}, |\varphi| \le \frac{\pi}{6}$
15. $0 \le r \le a\sin^{2}\varphi$

26. $r = ath\frac{\varphi}{2}, (0 \le \varphi \le 2\pi)$
27. $r = ae^{m\varphi} \ (m > 0), 0 < r < a$
28. $r = a\varphi, 0 \le \varphi \le 2\pi$
29. $r = a\varphi, 0 \le \varphi \le \pi$
30. $r = a\varphi, 0 \le \varphi \le 3\pi$

8.10. Aniq integral yordamida aylanma jismning sirt yuzini hisoblash.

1°. Aylanma sirt va uning yuzi tushunchasi. Ma'lumki, to`g`ri chiziq kesmasini biror o`q atrofida aylantirishdan silindrik, konus (kesik konus) sirtlar hosil bo`ladi. Bu sirtlar yuzaga egava ular ma`lum formulalar yordamida topiladi.

Aytaylik, $f(x) \in C[a,b]$ bo`lib, $\forall x \in [a,b]$ da $f(x) \ge 0$ bo`lsin. Bu funksiya grafigi $\breve{AB}$ yoyini tasvirlasin.



$\breve{AB}$ yoyni Ox o`qi atrofida aynaltirishdan hosil bo`lgan sirt aylanma sirt deyiladi. Uni Π deylik. $[a,b]$ segmentning ixtiyoriy

$$P = \{x_0, x_1, ..., x_n\} \qquad (a = x_0 < x_1 < ... < x_n = b)$$

bo`laklashini olaylik. Bu bo`laklashning har bir

$$x_k \ (k = 0,1,2,...,n)$$

bo`luvchi nuqtalari orqali $Oy$ o`qiga parallel to`g`ri chiziqlar o`tkazib, ularning $\breve{AB}$ yoyi bilan kesishish nuqtalarini $A_k = A_k(x_k, f(x_k))$ bilan belgilaylik.

$(A_0 = A,\ A_n = B;\ \ k = 0,1,2,...,n)$ Bu nuqtalarni o`zaro to`g`ri chiziq kesmalari bilan birlashtirib, $A\breve{B}$ yoyiga L siniq chiziq chizamiz.

$A\breve{B}$ yoyini Ox o`qi atrofida aylantirish bilan birga $L$ siniq chiziqni ham shu o`q atrofida aylantiramiz.Natijada kesik konus sirtlarining birlashmasidan tashkil topgan K sirt hosil bo`ladi.Bu K sirt yuzaga ega va uning yuzi

$$\mu(K) = 2\pi\sum_{k=0}^{n-1}\frac{f(x_k)+f(x_{k+1})}{2}\cdot\sqrt{(x_{k+1}-x_k)^2+[f(x_{k+1})-f(x_k)]^2}$$

ga teng.(Bunda kesik konusning yon sirtini toppish formulasidan foydalanildi).

Ravshanki, K sirt, binobarn uning yuzi $\mu(K)$ $[a,b]$ segmentning bo`lakashlariga bog`liq bo`ladi.

1-ta'rif. Agar $\forall\varepsilon > 0$ son olinganda ham shunday $\delta > 0$ son topilsaki, $[a,b]$ segmentning diametri $\lambda_p < \delta$ bo`lgan ixtiyoriy $P$ bo`laklashi uchun

$$|\mu(K) - S| < \varepsilon \quad (S \in R)$$

tengsizlik bajarilsa, S son $\mu(K)$ ning $\lambda_p \to 0$ dagi limiti deyiladi:

$$\lim_{\lambda_p\to 0}\mu(K) = S.$$

2-ta'rif. Agar $\lambda_p \to 0$ da $\mu(K)$ yig`indi chekli $S$ limitga ega bo`lsa, Π aylanma sirt yuzaga ega deyiladi.

Bunda S son Π aylanma sirtning yuzi deyiladi:

$$S = \mu(\Pi).$$

Demak,

$$\mu(\Pi) = \lim_{\lambda_p\to 0} 2\pi\sum_{k=0}^{n-1}\frac{f(x_k)+f(x_{k+1})}{2}\cdot\sqrt{(x_{k+1}-x_k)^2+[f(x_{k+1})-f(x_k)]^2}.$$

2^0. Aylanma sirtning yuzini hisoblash. Faraz qilaylik, $f(x) \in C[a,b]$ bo`lib, u $[a,b]$ segmentda uzluksiz $f'(x)$ hosilaga ega bo`lsin.

Bu funksiya grafigi $A\breve{B}$ yoyini Ox o`qi atrofida aylantirishdan hosil bo`lgan Π aylanma sirt yuzini topamiz.

$[a,b]$ segmentning ixtiyoriy P bo`laklashini olib,yuqoridagidek

$$\mu(K) = 2\pi\sum_{k=0}^{n-1}\frac{f(x_k)+f(x_{k+1})}{2}\cdot\sqrt{(x_{k+1}-x_k)^2+[f(x_{k+1})-f(x_k)]^2}$$

yig`indini topamiz.

Lagranj teoremasiga ko`ra

$$f(x_{k+1}) - f(x_k) = f'(\xi_k)(x_{k+1}-x_k) = f'(\xi_k)\cdot\Delta x_k$$

bo`ladi,bunda $\xi_k \in [x_k, x_{k+1}]$. Natijada

$$\mu(K) = 2\pi\sum_{k=0}^{n-1}\frac{f(x_k)+f(x_{k+1})}{2}\sqrt{1+f'^2(\xi_k)}\Delta x_k$$

bo`ladi.

Keyingi tenglikni quydagicha yozib olamiz:

$$\mu(K) = 2\pi\sum_{k=0}^{n-1}f(\xi_k)\sqrt{1+f'^2(\xi_k)}\Delta x_k + \pi\left\{\sum_{k=0}^{n-1}[(f(x_k) - f(\xi_k)) + (f(x_{k+1}) - f(\xi_k))]\sqrt{1+f'^2(\xi_k)}\Delta x_k\right\}. \quad (1)$$

$f'(x) \in C[a,b]$ bo`lgani sababli

$$f(x)\sqrt{1+f'^2(x)} \in R[a,b]$$

bo`ladi.Demak, $\lambda_p \to 0$ da

$$2\pi\sum_{k=0}^{n-1}f(\xi_k)\sqrt{1+f'^2(\xi_k)}\Delta x_k \to 2\pi\int_a^b f(x)\sqrt{1+f'^2(x)}dx \quad (2)$$

Ravshanki,

$$\sqrt{1+f'^2(x)} \in C[a,b].$$

Demak,bu funksiya $[a,b]$ da o`zining maksimum qiymatiga ega bo`ladi. Uni M deylik:

$$M = \max_{a\le x\le b}\sqrt{1+f'^2(x)}.$$

$f(x)$ funksiya $[a,b]$ segmentda tekis uzluksiz.Unda $\forall\varepsilon > 0$ olinganda ham, $\frac{\varepsilon}{2M(b-a)}$ ga ko`ra shunday $\delta > 0$ son topiladiki, $\lambda_p < \delta$ bo`lganda

$$|f(x_k) - f(\xi_k)| < \frac{\varepsilon}{2M(b-a)}, \quad |f(x_{k+1}) - f(\xi_k)| < \frac{\varepsilon}{2M(b-a)}$$

bo`ladi. Shularni e`tiborga olib topamiz:

$$\left|\left\{\sum_{k=0}^{n-1}[(f(x_k) - f(\xi_k)) + (f(x_{k+1}) - f(\xi_k))]\sqrt{1+f'^2(\xi_k)}\cdot\Delta x_k\right\}\right| \le$$

$$\le \sum_{k=0}^{n-1}\left[|f(x_k) - f(\xi_k)| + |f(x_{k+1}) - f(\xi_k)|\right]\sqrt{1+f'^2(\xi_k)}\cdot\Delta x_k <$$

$$< M\left[\frac{\varepsilon}{2M(b-a)} + \frac{\varepsilon}{2M(b-a)}\right]\cdot\sum_{k=0}^{n-1}\Delta x_k < \varepsilon.$$

Bundan $\lambda_p \to 0$ da

$$\sum_{k=0}^{n-1}[(f(x_k) - f(\xi_k)) + (f(x_{k+1}) - f(\xi_k))]\cdot\sqrt{1+f'^2(\xi_k)}\Delta x_k \to 0 \quad (3)$$

bo`lishi kelib chiqadi.

$\lambda_p \to 0$ da (1) tenglikda limitga o`tib,(bunda(2) va (3) munosabatlarni e'tiborga olib) aylanma sirt yuzi uchun

$$\mu(\Pi)=2\pi\int_a^b f(x)\sqrt{1+f'^2(x)}\,dx \qquad (4)$$

bo`lishini topamiz.

1-misol. Ushbu $f(x)=\frac{a}{2}(e^{\frac{x}{a}}+e^{\frac{-x}{a}}),\quad a>0,\ 0\le x\le a$

zanjir chizig`ini $Ox$ o`q atrofida aynaltirishdan hosil bo`lgan aylanma sirt yuzi topilsin.

Ravshanki,

$$f'(x)=\frac{1}{2}(e^{\frac{x}{a}}-e^{\frac{-x}{a}})$$

(4) formuladan foydalanib,izlanayotgan aylanma sirtning yuzini topamiz:

$$\mu(\Pi)=2\pi\int_0^a\frac{a}{2}(e^{\frac{x}{a}}+e^{-\frac{x}{a}})\sqrt{1+\frac{1}{4}(e^{\frac{x}{a}}-e^{-\frac{x}{a}})^2}\,dx=$$

$$=\frac{\pi a}{2}\int_0^a(e^{\frac{x}{a}}+e^{-\frac{x}{a}})^2dx=\frac{\pi a}{2}\int_0^a(e^{\frac{2x}{a}}+2+e^{-\frac{2x}{a}})dx=$$

$$=\frac{\pi a}{2}\left[\frac{a}{2}e^{\frac{2x}{a}}+2x-\frac{a}{2}e^{-\frac{2x}{a}}\right]_0^a=\frac{\pi a^2}{4}(e^2-e^{-2}+4).$$

Aytaylik, $A\bar{B}$ egri chiziq yuqori yarim tekislikda joylashgan bo`lib,u ushbu

$$\begin{cases}x=\varphi(t)\\ y=\psi(t)\end{cases}\quad(\alpha\le t\le\beta)$$

parametrik tenglamalar sistemasi bilan berilgan bo`lsin.Bunda $\varphi(t)$, $\psi(t)$ funksiyalari $[\alpha,\beta]$ da uzluksiz va uzluksiz $\varphi'(t),\psi'(t)$ hosilaga ega.Bu egri chiziqni $Ox$ o`qi atrofida aylantirishdan hosil bo`lgan aylanma sirtning yuzi

$$\mu(\Pi)=2\pi\int_\alpha^\beta\psi(t)\sqrt{\varphi'^2(t)+\psi'^2(t)}\,dt \qquad (5)$$

bo`ladi.

2-misol. Ushbu $x^2+(y-2)^2=1$

aylanani Ox o`qi atrofida aylantirishdan hosil bo`lgan aylanma sirtning (torning) yuzi topilsin.

Aylananing tenglamasini quydagicha

$$x=\varphi(t)=\cos t \qquad (0\le t\le 2\pi)$$
$$y=\psi(t)=2+\sin t$$

parametrik ko`rinishda yozamiz.

Izlanayotgan aylanma sirtning yuzi, (5) formulaga ko`ra

$$\mu(\Pi)=2\pi\int_0^{2\pi}(2+\sin t)\sqrt{(\cos t)'^2+(2+\sin t)'^2}\,dt=$$

$$=2\pi\int_0^{2\pi}(2+\sin t)dt=8\pi^2$$

bo`ladi.

Quydagi egri chiziqlarning aylanishidan hosil bo`lgan sirtlarning yuzalari aniqlansin.

1. $x^2+y^2=R^2$, Ox o`qi atrofida
2. $y=\frac{x^2}{2}$ ning $y=1.5$ to`g`ri chiziq bilan kesishgan qismini, Oy o`qi atrofida
3. $y=ach\frac{x}{a}$ ning $x=\pm a$ to`g`ri chiziqlar orasidagi qismini, Ox o`qi atrofida
4. $4x^2+y^2=4$, Oy o`q atrofida.(Ko`rsatma. y ni erkli o`zgaruvchi deb olsak,izlangan sirt yuzi $P=\pi\int_0^2\sqrt{16-3y^2}\,dy$ bo`ladi.So`ngra $y=\frac{4}{\sqrt{3}}\sin t$ o`rniga qo`yishni tatbiq etamiz.
5. $y=\sin x$ egri chiziqning bitta yarim to`lqini ,$Ox$ o`qi atrofida
6. $\begin{cases}x=a(t-\sin t)\\ y=a(1-\cos t)\end{cases}$ stikloidaning bir davri, Ox o`qi atrofida
7. $x=t^2$, $y=\frac{t}{3}(t^2-3)$ egri chiziq ilmog`i, $Ox$ o`qi atrofida
8. $x^2+y^2=a^2$, $x=b>a$ to`g`ri chiziq atrofida.
Ko`rsatma: $dP=2\pi(b+x)ds+2\pi(b-x)ds$
9. $y=\frac{x^3}{3}$ egri chiziqning $x=-2$ dan $x=2$ gacha bo`lgan yoyi
10. $y^2=4+x$ egri chiziqni $x=2$ to`g`ri chiziq bilan kesilgan qismi.
11. $x=a\cos^2 t$, $y=a\sin^2 t$ egri chiziqning barcha yoyi.
12. $x=\frac{t^3}{3}$, $y=4-\frac{t^2}{2}$ egri chiziqning koordinata o`qlari bilan kesishgan nuqtalari orasidagi qismi.

13. $y=x^3,\ x\in\left[0;\sqrt[4]{1/3}\right],$ Ox o`qi atrofida.

14. $9y^2=x(3-x)^2,\ x\in[0;3],$ Ox o`qi atrofida.

15. $x^2+y^2=9,\ x\in[-2;1],$ Ox o`qi atrofida.

Quydagi berilgan egri chiziqni Ox o`qi atrofida aylantirishdan hosil bo`lgan sirt yuzini toping.(16-30)

16. $y=\sqrt{x},\ \frac{5}{4}\le x\le 21$

17. $y=x^3,\ 0\le x\le 1$

18. $y=e^{-x},\ 0\le x\le a$

19. $y=x\sqrt{\frac{x}{a}},\ (0\le x\le a)$

20. $\pm x=a\ln\frac{a+\sqrt{a^2-y^2}}{y}-\sqrt{a^2-y^2}$

21. $y=a\cos\frac{\pi x}{2b},\ (|x|\le b)$

22. $2ay=a^2+x^2,\ 0\le x\le a$

23. $y=tgx,\ 0\le x\le\frac{\pi}{4}$

24. $y=\sqrt{x^2-1},\ 1\le x\le 5$

25. $y=\frac{1}{x},\ 1\le x\le a$

26. $y=\frac{2x^{\frac{3}{2}}}{3},\ 0\le x\le 1$

27. $y=\frac{1}{x},\ 1\le x\le a$

28. $y=\sqrt{x^2+1},\ 0\le x\le\frac{1}{4}$

29. $\frac{x^2}{2}+y^2=1,\ 1\le x\le 2,\ y\ge 0$

30. $x^2+y^2=2y$

8.11. Aniq integralning mexanik va fizik masalalarga tadbiqlari.

1^0. Inersiya momenti. Mexanikada moddiy nuqta harakati muhim tushunchalardan biri hisoblanadi.

Odatda,o`lchamlari yetarli darajada kichik va massaga ega bo`lgan jism moddiy nuqta deb qaraladi.

Aytaylik,tekislikda m massaga ega bo`lgan $A$ moddiy nuqta berilgan bo`lib, bu nuqtadan biror l o`qqacha (yoki $O$ nuqtagacha) bo`lgan masofa r ga teng bo`lsin.

Ushbu

$$J=mr^2$$

miqdor A moddiy nuqtanig l o`qqa (O nuqtaga) nisbatan inersiya momenti deyiladi.

Masalan, $A=A(x,y)$ moddiy nuqtaning koordinata o`qlari hamda koordinata boshiga nisbatan inersiya momentlari mos ravishda

$$J_x=my^2\ ,\ J_y=mx^2\ ,\ J_0=m\sqrt{x^2+y^2}$$

bo`ladi.

Tekislikda, har biri mos ravishda

$$m_0,m_1,m_2,\dots,m_{n-1}$$

massaga ega bo`lgan moddiy nuqtalar sistemasi

$$\{A_0,A_1,A_2,\dots,A_{n-1}\}$$

ning biror l o`qqa (O nuqtaga) nisbatan inersiya momenti ushbu

$$J_n=\sum_{k=0}^{n-1}m_k r_k^{\ 2}$$

yig`indi bilan ta'riflanadi,bunda $r_k-A_k$ nuqtadan $l$ o`qqacha (O nuqtagacha) bo`lgan masofa $(k=0,1,2,\dots,n-1)$.

Faraz qilaylik, $y=f(x)$ egri chiziq yoyi $A\breve{B}$ bo`ycha zichligi $\rho=1$ ga teng massa tarqalgan bo`lib,bunda $f(x)$ funksiya $[a,b]$ segmentda uzluksiz va uzluksiz $f'(x)$ hosilaga ega bo`lsin.

Ravshanki, bu holda massa yoy uzunligiga teng bo`ladi:

$$m=\int_a^b\sqrt{1+f'^2(x)}\,dx.$$

$[a,b]$ segmentning ixtiyoriy

$$P=\{x_0,x_1,\dots,x_n\}\quad(a=x_0<x_1<\dots<x_n=b)$$

bo`laklashini olamiz. Bu bo`laklash $A\breve{B}$ yoyni

$$A_k=A_k(x_k,f(x_k))\quad(k=0,1,2,\dots,n-1)$$

nuqtalar bilan n ta $A_k\breve{A}_{k+1}$ $(A_0=A,\ A_{n-1}=B)$ bo`lakka ajratadi.Bunda $A_k\breve{A}_{k+1}$ bo`lakning massasi

$$m_k=\int_{x_k}^{x_{k+1}}\sqrt{1+f'^2(x)}\,dx\qquad(k=0,1,2,\dots,n-1)$$

bo`ladi. O`rta qiymat haqidagi teoremadan foydalanib topamiz:

$$m_k=\sqrt{1+f'^2(\xi_k)}\cdot\Delta x_k,$$

bunda,

$$\xi_k\in[x_k,x_{k+1}]\ ,\Delta x_k=x_{k+1}-x_k.$$

Ma'lumki,

$$(\xi_k,f(\xi_k))\quad(k=0,1,2,\dots,n-1)$$

Moddiy nuqtaning koordinata o`qlariga hamda koordinata boshiga nisbatan inersiya momentlari mos ravishda

$$J'_{x_k}=m_k\cdot f^2(\xi_k)=f^2(\xi_k)\cdot\sqrt{1+f'^2(\xi_k)}\cdot\Delta x_k,$$
$$J'_{y_k}=m_k\cdot\xi_k^2=\xi_k^2\cdot\sqrt{1+f'^2(\xi_k)}\cdot\Delta x_k,$$
$$J'_0=m_k(\xi_k^2+f^2(\xi_k))=(\xi_k^2+f^2(\xi_k))\cdot\sqrt{1+f'^2(\xi_k)}\cdot\Delta x_k$$
bo`ladi. Unda ushbu
$$\{(\xi_0,f(\xi_0)),(\xi_1,f(\xi_1)),\dots,(\xi_{n-1},f(\xi_{n-1}))\}$$
moddiy nuqtalar sistemasining inersiya momenti mos ravishda
$$J_x^{(n)}=\sum_{k=0}^{n-1}f^2(\xi_k)\cdot\sqrt{1+f'^2(\xi_k)}\cdot\Delta x_k,$$
$$J_y^{(n)}=\sum_{k=0}^{n-1}\xi_k^2\cdot\sqrt{1+f'^2(\xi_k)}\cdot\Delta x_k,$$
$$J_0^{(n)}=\sum(\xi_k^2+f^2(\xi_k))\cdot\sqrt{1+f'^2(\xi_k)}\cdot\Delta x_k,$$
tengliklar bilan ifodalanadi.

Agar P bo`laklashning diametri λ_p nolga intila borsa, unda har bir $A_k\breve{A}_{k+1}$ yoyning uzunligi ham nolga intila borib,yuqoridagi
$$J_x^{(n)},\ J_y^{(n)},\ J_0^{(n)},$$
yig`indining limitini massaga ega bo`lgan $A\breve{B}$ egri chiziqning mos ravishda koordinata boshi hamda koordinata o`qlariga nisbatan inersiya momentlarini ifodalaydi deb qarash mumkin.

Ayni paytda,
$$\lim_{\lambda_p\to 0}J_x^{(n)}=\int_a^b f^2(x)\sqrt{1+f'^2(x)}\,dx\ ,$$
$$\lim_{\lambda_p\to 0}J_y^{(n)}=\int_a^b x^2\sqrt{1+f'^2(x)}\,dx\ ,$$
$$\lim_{\lambda_p\to 0}J_0^{(n)}=\int_a^b (x^2+f^2(x))\sqrt{1+f'^2(x)}\,dx$$
bo`ladi.

Demak,massaga ega bo`lgan $A\breve{B}$ egri chiziqning koordinata o`qlariga hamda koordinata boshiga nisbatan inersiya momentlari aniq integrallar yordamida topiladi:

$$J_x=\int_a^b f^2(x)\sqrt{1+f'^2(x)}\,dx\ ,$$
$$J_y=\int_a^b x^2\sqrt{1+f'^2(x)}\,dx\ ,$$
$$J_0=\int_a^b (x^2+f^2(x))\sqrt{1+f'^2(x)}\,dx.$$

**2^0. O`zgaruvchi kuchning bajargan ishi.** Biror jismni $Ox$ o`qi bo`ylab,shu o`q yo`nalishida bo`lgan $F=F(x)$ kuch ta'siri ostida a nuqtadan b nuqtaga $(a<b)$ o`tkazish uchun bajariladigan ishni topish lozim bo`lsin.

Ravshanki,jismga ta'sir etuvchi kuch o`zgarmas,ya'ni
$$F(x)=C-const$$
bo`lsa,unda jismni $a$ nuqtadan $b$ nuqtaga o`tkazish uchun bajarilgan ish
$$A=C\cdot(b-a)$$
ga teng bo`ladi

Aytaylik, jismga ta'sir etuvchi kuch x ga $(x\in[a,b])$ bog`liq bo`lib, u $[a,b]$ da uzluksiz bo`lsin:
$$F=F(x)\in C[a,b].$$
$[a,b]$ segmentning ixtiyoriy
$$P=\{x_0,x_1,\dots,x_n\}\quad (a=x_0<x_1<\dots<x_n=b)$$
bo`laklashini olib,bu bo`laklashning har bir
$$[x_k,x_{k+1}]\qquad (k=0,1,2,\dots,n-1)$$
bo`lakchasida ixtiyoriy ξ_k $\quad\xi_k\in[x_k,x_{k+1}];\ (k=0,1,2\dots,n-1)$ nuqta olamiz.

Agar har bir $[x_k,x_{k+1}]$ da jismga ta'sir etuvchi kuchni o`zgarmas va u $F(\xi_k)$ ga teng deyilsa,u holda $[x_k,x_{k+1}]$ oraliqda bajarilgan ish (kuch ta'sirida jismni $x_k$ nuqtadan $x_{k+1}$ nuqtaga o`tkazish uchun bajarilgan ish) taxminan
$$F(\xi_k)\cdot(x_{k+1}-x_k)$$
formula bilan, $[a,b]$ oraliqda bajarilgan ish esa,taxminan
$$A\approx\sum_{k=0}^{n-1}F(\xi_k)\cdot(x_{k+1}-x_k)=\sum_{k=0}^{n-1}F(\xi_k)\cdot\Delta x_k \qquad (1)$$
formula bilan ifodalanadi.

P bo`laklashning diametri $\lambda_p$ nolga intila borganda yuqoridagi yig`indining qiymati izlanayotgan ish miqdorini tobora aniqroq ifodalaydi. Bu hol $\lambda_p\to 0$ da (1) yig`indining chekli limitini bajarilgan ish deyilishi mumkinligini ko`rsatadi.

Demak,

$$A=\lim_{\lambda_p\to 0}\sum_{k=0}^{n-1}F(\xi_k)\cdot\Delta x_k.$$

Modomiki, $F(x)\in C[a,b]$ ekan,

$$\lim_{\lambda_p\to 0}\sum_{k=0}^{n-1}F(\xi_k)\cdot\Delta x_k=\int_a^b F(x)dx$$

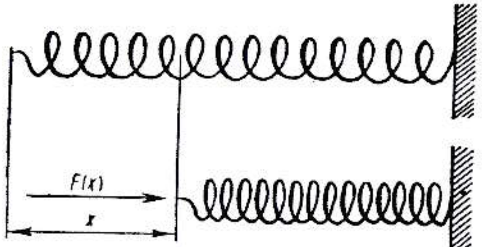
bo`ladi.

Shunday qilib,o`zgaruvchi $F(x)$ kuchning $[a,b]$ dagi bajargan ishi

$$A=\int_a^b F(x)dx \qquad (2)$$

formula bilan ifodalanadi.

Misol. Vintsimon prujinaning bir uchi mustahkamlangan,ikkinchi uchi esa $F=F(x)$ kuch ta'sir etib,prujina qisilgan.



Agar prujinaning qisilishi unga ta'sir etayotgan $F(x)$ kuchga proparsional bo`lsa,prujinani a birlikka qisish uchun $F(x)$ kuchning bajargan ishi topilsin.

Agar $F(x)$ kuch ta'sirida prujinaning qisilish miqdorini x orqali belgilasak, u holda

$$F(x)=kx$$

bo`ladi,bunda $k$-proporstionallik koeffisenti(qisilish koeffisenti).(2) formulaga ko`ra bajarilgan ish

$$A=\int_0^a kxdx=\frac{ka^2}{2}$$

bo`ladi.

Quydagi berilgan egri chiziqning M_x va M_y statik momentini toping.(1-6)

1. $x^2+y^2=a^2,\ y\geq 0$

4. $x=a\sin t, y=b\cos t, 0\leq t\leq\frac{\pi}{2}, a>b$

2. $y=ach\left(\frac{x}{a}\right), 0\leq x\leq a$

5. $y^2-p^2=2px,\ x\leq 0$

3. $r=2a\cos\varphi,\ 0\leq\varphi\leq 2\pi$

6. $r=a(1+\cos\varphi),\ -\pi\leq\varphi\leq\pi$

Egri chiziqning massa markazining x_c va y_c koordinatalarini toping.(1-10):

1. $x=R\cos\varphi, y=R\sin\varphi, |\varphi|\leq a\leq\pi$

6. $y=\sin x,\ 0\leq x\leq\pi, y=0$

2. $y=ach\left(\frac{x}{a}\right), |x|\leq b$

7. $y^2=2px,\ x^2=2py$

3. $x=\frac{y^2}{4}-\frac{\ln y}{2}, 1\leq y\leq 2$

8. $\sqrt{x}+\sqrt{y}=\sqrt{a},\ x=0, y=0$

4. $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}},\ x\geq 0,\ y\geq 0$

9. $y^2=ax^3-x^4$

5. $y=h\left(1-\frac{x^2}{a^2}\right), y=0,\ h>0,\ a>0$

10. $y=\cos x,\ 0\leq x\leq\pi,\ y=0$

1. $x=0,\ x=a,\ y=0$ va $y=b$ chiziqlar bilan chegaralangan to`g`ri to`rtburchakning $Ox$ va $Oy$ o`qlariga nisbatan inersiya momentlari aniqlansin.

2. $a^2y=bx^2,\ x=a$ va $y=0$ chiziqlar bilan chegaralangan yuzning og`irlik markazi topilsin.

3. Massasi m bo`lgan jism yerdan $h$ balandlikka ko`tarish uchun sarf etish kerak bo`lgan ish aniqlansin.

4. Qozonning asosining radiusi $R=0.4m$ chuqurligi esa $H=0.5\ m$ dan iborat aylanish paraboloid shaklida.Suv to`ldirilgan shunday qozondan barcha suvni tortib chiqarish uchun sarf etiladigan ish aniqlansin.

5. Uzunligi 1 m, kesim radiusi 2 mm bo`lgan mis simni 0.001 $m$ cho`zilishi uchun sarflanadigan ishni toping.($E=12000\,kg/mm^2$)

6. Asosi $S=420\ sm^2$,balandligi $H=40\ sm$ bo`lgan silindrik idishdagi suv silindir tubidagi yuzi $s=2\ sm^2$ bo`lgan teshikdan qancha vaqtda oqib tamom bo`ladi. $\mu=0.6$

7. Balandligi h, asosi a suv yuziga parallel,unga qarshi uchi esa suv yuzida bo`lgan uchburchakli vertikal yuzaga bo`lgan suv bosimi aniqlansin.

8. $x=a\cos t$ va $y=a\sin t$ doira chorak yuzining Ox o`qiga nisbatan inersiya momenti topilsin.

9. $y=4-x^2$ va $y=0$ chiziqlar bilan chegaralangan yuzning og`irlik markazi koordinatalari topilsin.

10.Balandligi h ga teng to`g`ri burchakli shlyuz shunday x chuqurlikda ikki gorizantal bo`lakka ajratilsinki,ularga bo`lgan suv bosimi bir xil bo`lsin.

11.$2R$ diametri suv yuzasida joylashgan vertikal yarim doira bo`lgan suv bosimi aniqlansin.

12.Asosi 4 m ga teng va suv yuziga parallel joylashgan parabolik segmentning uchi 4 m chuqurlikda yotadi.O`sha segmentga bo`lgan suv bosimi aniqlansin.

13.$x=2, y=x^2$ va $y=0$ chiziqlar bilan chegaralangan yuzning Oy o`qiga nisbatan inersiya momenti topilsin.

14. $x=0, y=0$ va $\frac{x}{a}+\frac{y}{b}=1$ chiziqlar bilan chegaralangan uchburchakning Ox va Oy o`qlariga nisbatan inersiya momentlari aniqlansin.

IX BOB
SONLI QATORLAR

9.1. Yaqinlashuvchi qatorlar va ularning yig`indisi.

Faraz qilaylik,sonlarning biror cheksiz ketma-ketligi berilgan bo`lsin:

$$a_1,\ a_2,\ a_3,...,\ a_n,....$$

Bu sonlardan tuzilgan ushbu

$$a_1+a_2+a_3+...+a_n+... \qquad (1)$$

ifoda cheksiz qator (qisqacha-qator) deyiladi.

$\{a_n\}$ ketma-ketlik hadlari qatorning hadlari deyiladi.(1) ifodada qo`shish amali qatnashganisababli qatorni $\sum_{n=1}^{\infty} a_n$ ko`rinishda ham yoziladi.

Agar n tayinlangan bo`lsa, $a_n$-qatorning $n$-hadi deyiladi,agar $n$ umumiy holda berilsa, $a_n$- qatorning umumiy hdi deyiladi.Umumiy had yordamida berilgan qatorning ixtiyoriy hadini yozib olish mumkin.Masalan,agar $a_n=\frac{1}{2^n}$ bo`lsa u holda qator

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+...+\frac{1}{2^n}+... \text{ yoki } \sum_{n=1}^{\infty}\frac{1}{2^n}$$

ko`rinishda bo`ladi.

$$S_n=a_1+a_2+a_3+...+a_n \qquad (2)$$

Bu yig`indini (1) qatorning $n$-xususiy yig`indisi deyiladi.Bunda S_1 deganda a_1 ni qarashga kelishamiz.

(2) da n ga 1,2,3, ... qiymatlar berib,quydagi xususiy yig`indilar ketma-ketligiga ega bo`lamiz:

$$S_1=a_1, S_2=a_1+a_2, S_3=a_1+a_2+a_3,..., S_n=a_1+a_2+a_3+....+a_n,.....$$

Yuqoridagi $\{S_n\}$ ketma-ketlik yaqinlashuvchi yoki uzoqlashuvchi bo`lishi mumkin.

Ta'rif. Agar (1) qatorning $\{S_n\}$ xususiy yig`indilari ketma-ketligi chekli limitga ega bo`lsa,ya'ni $\lim_{n\to\infty} S_n$ mavjud bo`lsa,u holda bu qator **yaqinlashuvchi qator** deyiladi. $\{S_n\}$ ketma-ketlik limiti

$$S = \lim_{n\to\infty} S_n \qquad (2^*)$$

qatorning yig`indisi deyiladi.

Bu holda $S = a_1 + a_2 + a_3 + ... + a_n + ...$ yoki $S = \sum_{n=1}^{\infty} a_n$ kabi yoziladi.

Agar (1) qatorning xususiy yig`indilari ketma-ketligi chekli limitga ega bo`lmasa,u holda **uzoqlashuvchi qator** deyiladi.

Agar $\lim_{n\to\infty} S_n = \infty$ bo`lsa,u holda $\sum_{n=1}^{\infty} a_n = \infty$ yoki $S = \infty$ kabi yoziladi.

Shunday qilib,qator yig`indisi ikkita amal (qo`shish va limitga o`tish) natijasida hosil qilinadi.Qo`shish amali xususiy yig`indilarini,ikkinchi amal esa ularning limitini topish uchun kerak bo`ladi.

1-misol: Ushbu qatorni yaqinlashishga tekshiring:

$$\frac{1}{1\cdot 3} + \frac{1}{2\cdot 4} + \frac{1}{3\cdot 5} + ... + \frac{1}{n(n+2)} + ...$$

Yechish. Berilgan qatorning n-xususiy yig`indisi

$$S_n = \frac{1}{1\cdot 3} + \frac{1}{2\cdot 4} + \frac{1}{3\cdot 5} + ... + \frac{1}{n(n+2)}.$$

$$S_n = \frac{1}{2}\left(\frac{1}{1} - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right) + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) + ... + \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n+1}\right) + \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right) =$$

$$\frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right)$$

Ravshanki, $\{S_n\}$ ketma-ketlik limiti mavjud va $\frac{3}{4}$ ga teng.

2-misol. Ushbu qatorni yaqinlashishga tekshiring:

$$1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + ... + \frac{1}{\sqrt[3]{n}} + ...$$

Yechish. Bu qatorning n-xususiy yig`indisi

$$S_n = 1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + ... + \frac{1}{\sqrt[3]{n}} \text{ va } S_n > \underbrace{\frac{1}{\sqrt[3]{n}} + \frac{1}{\sqrt[3]{n}} + ... + \frac{1}{\sqrt[3]{n}}}_{n\text{-ta}} = \frac{1}{\sqrt[3]{n}} \cdot n = \sqrt[3]{n^2}$$

bo`lgani sababli, $\lim_{n\to\infty} S_n = \infty$ bo`ladi.Demak,berilgan qator uzoqlashuvchi.

3-misol. Umumiy hadi $a_n = (-1)^{n-1}$ bo`lgan qatorni yaqinlashishga tekshiring.

Yechish. Bu qatorning n-xususiy yig`indisi

$S_n = 1 - 1 + 1 - 1 + ... + (-1)^{n-1}$ ga teng. Xususiy yig`indilar ketma-ketligi quydagi ko`rinishda bo`ladi:

$$1,\ 0,\ 1,\ 0,\ ...$$

Ma'lumki bu ketma-ketlik chekli limitga ega emas.Demak, $\sum_{n=1}^{\infty}(-1)^{n-1}$ qator uzoqlashuvchi ekan.

Ketma-ketlikning qismiy yig`indisi $S_n$ va yig`indisi S ni toping.

1. $\frac{2}{5} + \frac{2}{25} + ... + \frac{2}{5^n} + ...$
2. $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + ... \frac{(-1)^{n-1}}{3^{n-1}} + ...$
3. $\left(\frac{1}{3} + \frac{1}{5}\right) + \left(\frac{1}{3^2} + \frac{1}{5^2}\right) + ... + \left(\frac{1}{3^n} + \frac{1}{5^n}\right) + ...$
4. $\sum_{n=1}^{\infty} \frac{1}{36n^2 - 24n - 35}$
5. $\sum_{n=1}^{\infty} \frac{1}{16n^2 - 8n - 15}$
6. $\frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + ... + \frac{1}{(n+2)(n+3)} + ...$
7. $\frac{1}{1\cdot 4} + \frac{1}{4\cdot 7} + ... + \frac{1}{(3n-2)(3n+1)} + ...$
8. $\frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + ... + \frac{1}{n(n+1)(n+2)} + ...$
9. $\frac{1}{3\cdot 5\cdot 7} + \frac{1}{5\cdot 7\cdot 9} + ... + \frac{1}{(2n+1)(2n+3)(2n+5)} + ...$
10. $\sum_{n=1}^{\infty} \frac{1}{4n^2 + 4n - 3}$
11. $\sum_{n=1}^{\infty} \frac{1}{16n^2 - 8n - 3}$
12. $\sum_{n=1}^{\infty} \frac{1}{25n^2 + 5n - 6}$
13. $\sum_{n=1}^{\infty} \frac{1}{36n^2 - 24n - 5}$
14. $\sum_{n=1}^{\infty} \frac{1}{49n^2 + 7n - 12}$
15. $\sum_{n=1}^{\infty} \frac{1}{n^2 - 1}$
16. $\sum_{n=1}^{\infty} \frac{1}{4n^2 + 4n - 3}$
17. $\sum_{n=1}^{\infty} \frac{1}{16n^2 - 8n - 15}$
18. $\sum_{n=1}^{\infty} \frac{1}{36n^2 + 12n - 35}$
19. $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$
20. $\sum_{n=1}^{\infty} \frac{n}{(2n-1)^2(2n+1)^2}$
21. $\sum_{n=1}^{\infty} \frac{n - \sqrt{n^2 - 1}}{\sqrt{n(n+1)}}$
22. $\sum_{n=1}^{\infty} (\sqrt{n+2} - 2\sqrt{n+1} + \sqrt{n})$
23. $\sum_{n=2}^{\infty} \ln(1 - \frac{1}{n^2})$
24. $\sum_{n=2}^{\infty} \ln(1 - \frac{2}{n(n+1)})$
25. $\sum_{n=2}^{\infty} \ln\frac{n^3 - 1}{n^3 + 1}$
26. $\sum_{n=1}^{\infty} \ln\frac{n(2n+1)}{(n+1)(2n-1)}$
27. $\sum_{n=1}^{\infty} \sin\frac{1}{2^n}\cos\frac{3}{2^n}$
28. $\sum_{n=1}^{\infty} \sin\frac{\alpha}{2^{n+1}}\cos\frac{3\alpha}{2^{n+1}}$
29. $\sum_{n=0}^{\infty} \frac{1}{n!(n+2)}$
30. $\sum_{n=1}^{\infty} arctg\frac{1}{2n^2}$

9.2. Yaqinlashuvchi qatorlarning xossalari.

Yaqinlashuvchi qatorlarning xossaari. Faraz qilaylik, biror

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + ... + a_n + ... \qquad (1)$$

qator berilgan bo`lsin.

Ushbu

$$\sum_{n=m+1}^{\infty} a_n = a_{m+1} + a_{m+2} + ... \qquad (2)$$

qator (bunda $m-$ tayinlangan natural son) (1) qatorning qoldig`i deyiladi.

1-xossa. Agar (1) qator yaqinlashuvchi bo`lsa, (2) qator ham yaqinlashuvchi bo`ldi va aksincha; (2) qatorning yaqinlashuvchi bo`lishidan (1) qatorning yaqinlashuvchiligi kelib chiqadi.

(1) qatorning qismiy yig`indisi

$$S_n = a_1 + a_2 + ... + a_n \quad ,$$

(2) qatorning qismiy yig`indisi

$$M_k^{(m)} = a_{m+1} + a_{m+2} + ... + a_{m+k}$$

lar uchun

$$S_{m+n} = S_m + M_k^{(m)}, \qquad (3)$$

bo`ladi.

Aytaylik, (1) qator yaqinlashuvchi bo`lsin. Unda $k \to \infty$ da $S_{m+n}$ chekli limitga ega bo`lib, (3) munosabatga ko`ra $k \to \infty$ da $M_k^{(m)}$ ham chekli limitga ega bo`ladi.Demak, (2) qator yaqinlashuvchi.

Aytaylik, (2) qator yaqinlashuvchi bo`lsin. Unda $k \to \infty$ da $M_k^{(m)}$ chekli limitga ega bo`ladi. Yana (3) munosabatga ko`ra $k \to \infty$ da $S_{m+n}$ ham chekli limitga ega bo`ladi. Demak, (1) qator yaqinlashuvchi.

2-xossa. Agar

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + ... + a_n + ...$$

qator yaqinlashuvchi bo`lib, uning yig`indisi S ga teng bo`lsa,u holda

$$\sum_{n=1}^{\infty} c \cdot a_n = c \cdot a_1 + c \cdot a_2 + ... + c \cdot a_n + ...$$

qator ham yaqinlashuvchi va uning yig`indisi $c \cdot S$ ga teng bo`ladi,bunda $c \neq 0$ bo`lmagan o`zgarmas son.

3-xossa. Agar

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + ... + a_n + ... ,$$

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + ... + b_n + ...$$

qator yaqinlashuvchi bo`lib, ularning yig`indisi mos ravishda S_1 va S_2 ga teng bo`lsa,u holda

$$\sum_{n=1}^{\infty} (a_n + b_n) = (a_1 + b_1) + (a_2 + b_2) + ... + (a_n + b_n) + ...$$

qator ham yaqinlashuvchi va uning yig`indisi $S_1 + S_2$ ga teng bo`ladi.

2) va 3)-xossalarning isboti sonli qatorlar va ularning yaqinlashuvchiligi ta'rifidan bevosita kelib chiqadi.

4-xossa. Agar

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + ... + a_n + ...$$

qator yaqinlashuvchi bo`lsa, $n \to \infty$ da a_n nolga intiladi:

$$\lim_{n \to \infty} a_n = 0$$

Faraz qilaylik, $\sum_{n=1}^{\infty} a_n$ qator yaqinlashuvchi bo`lib, uning yig`indisi S ga teng bo`lsin: Ta'rifga binoan

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} (a_1 + a_2 + ... + a_n) = S .$$

Ravshanki,

$$a_n = S_n - S_{n-1}$$

bo`ladi. Keyingi tenglikdan topamiz:

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} (S_n - S_{n-1}) = S - S = 0 .$$

Eslatma. Qatorning umumiy hadi a_n ning $n \to \infty$ da nolga intilishidan uning yaqinlashuvchi bo`lishi har doim kelib chiqavermaydi. Maslan, ushbu

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{n}} + ...$$

qatorning umumiy hadi $a_n = \frac{1}{\sqrt{n}}$ bo`lib, u $n \to \infty$ da nolga intiladi. Ammo bu qator uzoqlashuvchi,chunki

$$S_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{n}} \geq n \frac{1}{\sqrt{n}} = \sqrt{n}$$

ketma-ketlik $n \to \infty$ da $+\infty$ ga intiladi:

$$\lim_{n\to\infty} S_n = \infty .$$

Yuqorida keltirilgan 4)- xossa qator yaqinlashuvchi bo`lishining zaruriy shartini ifodalaydi.

5-xossa. Bizga,

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots \qquad (1)$$

qator berilgan bo`lsin. Bu qatorning hadlarini guruhlab quydagi

$$(a_1 + a_2 + \ldots + a_{n_1}) + (a_{n_1+1} + a_{n_1+2} + \ldots + a_{n_2}) + \ldots \qquad (4)$$

qatorni hosil qilamiz, bunda

$$n_1 < n_2 < \ldots$$

bo`lib, $\{n_k\}$ ketma-ketlik natural sonlar ketma-ketligi $\{n\}$ ning qismiy ketma-ketligi.

Agar (1) qator yaqinlashuvchi bo`lib, uning yig`indisi S ga teng bo`lsa, u holda (4) qator ham yaqinlashuvchi va yig`indisi S bo`ladi.

(1) qator yaqinlashuvchi bo`lib, yig`indisi S ga teng bo`lsin. U holda

$$n \to \infty \text{ da } \quad S_n = a_1 + a_2 + \ldots + a_n \to S$$

bo`ladi.

Faraz qilaylik, (4) qatorning qismiy yig`indilaridan iborat ketma-ketlik $\{S_{n_k}\}$ bo`lsin $(k = 1,2,3,\ldots)$. Ravshanki, bu ketma-ketlik $\{S_n\}$ ketma-ketlikning qismiy ketma-ketligi bo`ladi. Ma'lum teoremaga ko`ra

$$k \to \infty \text{ da } S_{n_k} \to S$$

bo`ladi. Demak, (4) qator yaqinlashuvchi va uning yig`indisi S ga teng.

Yaqinlashishga tekshiring:

1. $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)} + \ldots$
2. $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \ldots + \frac{1}{(2n-1)(2n+1)} + \ldots$
3. $\frac{1}{1\cdot 4} + \frac{1}{4\cdot 7} + \ldots + \frac{1}{(3n-2)(3n+1)} + \ldots$
4. $\frac{1}{1\cdot 4} + \frac{1}{2\cdot 5} + \ldots + \frac{1}{n(n+3)} + \ldots$
5. $\frac{1}{1\cdot 7} + \frac{1}{3\cdot 9} + \ldots + \frac{1}{(2n-1)(2n+5)} + \ldots$
6. $\frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \ldots + \frac{1}{n(n+1)(n+2)} + \ldots$
7. $\frac{5}{6} + \frac{13}{36} + \ldots + \frac{3^n + 2^n}{6^n} + \ldots$
8. $\frac{3}{4} + \frac{5}{36} + \ldots + \frac{2n+1}{n^2(n+1)^2} + \ldots$
9. $\frac{1}{9} + \frac{2}{225} + \ldots + \frac{1}{(2n-1)^2(2n+1)^2} + \ldots$
10. $arctg\frac{1}{2} + arctg\frac{1}{8} + \ldots + arctg\frac{1}{2n^2} + \ldots$
11. $\sum_{n=1}^{\infty} \frac{1}{n(n+2)(n+3)}$
12. $\sum_{n=1}^{\infty} \frac{2-n}{n(n+1)(n+3)}$
13. $\sum_{n=1}^{\infty} \frac{3n+4}{n(n+1)(n+4)}$
14. $\sum_{n=1}^{\infty} \left(\frac{(-1)^n}{3^n} + \frac{i}{5^n}\right)$
15. $\sum_{n=1}^{\infty} \frac{(1+i)^n}{4^n}$
16. $\sum_{n=1}^{\infty} \frac{n+2+in}{n(n+1)(n+2)}$
17. $\sum_{n=1}^{\infty} \frac{n}{(1-i)^n}$
18. $\sum_{n=1}^{\infty} a^n \cos na,\ \alpha \in R, a \in R, |a| < 1$
19. $\sum_{n=1}^{\infty} a^n \sin na,\ \alpha \in R, a \in R, |a| < 1$
20. $\sum_{n=1}^{\infty} \frac{1}{n^2 - 4n + 5}$
21. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2 + 2n}}$
22. $\sum_{n=1}^{\infty} \left(\sqrt{n} - \sqrt{n-1}\right)$
23. $\sum_{n=1}^{\infty} \frac{1}{n}\left(\sqrt{n+1} - \sqrt{n-1}\right)$
24. $\sum_{n=1}^{\infty} \frac{1}{n}\left(\sqrt{n^2+n+1} - \sqrt{n^2-n+1}\right)$
25. $\sum_{n=1}^{\infty} \left(\frac{1+n^2}{1+n^3}\right)^2$
26. $\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt[4]{n^5}}$
27. $\sum_{n=1}^{\infty} \sqrt{\frac{1}{n^4+1}}$
28. $\sum_{n=0}^{\infty} \frac{1}{(a+n)(a+n+1)(a+n+2)(a+n+3)}$
29. $\sum_{n=1}^{\infty} \left(\frac{1}{10^n} + \frac{2}{10^{n+1}} + \frac{5}{10^{n+2}}\right)$
30. $\sum_{n=1}^{\infty} \left(\frac{3}{2^{n-1}} + \frac{(-1)^{n-1}}{2\cdot 3^{n-1}}\right)$

9.3. Musbat hadli qatorlarning yaqinlashuvchi bo`lish sharti.

Musbat hadli qatorlar va ularning yaqinlashuvchiligi:

Faraz qilaylik,

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots \qquad (1)$$

qator berilgan bo`lsin.

Agar bu qatorda $a_n \ge 0 \quad (\forall n \in N)$ bo`lsa,(1) musbat hadli qator deyiladi.

Musbat hadli qatorlarda, ularning qismiy yig`indilaridan iborat $\{S_n\}$ ketma-ketlik o`suvchi ketma-ketlik bo`ladi. Haqiqatdan ham,

$$S_{n+1} = a_1 + a_2 + \ldots + a_n + a_{n+1} = S_n + a_{n+1} \ge S_n .$$

1-teorema. Musbat hadli

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots$$

qatorning yaqinlashuvchi bo`lishi uchun

$$\{S_n\} = \{a_1 + a_2 + \ldots + a_n\} \qquad (n = 1,2,3,\ldots)$$

ketma-ketlikning yuqoridan chegaralangan bo`lishi zarur vayetarli.

Isbot. Zarurligi. (1) qator yaqinlashuvchi bo`lsin.Unda $n\to\infty$ da $\{S_n\}$ ketma-ketlik chekli limitga ega bo`ladi. Yaqinlashuvchi ketma-ketlikning xossasiga ko`ra $\{S_n\}$ chegaralangan, jumladan yuqoridan chegaralangan bo`ladi.

Yetarliligi. $\{S_n\}$ ketma-ketlik yuqoridan chegaralangan bo`lsin. Unda monoton ketma-ketlikning limiti haqidagi teoremaga ko`ra $\{S_n\}$ ketma-ketlik $n\to\infty$ da chekli limitga ega bo`ladi.Demak, (1) qator yaqinlashuvchi

Eslatma. Agar

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots$$

musbat hadli qatorda, uning qisiy yig`indilaridan iborat $\{S_n\}$ ketma-ketlik yuqoridan chegaralanmagan bo`lsa, u holda qator uzoqlashuvchi bo`ladi.

Qator yaqinlashuvchi bo`ladigan α ning barcha qiymatlarini toping.

1. $a_n = \left(1 - n\sin\left(\frac{1}{n}\right)\right)^{\alpha}$

2. $a_n = \left(sh\left(\frac{1}{n}\right) - \sin\left(\frac{1}{n}\right)\right)^{\alpha}$

3. $a_n = \left(nsh\left(\frac{1}{n}\right) - ch\left(\frac{1}{n}\right)\right)^{\alpha}$

4. $a_n = \left(-\cos\left(\frac{1}{n}\right) + ch\left(\frac{1}{n}\right)\right)^{\alpha}$

5. $a_n = \left(arctg\left(\frac{1}{n}\right) - \ln\left(1+\frac{1}{n}\right)\right)^{\alpha}$

6. $a_n = \left(e^{tg\left(\frac{1}{n}\right)} - 1\right)^{\alpha}$

7. $a_n = \left(\exp\left\{\frac{1}{n}\cos\frac{1}{n}\right\} - 1 - \frac{1}{n}\right)^{\alpha}$

8. $a_n = \left(\cos\frac{1}{\sqrt{n}} - \frac{\sqrt{n^2-n}}{n}\right)^{\alpha}$

9. $a_n = n\sin^{\alpha}\left(\frac{1}{n} - arctg\frac{1}{n}\right)$

16. $a_n = (n+1)\left(e^{tg\left(\sqrt{n+1}-\sqrt{n}\right)} - 1\right)^{\alpha}$

17. $a_n = \left(n\sin\frac{1}{n} - \cos\frac{1}{n\sqrt{3}}\right)^{\alpha}$

18. $a_n = n^{\alpha}\left(\ln\left(n^2+1\right) - 2\ln n\right)$

19. $a_n = \frac{\sqrt[3]{n^2+1} - \sqrt[3]{n^2-1}}{n^{\alpha}}$

20. $a_n = \alpha^{-\ln n}, \alpha > 0$

21. $a_n = \left(\cos\cos\frac{1}{n} - \cos ch\frac{1}{n}\right)^{\alpha}$

22. $a_n = \left(e^{\frac{1}{2n}} - \left(1 + sh\frac{1}{n}\right)^{\frac{1}{2}}\right)^{\alpha}$

23. $a_n = \left|\ln arctg\frac{1}{n} - \ln tg\frac{1}{n}\right|^{\alpha}$

24. $a_n = \left(1 - \sin\frac{\pi n^2}{2n^2+1}\right)^{\alpha}$

10. $a_n = \left(\exp\left\{\frac{-1}{2n^2}\right\} - \cos\frac{1}{n}\right)^{\alpha}$

11. $a_n = \left(\ln sh\frac{1}{n} - \ln\frac{1}{n}\right)^{\alpha}$

12. $a_n = \left|\ln n + \ln\sin\frac{1}{n}\right|^{\alpha}$

13. $a_n = \left(\exp\left\{1 - \cos\frac{1}{n}\right\} - 1\right)^{\alpha}\sin\frac{1}{\sqrt{n}}$

14. $a_n = \left|\ln\frac{n+1}{n-1} - \frac{2}{n-1}\right|^{\alpha} \quad n \ge 2$

15. $a_n = \frac{\ln^{\alpha}\left(1 + \sqrt{arctg\left(\frac{1}{n}\right)}\right)}{\sin\left(\sqrt{n+1} - \sqrt{n}\right)}$

25. $a_n = \left(\frac{e^2 - \left(1+\frac{2}{n}\right)}{n}\right)^{\alpha}$

26. $a_n = \left(\left(\frac{sh\frac{1}{n}}{\sin\frac{1}{n}}\right)^{3n} - 1\right)^{\alpha}$

27. $a_n = \left(1 - \left(\cos\frac{1}{n}\right)^{\frac{1}{n}}\right)^{\alpha}$

28. $a_n = \ln^{\alpha}\frac{ch\left(\frac{1}{n}\right)}{\cos\left(\frac{1}{n}\right)}$

29. $a_n = \exp\left\{n\sin\frac{1}{n^2}\right\} - n^{\alpha}$

30. $a_n = \alpha^{\frac{1}{n}} - \alpha^{\frac{1}{n+1}}, \ \alpha > 0$

9.4. Musbat hadli qatorlarning taqqoslash haqidagi teoremalar

Musbat hadli qatorlarda taqqoslash teoremalari.

Ikkita

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots$$

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + \ldots + b_n + \ldots$$

musbat hadli qatorlar berilgan bo`lsin.

1-teorema. Musbat hadli

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots$$

qatorning yaqinlashuvchi bo`lishi uchun

$$\{S_n\} = \{a_1 + a_2 + \ldots + a_n\} \quad (n = 1, 2, 3, \ldots)$$

ketma-ketlikning yuqoridan chegaralangan bo`lishi zarur va yetarli.

2-teorema. Faraz qilaylik $\sum_{n=1}^{\infty} a_n$ va $\sum_{n=1}^{\infty} b_n$ qatorlar uchun $\forall n \in N$ da

$$a_n \le b_n \qquad (2)$$

tengsizlik bajarilsin.

U holda:

1) $\sum_{n=1}^{\infty} b_n$ qator yaqinlashuvchi bo`lsa, $\sum_{n=1}^{\infty} a_n$ qator ham yaqinlashuvchi bo`ladi

2) $\sum_{n=1}^{\infty} a_n$ qator uzoqlashuvchi bo`lsa, $\sum_{n=1}^{\infty} b_n$ qator ham uzoqlashuvchi bo`ladi.

$\sum_{n=1}^{\infty} a_n$ va $\sum_{n=1}^{\infty} b_n$ qatorning qismiy yig`indilari mos ravishda

$$S_n = a_1 + a_2 + \ldots + a_n,$$
$$S'_n = b_1 + b_2 + \ldots + b_n$$

bo`lsin. U holda (2) munosabatga ko`ra

$$S_n \le S'_n \qquad (3)$$

bo`ladi.

Faraz qilaylik, $\sum_{n=1}^{\infty} b_n$ qator yaqinlashuvchi bo`lsin. Unda 1-teoremaga binoan $\{S'_n\}$ ketma-ketlik yuqoridan chegaralangan bo`ladi.Ayni paytda, (3) munosabatni e'tiborga olib, $\{S_n\}$ ketma-ketlikning ham yuqoridan chegaralangan bo`lishini topamiz. Yana 2-teoremaga ko`ra $\sum_{n=1}^{\infty} a_n$ qator yaqinlashuvchi bo`ladi.

Faraz qilaylik, $\sum_{n=1}^{\infty} a_n$ qator uzoqlashuvchi bo`lsin. Unda (3) munosabat va eslatmadan foydalanib, $\sum_{n=1}^{\infty} b_n$ qatorning uzoqlashuvchi bo`lishini topamiz.

1-misol. Ushbu

$$\sum_{n=1}^{\infty} \sin\frac{\pi}{2^n} = \sin\frac{\pi}{2} + \sin\frac{\pi}{2^2} + \ldots + \sin\frac{\pi}{2^n} + \ldots$$

qator yaqinlashuvchilikka tekshirilsin.

Ravshanki,bu qator hadlari uchun

$$0 < \sin\frac{\pi}{2^n} < \frac{\pi}{2^n} \qquad (n = 1,2,3,\ldots)$$

tengsizlik o`rinli bo`ladi.

Natijada berilan qatorning har bir hadi yaqinlashuvchi $\sum_{n=1}^{\infty} \frac{1}{2^n}$ qatorning (geometrik qatorning) mos hadidan kichik. 2-teoremaga muvofiq berilgan qator yaqinlashuvchi bo`ladi.

3-teorema. Faraz qilaylik, musbat hadli $\sum_{n=1}^{\infty} a_n$ va $\sum_{n=1}^{\infty} b_n$ qatorning umumiy hadlari uchun

$$\lim_{n\to\infty} \frac{a_n}{b_n} = K \qquad (b_n > 0, \quad n = 1,2,\ldots)$$

bo`lsin,u holda:

1) $K < +\infty$ bo`lib, $\sum_{n=1}^{\infty} b_n$ qator yaqinlashuvchi bo`lsa, $\sum_{n=1}^{\infty} a_n$ qator ham yaqinlashuvchi bo`ladi.

2) $K > 0$ bo`lib, $\sum_{n=1}^{\infty} b_n$ qator uzoqlashuvchi bo`lsa, $\sum_{n=1}^{\infty} a_n$ qator ham uzoqlashuvchi bo`ladi.

Isbot. Faraz qilaylik, $K < +\infty$ bo`lib, $\sum_{n=1}^{\infty} b_n$ qator yaqinlashuvchi bo`lsin.

Ravshanki,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = K \Rightarrow \quad \forall \varepsilon > 0 \quad \exists n_0 \in N, \quad \forall n > n_0:$$

$$\left|\frac{a_n}{b_n} - K\right| < \varepsilon \quad \Rightarrow \quad (K-\varepsilon)b_n < a_n < (K+\varepsilon)b_n.$$

Bundan esa, $\sum_{n=1}^{\infty} (K+\varepsilon)b_n$ qator yaqinlashuvchi bo`lgani uchun 1-teoremaga ko`ra $\sum_{n=1}^{\infty} a_n$ qatorning yaqinlashuvchiligi kelib chiqishini topamiz.

Faraz qilaylik, $K > 0$ bo`lib, $\sum_{n=1}^{\infty} b_n$ qator uzoqlashuvchi bo`lsin.

Ravshaki, $\lim_{n\to\infty} \frac{a_n}{b_n} = K$ va $0 < K_1 < K$ bo`lishidan $\forall n > n_0 \in N$ uchun $\frac{a_n}{b_n} > K_1$ ya'ni bo`lishi kelib chiqadi.1-teoremadan foydalanib $\sum_{n=1}^{\infty} a_n$ qatorning uzoqlashuvchi bo`lishini topamiz.

Natija. Musbat hadli $\sum_{n=1}^{\infty} a_n$ va $\sum_{n=1}^{\infty} b_n$ qatorlar uchun

$$\lim_{n\to\infty} \frac{a_n}{b_n} = K \ , \qquad (0 < K < +\infty)$$

bo`lsa u holda $\sum_{n=1}^{\infty} a_n$ va $\sum_{n=1}^{\infty} b_n$ qatorlar bir vaqtda yoki yaqinlashuvchi yoki uzoqlashuvchi bo`ladi.

2-misol. Ushbu $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$

qator yaqinlashuvchilikka tekshirilsin.

Berilgan qator bilan birga uzoqlashuvchiligi ma'lum bo`lgan $\sum_{n=1}^{\infty} \frac{1}{n}$ garmonik qatorni qaraymiz. Bu qatorning umumiy hadlari uchun

$$\lim_{n\to\infty} \frac{\frac{1}{n^{1+\frac{1}{n}}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n^{1+\frac{1}{n}}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = 1$$

bo`ladi. Demak berilgan qator uzoqlashuvchi.

4-teorema. Aytaylik, musbat hadli $\sum_{n=1}^{\infty} a_n$ va $\sum_{n=1}^{\infty} b_n$ qatorlar uchun

$$\frac{a_{n+1}}{a_n} \le \frac{b_{n+1}}{b_n}$$

bo`lsin $(a_n > 0, \quad b_n > 0, \quad n = 1,2,3,...)$

U holda:

1) $\sum_{n=1}^{\infty} b_n$ qator yaqinlashuvchi bo`lsa, $\sum_{n=1}^{\infty} a_n$ qator ham yaqinlashuvchi bo`ladi.

2) $\sum_{n=1}^{\infty} a_n$ qator uzoqlashuvchi bo`lsa, $\sum_{n=1}^{\infty} b_n$ qator ham uzoqlashuvchi bo`ladi.

Isbot. Faraz qilaylik, $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ qatorlar $(a_n > 0, b_n > 0, \ n = 1,2,3,...)$ uchun

$$\frac{a_{n+1}}{a_n} \le \frac{b_{n+1}}{b_n} \qquad (n = 1,2,3,....)$$

tengsizliklar bajarilsin. Bu shartdan quydagi munosabat kelib chiqadi:

$$\frac{a_2}{a_1} \cdot \frac{a_3}{a_2} \cdot ... \cdot \frac{a_n}{a_{n-1}} \le \frac{b_2}{b_1} \cdot \frac{b_3}{b_2} \cdot ... \cdot \frac{b_n}{b_{n-1}}.$$

Keyingi tengsizlikdan topamiz:

$$a_n \le \frac{a_1}{b_1} b_n. \qquad (n = 1,2,3,...)$$

Faraz qilaylik, $\sum_{n=1}^{\infty} b_n$ qator yaqinlashuvchi bo`lsin. Ravshanki, $\sum_{n=1}^{\infty} \frac{a_1}{b_1} b_n$ qator ham yaqinlashuvchi bo`ladi. 1-teoremadan foydalanib, $\sum_{n=1}^{\infty} a_n$ qator yaqinlashuvchi bo`lishini topamiz.Xuddi shunga o`xshash $\sum_{n=1}^{\infty} a_n$ qator uzoqlashuvchi bo`lishidan $\sum_{n=1}^{\infty} b_n$ qatorning uzoqlashuvchi bo`lishi ko`rsatiladi.

Yuqorida keltirilgan teorema va misollardan ko`rinadiki,musbat hadli qatorning yaqinlashuvchiligi yoki uzoqlashuvchiligini bilgan holda, hadlari bu qator hadlari bilan ma'lum munosabatda bo`lgan (taqqoslangan) ikkinchi musbat hadli qatorning yaqinlashuvchiligi yoki uzoqlashuvchiligini aniqlash mumkin bo`lar ekan.

Izoh. Yuqorida keltirilgan teoremalar n ning biror n_0 qiymatidan boshlab bajarilganida ham o`rinli bo`ladi.

Taqqoslash alomatlaridan foydalanib,qatorni yaqinlashishga tekshiring.

1. $\sum_{n=1}^{\infty} \frac{5+3\cdot(-1)^{n+1}}{2^n}$
2. $\sum_{n=1}^{\infty} \frac{arctgn}{n^2+1}$
3. $\sum_{n=1}^{\infty} \frac{\sin^2 3n}{n\sqrt{n}}$
4. $\sum_{n=1}^{\infty} \frac{\cos\left(\frac{\pi}{4n}\right)}{\sqrt[5]{2n^5-1}}$
5. $\sum_{n=1}^{\infty} \sin\frac{3+(-1)^n}{n^2}$
6. $\sum_{n=1}^{\infty} \frac{\ln n+\sin n}{n^2+2\ln n}$

16. $\sum_{n=1}^{\infty} \frac{\left(3-2\cos^2\left(\frac{\pi n}{3}\right)\right)e^n}{n^2 2^n}$
17. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{(2n+1)(2n+3)}}$
18. $\sum_{n=1}^{\infty} 1-\cos\frac{2\pi}{n}$
19. $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n\sqrt[3]{n}}\right)$
20. $\sum_{n=1}^{\infty} \frac{3n+1}{(2n+1)^2}$
21. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} arctg\frac{1}{2\sqrt{n}}$

7. $\sum_{n=1}^{\infty}\frac{n+2}{n^2\left(4+3\sin\left(\frac{\pi n}{3}\right)\right)}$

8. $\sum_{n=1}^{\infty}\frac{\ln\left(1+(3+(-1)^n arctg2n/n)\right)}{\ln^2 n}$ $n\geq 2$

9. $\sum_{n=1}^{\infty}\frac{\arcsin\left(\frac{n-1}{n+1}\right)}{n\sqrt{\ln(n+1)}}$

10. $\sum_{n=1}^{\infty}\frac{arctg\left(n^2+2n\right)}{3^n+n^2}$

11. $\sum_{n=1}^{\infty}\frac{\cos^4\left(\frac{2n}{n+1}\right)}{\sqrt{n^2+4}-\sqrt{n^2+1}}$

12. $\sum_{n=1}^{\infty}\frac{n^5\left(\sqrt{2}+\sin\sqrt{n}\right)}{2^n+n^2}$

13. $\sum_{n=1}^{\infty}\frac{\ln(1+\ln n)}{\ln^3(n+2)\sqrt[4]{n^4+3n^2+1}}$

14. $\sum_{n=1}^{\infty}n^2e^{-n}$

15. $\sum_{n=1}^{\infty}\left(3n+n^3\right)e^{-\sqrt{n}}\ln n$

22. $\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n}}\arcsin\frac{1}{\sqrt[5]{n^4}}$

23. $\sum_{n=1}^{\infty}\left(e^{\frac{1}{n}}-1\right)\sin\frac{1}{\sqrt{n+1}}$

24. $\sum_{n=1}^{\infty}\frac{n^3+3n^2+5}{n\sqrt[5]{n^{16}+n^4+1}}$

25. $\sum_{n=1}^{\infty}\sin\frac{2n+1}{n^3+5n+3}$

26. $\sum_{n=1}^{\infty}ntg\frac{n+2}{n^2+3}$

27. $\sum_{n=1}^{\infty}\frac{\ln\left(1+\sin\left(\frac{1}{n}\right)\right)}{n+\ln^2 n}$

28. $\sum_{n=1}^{\infty}\ln\frac{n+3}{n^2+4}$

29. $\sum_{n=1}^{\infty}\ln\frac{n^2+4}{n^2+3}$

30. $\sum_{n=1}^{\infty}\sqrt{n}\left(ch\left(\frac{\pi}{n}\right)-1\right)$

9.5.Musbat hadli qatorlar uchun yaqinlashuvchilik alomati.

Musbat hadli qatorlar mavzusida bayon etilgan taqqoslash teoremalaridan foydalanib, yaqinlashish alomatlarini keltiramiz.

1^0. Koshi alomati. Agar musbat hadli

$$\sum_{n=1}^{\infty}a_n=a_1+a_2+...+a_n+... \qquad (1)$$

qatorda barcha $n\geq 1$ uchun

$$\sqrt[n]{a_n}\leq q<1 \qquad (2)$$

bo`lsa, (1) qator yaqinlashuvchi bo`ladi;

$$\sqrt[n]{a_n}\geq 1 \qquad (3)$$

bo`lsa, (1) qator uzoqlashuvchi bo`ladi.

Isbot. Aytaylik, (1) qator hadlari uchun

$$\sqrt[n]{a_n}\leq q<1$$

bo`lsin. Ravshanki, bu tengsizlikdan

$$a_n\leq q^n$$

bo`lishi kelib chiqadi.

Demak,berilgan qatorning har bir hadi yaqinlashuvchi geometrik qatorning mos hadidan katta emas.Demak berilgan qator yaqinlashuvchi bo`ladi.

Faraz qilaylik, (1) qator hadlari uchun

$$\sqrt[n]{a_n}\geq 1,\text{ ya'ni } a_n\geq 1$$

bo`lsin. Bu munosabat berilgan qatorning har bir hadini uzoqlashuvchi

$$\sum_{n=1}^{\infty}1=1+1+...+1+...$$

qatorning mos hadidan kichik emasligini ko`satadi. Bunda yuqoridagi qator uzoqlashuvchi bo`ladi.

Ko`pincha Koshi alomatining quydagi keltirilgan limit ko`rinishdagi tasdig`idan foydalaniladi.

Faraz qilaylik, musbat hadli (1) qatorda

$$\lim_{n\to\infty}\sqrt[n]{a_n}=k$$

mavjud bo`lsin. U holda :

1) $k<1$ bo`lganda (1) qator yaqinlashuvchi bo`ladi,
2) $k>1$ bo`lganda (1) qator uzoqlashuvchi bo`ladi.

1-misol. Ushbu $\sum_{n=1}^{\infty}\left(\frac{n+1}{n+2}\right)^{n^2}$ qator yaqinlashuvchilikka tekshirilsin.

Bu qatorning umumiy hadi

$$a_n=\left(\frac{n+1}{n+2}\right)^{n^2}$$

bo`lib, uning uchun

$$\sqrt[n]{a_n}=\left(\frac{n+1}{n+2}\right)^n=\frac{(1+\frac{1}{n})^n}{(1+\frac{2}{n})^n}$$

bo`ladi. Ravshanki,

$$\lim_{n\to\infty}\frac{(1+\frac{1}{n})^n}{(1+\frac{2}{n})^n}=\frac{1}{e}.$$

Demak, $k = \frac{1}{e} < 1$, berilgan qator yaqinlashuvchi

1-eslatma. Koshi alomatidagi (2) va (3) tengsizliklar n ning biror n_0 qiymatidan boshlab bajarilganda ham tasdiq o`rinli bo`ladi.

2-eslatma. Koshi alomatining limit ko`rinishdagi ifodasida $k = 1$ bo`lsa,u holda (1) qator yaqinlashuvchi ham, uzoqlashuvchi ham bo`ladi.

2^0. Dalamber alomati. Agar musbat hadli

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots \qquad (1)$$

qatorda barcha $n \geq 1$ uchun

$$\frac{a_{n+1}}{a_n} \leq q < 1 \qquad (a_n > 0\,,\ n = 1,2\ldots) \qquad (4)$$

bo`lsa, (1) qator yaqinlashuvchi bo`ladi;

$$\frac{a_{n+1}}{a_n} \geq 1 \qquad (a_n > 0\,,\ n = 1,2\ldots) \qquad (5)$$

bo`lsa, (1) qator uzoqlashuvchi bo`ladi.

Isbot. Aytaylik, (1) qator hadlari uchun

$$\frac{a_{n+1}}{a_n} \leq q < 1$$

bo`lsin. Bu tengsizlikni quydagicha

$$\frac{a_{n+1}}{a_n} \leq \frac{q^{n+1}}{q^n} \qquad (q < 1)$$

yozish mumkin.

Ravshanki,

$$\sum_{n=1}^{\infty} q^n \qquad (0 < q < 1)$$

qator (geometrik qator) yaqinlashuvchi.

(1) qator hadlari uchun

$$\frac{a_{n+1}}{a_n} \geq 1$$

bo`lganda (1) qatorning uoqlashuvchi bo`lishini aniqlash qiyin emas.

Dalamber alomatining quydagi limit ko`rinishdagi tasdig`idan foydalaniladi.

Faraz qilaylik, musbat hadli (1) qatorda

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = d$$

limiti mavjud bo`lsin. U holda :

1) $d < 1$ bo`lganda (1) qator yaqinlashuvchi bo`ladi,
2) $d > 1$ bo`lganda (1) qator uzoqlashuvchi bo`ladi.

2-misol. Ushbu $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ qator yaqinlashuvchilikka tekshirilsin.

Berilgan qator uchun

$$a_n = \frac{n!}{n^n}\ , \qquad a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$$

bo`lib,

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!\cdot n^n}{(n+1)^{n+1}\cdot n!} = \frac{1}{(1+\frac{1}{n})^n}$$

bo`ladi. Ravshanki,

$$\lim_{n\to\infty} = \frac{1}{(1+\frac{1}{n})^n} = \frac{1}{e}.$$

Demak, $d = \frac{1}{e} < 1$, berilgan qator yaqinlashuvchi.

3-eslatma. Dalamber alomatidagi (4) va (5) tengsizliklar n ning biror n_0 qiymatidan boshlab bajarilganda ham tasdiq o`rinli bo`ladi.

4-eslatma. Dalamber alomatining limit ko`rinishdagi ifodasida $d = 1$ bo`lsa, u holda (1) qator yaqinlashuvchi ham, uzoqlashuvchi ham bo`lishi mumkin.

3^0. Integral alomat. Faraz qilaylik, musbat hadli

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots$$

qator berilgan bo`lsin. Ayni paytda, $[1,+\infty)$ oráliqda berilgan $f(x)$ funksiya quydagi shartlarni qanotlantirsin:

1) $f(x)$ funksiya $[1,+\infty)$ da uzluksiz,
2) $f(x)$ funksiya $[1,+\infty)$ da kamayuvchi,
3) $\forall x \in [1,+\infty)$ da $f(x) \geq 0$,
4) $f(n) = a_n \quad (n = 1,2,3,\ldots)$.

Bunda berilgan qator ushbu

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} f(n)$$

ko`rinishga keladi.

Yuqoridagi shartlargan foydalanib, $n < x < n+1$ $(n \in N)$ bo`lganda

$$f(n) \ge f(x) \ge f(n+1), \text{ ya'ni } a_n \ge f(x) \ge a_{n+1}$$

bo`lishini topamiz. Keyingi tengsizlikni $[n, n+1]$ oraliq bo`ycha integrallash natijasida

$$a_{n+1} \le \int_n^{n+1} f(x)dx \le a_n \qquad (6)$$

bo`lishi kelib chiqadi.

Endi berilgan

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} f(n)$$

qator bilan birga ushbu

$$\sum_{n=1}^{\infty} \int_n^{n+1} f(x)dx \qquad (7)$$

qatorni qaraymiz. Bu qatorning qismiy yig`indisi

$$\sum_{k=1}^{n} \int_k^{k+1} f(x)dx = \int_1^{n+1} f(x)dx$$

bo`ladi.

Aytaylik, $F(x)$ funksiya $[1,+\infty]$ oraliqda $f(x)$ funksiyaning boshlang`ich funksiyasi bo`lsin: $F'(x) = f(x)$.

Uni qauydagicha

$$F(x) = \int_1^x f(t)dt \ , \qquad F(1) = 0$$

ifodalash mumkin. Natijada

$$\sum_{k=1}^{n} \int_k^{k+1} f(x)dx = F(n+1)$$

bo`ladi.

Agar $n \to \infty$ da $F(n+1)$ chekli songa intilsa, (bu holda (7) qatorning qismiy yig`indisi chekli limitga ega bo`ladi) unda (7) qator yaqinlashuvchi.

Binobarin, $\int_1^n f(x)dx$ $(n = 1,2,3,...)$ ketma-ketlik yuqoridan chegaralangan bo`ladi. (6) munosabatga ko`ra berilgan qatorning qismiy yig`indilaridan iborat ketma-ketlik yuqoridan chegaralangan bo`lib, musbat hadli qatorning

yaqinlashuvchiligi haqidagi teoremaga oid berilgan $\sum_{n=1}^{\infty} a_n$ qator yaqinlashuvchi bo`ladi.

Agar $n \to \infty$ da $F(n+1) \to \infty$ bo`lsa, berilgan qator uzoqlashuvchi bo`ladi.

Shunday qilib, quydagi integral alomatiga kelamiz.

Integral alomati. Agar

$$\lim_{x \to +\infty} F(x) = b$$

bo`lib, $b$ chekli son bo`lsa, $\sum_{n=1}^{\infty} a_n$ qator yaqinlashuvchi bo`ladi, $b = \infty$ bo`lsa, $\sum_{n=1}^{\infty} a_n$ qator uzoqlashuvchi bo`ladi.

3-misol. Ushbu $\sum_{n=1}^{\infty} \frac{1}{n^\alpha} = 1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + ... + \frac{1}{n^\alpha} + ...$ $\qquad (\alpha > 0)$ qator yaqinlashuvchilikka tekshirilsin.

Agar $f(x) = \frac{1}{x^\alpha}$ $(\alpha > 0)$ deyilsa, unda bu funksiya $[1,+\infty)$ oraliqda integral alomatida keltirilgan barcha shartlarni qanotlantiradi. Bu funksiyaning boshlang`ich funksiyasi

$$F(x) = \int_1^x f(t)dt = \int_1^x \frac{dt}{t^\alpha} = \frac{1}{1-\alpha}\left(\frac{1}{x^{\alpha-1}} - 1\right) \qquad (\alpha \ne 1)$$

bo`ladi.

Ravshanki,

$$\lim_{x \to +\infty} F(x) = \lim_{x \to \infty} \frac{1}{1-\alpha}\left(\frac{1}{x^{\alpha-1}} - 1\right) =$$

$$= \begin{cases} \frac{1}{\alpha - 1} \ , & \text{agar } \alpha > 1 \text{ bo'lsa,} \\ \infty \ , & \text{agar } \alpha < 1 \text{ bo'lsa,} \end{cases}$$

bo`lib, $\alpha = 1$ bo`lganda

$$\lim_{x \to +\infty} F(x) = \lim_{x \to \infty} \int_1^x \frac{dt}{t} = \infty$$

bo`ladi.

Demak, integral alomatiga ko`ra

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$$

qator $\alpha > 1$ bo`lganda yaqinlashuvchi, $\alpha \le 1$ bo`lganda uzoqlashuvchi bo`ladi.

Odatda, $\sum_{n=1}^{\infty}\frac{1}{n^{\alpha}}$ qator umumlashgan garmonik qator deyiladi.

4^0. Raabi alomati. Agar musbat hadli

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots \qquad (1)$$

qatorda $n \in N$ ning biror $n_0 (n_0 \ge 1)$ qiymatidan boshlab, $n > n_0$ uchun

$$n(1 - \frac{a_{n+1}}{a_n}) \ge r > 1$$

bo`lsa, (1) qator yaqinlashuvchi bo`ladi,

$$n(1 - \frac{a_{n+1}}{a_n}) \le 1$$

bo`lsa, (1) qator uzoqlashuvchi bo`ladi.

Isbot. Faraz qilaylik, (1) qator hadlari uchun

$$n(1 - \frac{a_{n+1}}{a_n}) \ge r > 1$$

bo`lsin. Bu tengsizlikdan

$$\frac{a_{n+1}}{a_n} \le 1 - \frac{r}{n} \qquad (8)$$

bo`lishi kelib chiqadi.

Endi $r > \alpha > 1$ tengsizlikni qanotlantiruvchi α sonni olib,uni

$$\alpha = \lim_{n\to\infty} \frac{(1 - \frac{1}{n})^{\alpha} - 1}{-\frac{1}{n}}$$

kabi ifodalaymiz. Limit xossasiga ko`ra shunday $n_0' \in N$ topiladiki,barcha $n > n_0'$ lar uchun

$$\frac{(1 - \frac{1}{n})^{\alpha} - 1}{-\frac{1}{n}} \le r,$$

ya'ni

$$(1 - \frac{1}{n})^{\alpha} \ge 1 - \frac{r}{n} \qquad (9)$$

tengsizlik o`rinli bo`ladi.

(8) va (9) munosabatlardan barcha $n > \bar{n}_0 \quad (\bar{n}_0 = \max\{n_0, n_0'\})$ lar uchun

$$\frac{a_{n+1}}{a_n} \le (1 - \frac{1}{n})^{\alpha} = \frac{\frac{1}{n^{\alpha}}}{\frac{1}{(n-1)^{\alpha}}}$$

bo`lishi kelib chiqadi.

Bu tengsizlikni va $\alpha > 1$ bo`lganda $\sum_{n=1}^{\infty}\frac{1}{n^{\alpha}}$ qatorning yaqinlashuvchiligini e'tiborga olib, berilgan $\sum_{n=1}^{\infty} a_n$ qatorningyaqinlashuvchi bo`lishini topamiz.

Endi (1) qatorning hadlari uchun $n > n_0$ bo`lganda

$$n(1 - \frac{a_{n+1}}{a_n}) \le 1$$

bo`lsin. Bu tengsizlikni quydagicha:

$$\frac{a_{n+1}}{a_n} \ge \frac{\frac{1}{n}}{\frac{1}{n-1}}$$

yozish mumkin.

Bu tengsizlikni va $\sum_{n=1}^{\infty}\frac{1}{n}$ qatorning uzoqlashuvchiligini e'tiborga olib, berilgan $\sum_{n=1}^{\infty} a_n$ qatorning uzoqlashuvchi bo`lishini topamiz.

Ko`p hollarda Raabe alomatining quydagi limit ko`rinishidan foydalaniladi:

Faraz qilaylik, musbat hadli (1) qator hadllari uchun

$$\lim_{n\to\infty} n(1 - \frac{a_{n+1}}{a_n}) = \rho$$

mavjud bo`lsin. U holda:

1) $\rho > 1$ bo`lganida (1) qator yaqinlashuvchi bo`ladi,
2) $\rho < 1$ bo`lganida (1) qator uzoqlashuvchi bo`ladi.

4-misol. Ushbu $\sum_{n=2}^{\infty} a^{-(1+\frac{1}{2}+\ldots+\frac{1}{n-1})} \quad (a > 0)$ qator yaqinlashuvchilikka tekshirilsin.

Bu qator uchun

$$n\left(1-\frac{a_{n+1}}{a_n}\right)=n\left(1-\frac{a^{-(1+\frac{1}{2}+\ldots+\frac{1}{n})}}{a^{-(1+\frac{1}{2}+\ldots+\frac{1}{n-1})}}\right)=n\left(1-a^{-\frac{1}{n}}\right)$$

bo`lib,

$$\lim_{n\to\infty} n\left(1-\frac{a_{n+1}}{a_n}\right)=\lim_{n\to\infty}\frac{a^{-\frac{1}{n}}-1}{-\frac{1}{n}}=\ln a$$

bo`ladi.

Agar $\ln a>1$, ya'ni $a>e$ bo'lsa, berilgan qator yaqinlashuvchi bo`ladi.

Agar $\ln a<1$, ya'ni $a<e$ bo'lsa, berilgan qator uzoqlashuvchi bo`ladi.

Agar $a=e$ bo'lsa, Raabi alomati berilgan qatorning yaqinlashuvchiligi yoki uzoqlashuvchiligi haqida xulosa qilolmaydi.

Berilgan sonli qatorlarni yaqinlashish alomatlaridan foydalanib tekshiring:

1. $\frac{1}{3!}+\frac{1}{5!}+\ldots+\frac{1}{(2n+1)!}+\ldots$

2. $\frac{1}{2}+\frac{2}{2^2}+\ldots+\frac{n}{2^n}+\ldots$

3. $\sin\frac{\pi}{2}+4\sin\frac{\pi}{4}+\ldots+n^2\sin\frac{\pi}{2^n}$

4. $\frac{1}{\ln 2}+\frac{1}{\ln^2 3}+\ldots+\frac{1}{\ln^n(n+1)}+\ldots$

5. $\frac{1}{3}+\left(\frac{2}{5}\right)^2+\ldots+\left(\frac{n}{2n+1}\right)^n+\ldots$

6. $\arcsin 1+\arcsin^2\frac{1}{2}+\ldots+\arcsin^n\frac{1}{n}+\ldots$

7. $\frac{2}{3}+\frac{\left(\frac{3}{2}\right)^4}{9}+\ldots+\frac{\left(\frac{n+1}{n}\right)^{n^2}}{3^n}+\ldots$

8. $\frac{1}{2\ln^2 2}+\frac{1}{3\ln^2 3}+\ldots+\frac{1}{(n+1)\ln^2(n+1)}+\ldots$

9. $\left(\frac{1+1}{1+1^2}\right)^2+\left(\frac{1+2}{1+2^2}\right)^2+\ldots+\left(\frac{1+n}{1+n^2}\right)^2+\ldots$

16. $a_n=\left(\frac{\sqrt{n}+2}{\sqrt{n}+3}\right)^{n^{\frac{3}{2}}}$

17. $a_n=\frac{n!e^n}{n^{n+\alpha}}$

18. $a_n=\left(\frac{n-1}{n+1}\right)^{n^2+4n+5}$

19. $a_n=\left(\cos\frac{1}{\sqrt{n}}\right)^{n^2}$

20. $a_n=\left(n\sin\frac{1}{n}\right)^{n^3}$

21. $a_n=\left(\frac{1}{ch\left(\frac{1}{\sqrt{n}}\right)}\right)^{n^2}$

22. $a_n=\left(n\,sh\frac{1}{n}\right)^{-n^3}$

23. $a_n=\frac{1}{3^n}\left(\frac{n+2}{n}\right)^{n^2}$

24. $a_n=\left(n\arcsin\frac{1}{n}\right)^{n^3}$

10. $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\ln\frac{n+1}{n-1}$

11. $a_n=\frac{1}{(\ln n)^n}$, $n\geq 2$

12. $a_n=\left(\frac{3}{n}\right)^n$

13. $a_n=2^n\left(\frac{n}{n+1}\right)^{n^2}$

14. $a_n=3^{n+1}\left(\frac{n+2}{n+3}\right)^{n^2}$

15. $a_n=\left(\frac{an}{n+2}\right)^n$, $a>0$

25. $a_n=3^{-n}\left(\frac{n+1}{n}\right)^{n^2}$

26. $a_n=\left(\frac{2n-1}{2n+1}\right)^{n(n-1)}$

27. $a_n=n!\,arctg\frac{2^n}{n^n}$

28. $a_n=\frac{1}{(\ln n)^{\ln n}}$, $n\geq 2$

29. $a_n=\frac{9^{2n}n!}{(2n)!}$

30. $a_n=\sin\frac{n^2}{\left(3+\frac{1}{n}\right)^n}$

9.6. Ixtiyoriy ishorali qatorlar va ularning yaqinlashuvchanligi, qatorning absolyut va shartli yaqinlashishi.

1^0. Leybnist alomati. Ushbu

$$\sum_{n=1}^{\infty}(-1)^{n-1}c_n=c_1-c_2+c_3-c_4+\ldots+(-1)^{n-1}c_n+\ldots \qquad (1)$$

qatorni qaraymiz, bunda $c_n>0$ $\quad(n=1,2,3,\ldots)$.

Odatda, bunday qator hadlarining ishoralari navbat bilan o`zgarib keladigan qator deyiladi.

Ravshanki, (1) qator ixtiyoriy hadli qatorning bitta holidir.

Masalan, ushbu

$$\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{1}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots+(-1)^{n-1}\frac{1}{n}+\ldots$$

qator hadlarning ishoralari bilan o`zgarib keladigan qator bo`ladi.

1-teorema (Leybnist alomati). Agar hadlarining ishoralari navbat bilan o`zgarib keladigan (1) qatorda:

1) $c_{n+1}<c_n$, $\quad(n=1,2,3,\ldots)$

2) $\lim_{n\to\infty}c_n=0$

bo`lsa, u holda (1) qator yaqinlashuvchi bo`ladi.

Isbot. (1) qatorning dastlabki $2m$ ta ($m\in N$) hadidan iborat qismiy yig`indisi

$$S_{2m}=c_1-c_2+c_3-c_4+\ldots+c_{2m-1}-c_{2m}$$

ni olaylik. Unda $S_{2(m+1)}$ uchun

$$S_{2(m+1)} = S_{2m} + (c_{2m+1} - c_{2m+2})$$

bo`lib, $c_{2m+2} < c_{2m+1}$ bo`lganligi sababli (bunda $c_{2m+1} - c_{2m+2} > 0$ bo`ladi)

$$S_{2(m+1)} > S_{2m} \qquad (m = 1,2,3,...)$$

bo`ladi.Demak, $\{S_{2m}\}$ ketma-ketlik o`suvchi.

Endi S_{2m} yig`indini quydagicha yozamiz:

$$S_{2m} = c_1 - (c_2 - c_3) - (c_4 - c_5) - ... - (c_{2m-2} - c_{2m-1}) - c_{2m}.$$

Bu tenglikning o`ng tomonidagi ifodada qatnashgan qavs ichidagi ayirmalarning, shuningdek $c_{2m}$ ning musbat bo`lishini e'tiborga olib,

$$S_{2m} < c_1$$

bo`lishini topamiz. Demak, $\{S_{2m}\}$ ketma-ketlik yuqoridan chegaralangan.

Monoton ketma-ketlikning limiti haqidagi teoremaga ko`ra

$$\lim_{m\to\infty} S_{2m} = S \qquad (S - \text{chekli son}) \qquad (2)$$

mavjud.

Endi (1) qatorning dastlabki $2m-1$ ta ($m \in N$) sondagi hadidan iborat ushbu

$$S_{2m-1} = c_1 - c_2 + c_3 - c_4 + ... + c_{2m-1}$$

qismiy yig`indisini olaylik. Ravshanki,

$$S_{2m-1} = S_{2m} + c_{2m}.$$

Teoremaning $n \to \infty$ da $c_n \to 0$ bo`lishi sharti hamda (2) munosabatdan foydalanib topamiz:

$$\lim_{m\to\infty} S_{2m-1} = \lim_{m\to\infty}(S_{2m} + c_{2m}) = S.$$

Shunday qilib, berilgan (1) qatorning qismiy yig`indilaridan iborat ketma-ketlik chekli limitga ega ekani ko`rsatildi. Demak,(1) qator yaqinlashuvchi.

Masalan,

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... + \frac{(-1)^{n-1}}{n} + ... \qquad (3)$$

qator hadlari keltirilgan teoremaning barcha shartlarini qanotlantiradi. Teoremaga ko`ra (3) qator yaqinlashuvchi bo`ladi ((3) qatorning yaqinlashuvi va yig`indisi ln2 ga teng bo`lishi ko`rsatilgan edi).

2^0. Dirixle-Abel alomati. Faraz qilaylik,

$$a_1\ ,\ a_2\ ,\ a_3\ ,...a_n\ ,...,$$

$$b_1\ ,b_2\ ,\ b_3\ ,...,b_n\ ,...$$

ixtiyoriy haqiqiy sonlar ketma-ketliklari bo`lib,

$$S_n = a_1 + a_2 + ... + a_n$$

bo`lsin. U holda $\forall n \in N$, $\forall m \in N$ uchun

$$\sum_{k=n}^{n+m} a_k b_k = \sum_{k=n}^{n+m-1} S_k(b_k - b_{k-1}) + S_{n+m}\cdot b_{n+m} - S_{n-1}b_n \qquad (4)$$

munosabat o`rinli bo`ladi.

Odatda, (4) munosabat Abel ayniyati deyiladi.

Isbot. Ravshanki, $a_k = S_k - S_{k-1}$ bo`ladi. Unda

$$\sum_{k=n}^{n+m} a_k b_k$$

yig`indi ushbu ko`rinishga

$$\sum_{k=n}^{n+m} a_k b_k = \sum_{k=n}^{n+m} S_k b_k - \sum_{k=n}^{n+m} S_{k-1} b_k \qquad (5)$$

keladi. Bu tenglikning o`ng tomonidagi birinchi qo`shiluvchini quydagicha:

$$\sum_{k=n}^{n+m} S_k b_k = \sum_{k=n}^{n+m-1} S_k b_k + S_{n+m} b_{n+m} \ ,$$

ikkinchi qo`shiluvchini esa quydagicha

$$\sum_{k=n}^{n+m} S_{k-1} b_k = \sum_{k=n-1}^{n+m-1} S_k b_{k+1} = S_{n-1}b_n + \sum_{k=n}^{n+m-1} S_k b_{k+1}$$

yozib olamiz.

Natijada (5) tenglik quydagicha:

$$\sum_{k=n}^{n+m} S_k b_k = \sum_{k=n}^{n+m-1} S_k b_k + S_{n+m} b_{n+m} - \sum_{k=n}^{n+m-1} S_k b_{k+1} - S_{n-1} b_n =$$

$$= \sum_{k=n}^{n+m-1} S_k (b_k - b_{k+1}) - S_{n+m} b_{n+m} - S_{n-1} b_n$$

bo`ladi.

2-teorema. (Dirixe-Abel alomati). Aytaylik,

$$\sum_{k=1}^{\infty} a_k b_k = a_1 b_1 + a_2 b_2 + ... + a_k b_k + ... \qquad (6)$$

qator berilgan bo`lsin. Agar:

1) $\{b_k\}$ ketma-ketlik kamayuvchi va u cheksiz kichik miqdor,

2) $\sum_{k=1}^{\infty} a_k$ qatorning qismiy yig`indilari ketma-ketligi chegaralangan bo`lsa, (6) qator yaqinlashuvchi bo`ladi.

Isbot. Agar $\sum_{k=1}^{\infty} a_k$ qatorning qismiy yig`indisini

$$S_n = a_1 + a_2 + \ldots + a_n$$

desak, unda teoremaning shartiga ko`ra, shunday $M > 0$ son topiladiki, barcha $n \in N$ uchun

$$|S_n| \le M \qquad (7)$$

bo`ladi.

Shartga ko`ra $\{b_k\}$ ketma-ketlik kamayuvchi va u cheksiz kichik miqdor. Unda $\forall \varepsilon > 0$ ga ko`ra shunday $n_0 \in N$ topiladiki, $\forall n > n_0$ da

$$0 \le b_n < \frac{\varepsilon}{2M} \qquad (8)$$

bo`ladi.

Endi

$$\sum_{k=n}^{n+m} a_k b_k$$

yig`indiga Abel ayniyatini qo`llaymiz:

$$\sum_{k=n}^{n+m} a_k b_k = \sum_{k=n}^{n+m-1} S_k (b_k - b_{k+1}) + S_{n+m} b_{n+m} - S_{n-1} b_n .$$

Natijada

$$\left|\sum_{k=n}^{n+m} a_k b_k\right| \le \sum_{k=n}^{n+m-1} |S_k (b_k - b_{k+1})| + |S_{n+m} b_{n+m}| + |S_{n-1} b_n| =$$
$$= \sum_{k=n}^{n+m-1} |S_k| \cdot (b_k - b_{k+1}) + |S_{n+m}| b_{n+m} + |S_{n-1}| \cdot b_n$$

bo`ladi.

(7) tengsizlikdanfoydalanib topamiz:

$$\left|\sum_{k=n}^{n+m} a_k b_k\right| \le M\left[\sum_{k=n}^{n+m-1} (b_k - b_{k+1}) + b_{n+m}\right] + M \cdot b_n .$$

Agar

$$\sum_{k=n}^{n+m-1} (b_k - b_{k+1}) + b_{n+m} = (b_n - b_{n+1}) + (b_{n+1} - b_{n+2}) + \ldots + (b_{n+m-1} - b_{n+m}) + b_{n+m} = b_n$$

bo`lishini e'tiborga olsak,unda

$$\left|\sum_{k=n}^{n+m} a_k b_k\right| \le 2M \cdot b_n$$

bo`lib, (8) munosabatga ko`ra

$$\left|\sum_{k=n}^{n+m} a_k b_k\right| < \varepsilon$$

bo`ladi. Bundan Koshi teoremasiga ko`ra $\sum_{k=n}^{n+m} a_k b_k$ qatorning yaqinlashuvchiligi kelib chiqadi.

Misol. Ushbu $\sum_{n=1}^{\infty} \frac{\cos kx}{k} = \frac{\cos x}{1} + \frac{\cos 2x}{2} + \ldots + \frac{\cos kx}{k} + \ldots$

qator yaqinlashuvchilikka tekshirilsin, bunda x – tayinlangan haqiqiy son.

Agar $x = 2\pi$ bo`lsa, berilgan qator

$$\sum_{k=1}^{\infty} \frac{\cos kx}{k} = \sum_{k=1}^{\infty} \frac{\cos 2\pi \cdot k}{k} = \sum_{k=1}^{\infty} \frac{1}{k}$$

garmonik qator bo`lib, u uzoqlashuvchi bo`ladi.

Aytaylik, $x \ne 2\pi$ bo`lsin. Berilgan qatorda

$$a_k = \cos kx \ , \ b_k = \frac{1}{k}$$

belgilashlarni bajaramiz.

Ravshanki, $\{b_k\} = \left\{\frac{1}{k}\right\}$ ketma-ketlik kamayuvchi va cheksizkichik miqdor bo`ladi ($k \to \infty$ da $\frac{1}{k} \to 0$).

Endi $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \cos kx$ qatorning qismiy yig`indisi S_n ni topamiz:

$$S_n = \sum_{k=1}^{n} \cos kx = \frac{1}{2\sin\frac{x}{2}} \sum_{k=1}^{n} 2\sin\frac{x}{2}\cos kx =$$

$$= \frac{1}{2\sin\frac{x}{2}} \sum_{k=1}^{n}\left[\sin(k+\frac{1}{2})x - \sin(k-\frac{1}{2})x\right] = \frac{\sin(n+\frac{1}{2})x - \sin\frac{x}{2}}{2\sin\frac{x}{2}}.$$

Keyingi munosabatdan, 2π ga karrali bo`lmagan x lar uchun

$$|S_n| \le \frac{1}{\left|\sin\frac{x}{2}\right|}$$

bo`lishi kelib chiqadi. Demak, $\{S_n\}$ ketma-ketlik chegaralangan. Unda berilgan qator 2-teoremaga ko`ra yaqinlashuvchi bo`ladi

1^0. Absolyut va shartli yaqinlashuvchi qatorlar tushunchasi.

Faraz qilaylik,

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + ... + a_n + ... \quad (1)$$

qator berilgan bo`lsin. Bu qatorning har bir hadi ixtiyoriy ishorali haqiqiy sonlardan iborat. (Odatda, bunday qator ixtiyoriy qator deyiladi.)

(1) qator hadlarining absolyut qaiymatlaridan ushbu

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_n| + ... + |a_n| + ... \quad (2)$$

qatorni tuzamiz.

1-teorema. Agar (2) qator yaqinlashuvchi bo`lsa, u holda (1) qator ham yaqinlashuvchi bo`ladi.

Isbot. Faraz qilaylik, (2) qator yaqinlashuvchi bo`lsin. Unda qator yaqinlashuvchiligi haqidagi Koshi teoremasiga ko`ra

$$\forall \varepsilon > 0, \quad \exists n_0 \in N, \quad \forall n > n_0 \quad m = 1,2,3... \text{ da}$$

$$|a_{n+1}| + |a_{n+2}| + ... + |a_{n+m}| < \varepsilon$$

bo`ladi. Ravshanki,

$$|a_{n+1} + a_{n+2} + ... + a_{n+m}| \le |a_{n+1}| + |a_{n+2}| + ... + |a_{n+m}|.$$

keyingi ikki munosabatdan

$$\forall \varepsilon > 0, \quad \exists n_0 \in N, \forall n > n_0, \quad m = 1,2,3,... \text{ da}$$

$$|a_{n+1} + a_{n+2} + ... + a_{n+m}| < \varepsilon$$

bo`lishi kelib chiqadi. Koshi teoremasiga muvofiq (1) qator yaqinlashuvchi bo`ladi.

1-ta'rif. Agar $\sum_{n=1}^{\infty} |a_n|$ qator yaqinlashuvchi bo`lsa, $\sum_{n=1}^{\infty} a_n$ qator absolyut yaqinlashuvchi deyiladi.

Masalan,ushbu $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}(-1)^{n-1} = 1 - \frac{1}{2^\alpha} + \frac{1}{3^\alpha} - \frac{1}{4^\alpha} + ... + \frac{(-1)^{n-1}}{n^\alpha} + ...$

qator $\alpha > 1$ bo`lganda absolyut yaqinlashuvchi qator bo`ladi, chunki

$$\sum_{n=1}^{\infty} \left| \frac{1}{n^\alpha}(-1)^{n-1} \right| = 1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} - \frac{1}{4^\alpha} + ... + \frac{1}{n^\alpha} + ...$$

umumlashgan garmonik qator $\alpha > 1$ bo`lganda yaqinlashuvchi.

2-ta'rif. Agar $\sum_{n=1}^{\infty} a_n$ qator yaqinlashuvchi bo`lib, $\sum_{n=1}^{\infty} |a_n|$ qator uzoqlashuvchi bo`lsa, $\sum_{n=1}^{\infty} a_n$ qator shartli yaqinlashuvchi qator deyiladi.

Misol. Ushbu $\sum_{n=1}^{\infty} \frac{1}{n}(-1)^{n-1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... + \frac{(-1)^{n-1}}{n} + ...$

qator shartli yaqinlashuvchi qator bo`ladi.

Ravshanki,berilgan qatorning qismiy yig`indisi

$$S_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... + \frac{(-1)^{n-1}}{n} \quad (3)$$

bo`ladi.

Ma'lumki, $\ln(1+x)$ funksiyaning Makloren formulasiga ko`ra yoyilmasi

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... + \frac{(-1)^{n-1}x^n}{n} + R_{n+1}(x),$$

bo`lib, $0 \le x \le 1$ bo`lganda

$$|R_{n+1}(x)| < \frac{1}{n+1}$$

bo`lar edi.

Xullas, $x = 1$ bo`lganda

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... + \frac{(-1)^{n-1}}{n} + R_{n+1}(1)$$

$$|R_{n+1}(1)| < \frac{1}{n+1} \quad (4)$$

bo`ladi.

(3) va (4) munosabatlardan

$$\ln 2 = S_n + R_{n+1}(1)$$

va undan

$$|S_n - \ln 2| < \frac{1}{n+1}$$

bo`lishi kelib chiqadi.

Demak, $n \to \infty$ da $S_n \to \ln 2$.Bu esa qaralayotgan qatorning yaqinlashuvchi ekanligini bildiradi.

Ayni paytda, berilgan qator hadlarning absolyut qiymatlaridan tuzilgan qator

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} + ...$$

garmonik qator bo`lib, uning uzoqlashuvchiligi ma'lum. Demak,berilgan qator shartli yaqinlashuvchi qator.

Endi

$$\sum_{n=1}^{\infty}|a_n|=|a_1|+|a_2|+...+|a_n|+...$$

qatorning musbat hadli qator ekanligini e'tiborga olib, $\sum_{n=1}^{\infty}a_n$ qatorning absolyut yaqinlashuvchiligi ifodalovchi alomatlarni keltiramiz.

Dalamber alomati. Faraz qilaylik,

$$\sum_{n=1}^{\infty}a_n=a_1+a_2+...+a_n+... \qquad (a_n\neq 0,\quad n=1,2,...)$$

qator hadlari uchun

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=d$$

limit mavjud bo`lsin. U holda:

1) $d<1$ bo`lganda, $\sum_{n=1}^{\infty}a_n$ qator absolyut yaqinlashuvchi bo`ladi,

2) $d>1$ bo`lganda, $\sum_{n=1}^{\infty}a_n$ qator uzoqlashuvchi bo`ladi.

Koshi alomati. Faraz qilaylik,

$$\sum_{n=1}^{\infty}a_n=a_1+a_2+...+a_n+...$$

qator hadlari uchun

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}=K$$

limiti mavjud bo`lsin. U holda:

1) $K<1$ bo`lganda, $\sum_{n=1}^{\infty}a_n$ qator absolyut yaqinlashuvchi bo`ladi.

2) $K>1$ bo`lganda, $\sum_{n=1}^{\infty}a_n$ qator uzoqlashuvchi bo`ladi.

2^0. Absolyut yaqinlashuvchi qatorlarning xossalari.

Absolyut yaqinlashuvchi qatorlarning gossalarini keltiramiz.

1) Agar qator absolyut yaqinlanlashuvchi bo`lsa,u holda bu qator yaqinlashuvchi bo`ladi.

Bu xossaning isboti 1-teoremadan kelib chiqadi.

2) Agar

$$\sum_{n=1}^{\infty}a_n=a_1+a_2+...+a_n+... \qquad (1)$$

qator absolyut yaqinlashuvchi bo`lib, $\{b_n\}$ sonlar ketma-ketligi chegaralangan bo`lsa, u holda

$$\sum_{n=1}^{\infty}a_nb_n=a_1b_1+a_2b_2+...+a_nb_n+... \qquad (5)$$

qator absolyut yaqinlashuvchi bo`ladi.

Isbot. Shartga ko`ra $\{b_n\}$ sonlar ketma-ketligi chegaralangan. Demak,

$$\exists M>0,\ \forall n\in N \text{ да } |b_n|\le M \qquad (6)$$

bo`ladi.

(1) qator absolyut yaqinlashuvchi.Unda Koshi teoremasiga ko`ra $\forall\varepsilon>0$ son olinganda ham $\frac{\varepsilon}{M}$ ga ko`ra shunday $n_0\in N$ topiladiki, $\forall n>n_0$ va $m=1,2,3,...$ bo`lganda

$$|a_{n+1}|+|a_{n+2}|+...+|a_{n+m}|<\frac{\varepsilon}{M} \qquad (7)$$

bo`ladi.

(6) va (7) munosabatlardan foydalanib topamiz:

$$|a_{n+1}b_{n+1}|+|a_{n+2}b_{n+2}|+...+|a_{n+m}b_{n+m}|\le$$
$$\le M(|a_{n+1}|+|a_{n+2}|+...+|a_{n+m}|)<\varepsilon.$$

Yana Koshi teoremasidan foydalanib, $\sum_{n=1}^{\infty}a_nb_n$ qaqtorning absolyut yaqinlashuvchi ekanligini topamiz.

3) Faraz qilaylik,

$$\sum_{n=1}^{\infty}a_n=a_1+a_2+...+a_n+... \qquad (1)$$

qator hadlar hadlarining o`rinlarini almashtirish natijasida ushbu

$$\sum_{j=1}^{\infty}a'_j=a'_1+a'_2+...+a'_j+... \qquad (8)$$

qator hosil qilingan bo`lsin.

Ravshanki, (8) qatorning har bir a'_j hadi $(j=1,2,...)$ (1) qatoning tayin bir a_{k_j} hadining aynan o`zidir,ya'ni $\forall j\in N,\ \exists k_j\in N, a_{k_j}=a'_j$ bo`ladi.

Agar (1) qator absolyut yaqinlashuvchi bo`lib,uning yig`indisi S ga teng bo`lsa, u holda bu qator hadlarining o`rninlarini ixtiyoriy ravishda almashtirishdan hosil bo`lgan (8) qator absolyut yaqinlashuvchi va uning yig`indisi ham S ga teng bo`ladi.

Isbot. Aytaylik, (1) qator absolyut yaqinlashuvchi bo`lib, uning yig`indisi S ga teng bo`lsin.

(8) qator hadlarining absolyut qiymatlaridan tuzilgan $\sum_{j=1}^{\infty}|a'_j|$ qatorning qismiy yig`indisini σ'_n bilan belgilaylik:

$$\sigma'_n = \sum_{j=1}^{n}|a'_j|. \qquad (a'_j = a_{k_j})$$

Agar $n' = \max\limits_{1\le j\le n} k_j$ deyilsa,unda $n' \ge n$ va $\forall n \in N$ bo`lganda

$$\sigma'_n \le \sum_{k=1}^{n'}|a_k|$$

bo`ladi. (1) qator absolyut yaqinlashuvchi bo`lgani sababli uning qismiy yig`indilari ketma-ketligi yuqoridan chegaralangandir.Binobarin, $\sigma'_n$ yig`indi ham yuqoridan chegaralangan bo`ladi.Unda musbat hadli qatorning yaqinlashuvchiligi haqidagi teoremaga ko`ra $\sum_{j=1}^{\infty}|a'_j|$ qator va ayni paytda $\sum_{j=1}^{\infty}a'_j$ qator ham yaqinlashuvchi bo`ladi. Demak, $\sum_{j=1}^{\infty}a'_j$ qator absolyut yaqinlashuvchi.Uning yig`indisini S' deylik.

Endi berilgan $\sum_{n=1}^{\infty}a_n$ qator hadlarining o`rinlarini ixtiyoriy ravishda almashtirishdan hosil bo`lgan

$$\sum_{n=1}^{\infty}a'_k = a'_1 + a'_2 + \ldots + a'_k + \ldots$$

qator yig`indisini $S$ ga teng ekaninni istbotlaymiz. Buning uchun $\forall\varepsilon > 0$ ga ko`ra shunday $\bar{n} \in N$ topilib, $\forall n > \bar{n}$ da

$$\left|\sum_{k=1}^{n}a'_k - S\right| < \varepsilon \qquad (9)$$

bo`lishini ko`ratish yetarli bo`ladi.

Ixtiyoriy musbat ε sonni tanlab olamiz. Modomiki, $\sum_{k=1}^{\infty}a_k$ qator absolyut yaqinlashuvchi ekan, unda Koshi teoremasiga binoan olingan $\varepsilon > 0$ songa ko`ra shunday n_0 nomer topiladiki,

$$\sum_{k=n_0+1}^{n_0+m}|a_k| < \frac{\varepsilon}{2} \qquad (m = 1,2,3,\ldots) \qquad (10)$$

shuningdek, qatorning yaqinlashish ta'rifiga ko`ra

$$\left|\sum_{k=1}^{n_0}a_k - S\right| < \frac{\varepsilon}{2} \qquad (11)$$

bo`ladi.

Yuqoridagi natural son $\bar{n}$ ni shunday katta qilib olamizki, $\sum_{n=1}^{\infty}a'_k$ qatorning $\bar{n}$ dan katta bo`lmagan $n$ nomerli ixtiyoriy qismiy yig`indisi

$$S'_n = \sum_{k=1}^{n}a'_k \text{ da } \sum_{k=1}^{\infty}a_k$$

qatorning barcha dastlabki n_0 ta hadi qatnashsin.

Ravshanki,

$$\sum_{k=1}^{n}a'_k - S = \left(\sum_{k=1}^{n}a'_k - \sum_{k=1}^{n_0}a_k\right) + \left(\sum_{k=1}^{n_0}a_k - S\right).$$

Keyingi munosabatdan va (11) tengsizlikni e'tiborga olib topamiz.

$$\left|\sum_{k=1}^{n}a'_k - S\right| \le \left|\sum_{k=1}^{n}a'_k - \sum_{k=1}^{n_0}a_k\right| + \left|\sum_{k=1}^{n_0}a_k - S\right| < \left|\sum_{k=1}^{n}a'_k - \sum_{k=1}^{n_0}a_k\right| + \frac{\varepsilon}{2} \qquad (12)$$

Ma'lumki, $n > \bar{n}$ bo`lganda $\sum_{k=1}^{\infty}a'_k$ qatorda $\sum_{k=1}^{\infty}a_k$ qatorning barcha dastlabki n_0 ta hadi qatnashadi.Binobarin,

$$\sum_{k=1}^{n}a'_k - \sum_{k=1}^{n_0}a_k$$

ayirma $\sum_{k=1}^{\infty}a_k$ qatorning,har bir hadining nomeri n_0 dan katta bo`lgan $n - n_0$ ta hadining yig`indisidan iborat.

Endi natural m sonni shunday katta qilib olamizki,bunda $n_0 + m$ son yuqorida aytilgan barcha $n - n_0$ ta hadlarning nomerlaridan katta bo`lsin.

Unda

$$\left|\sum_{k=1}^{n}a'_k - \sum_{k=1}^{n_0}a_k\right| \le \sum_{k=n_0+1}^{n_0+m}|a_k| \qquad (13)$$

bo`ladi.

(12), (13) va (10) munosabatlardan foydalanib, (9) tensizlikning, ya'ni

$$\left|\sum_{k=1}^{n}a'_k - S\right| < \varepsilon$$

tengsizlikni bajarilishini topamiz.

Berilgan qatorlarni absolyut yaqinlashishga tekshiring (1-11).Berilgan qatorlar α va β ning qanday qiymatida absolyut yaqinlashuvchi bo`ladi (12-30):

1. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2n-1}$

2. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\ln(n+1)}$

3. $\sum_{n=1}^{\infty}\frac{\sin n\alpha}{n^2}$

4. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}\cdot\frac{1}{2^n}$

5. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n+1}{n}$

6. $\sum_{n=1}^{\infty}(-1)\frac{1}{\sqrt{n}}$

7. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^3}{2^n}$

8. $\sum_{n=1}^{\infty}\frac{(-1)^n}{n-\ln n}$

9. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^{n^2}}{n!}$

10. $\sum_{n=1}^{\infty}(-1)^n\ln\cos\frac{\sqrt{n}}{n+2}$

11. $\sum_{n=1}^{\infty}\frac{(-1)^{[\ln n]}}{n}$

12. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\alpha}}$

13. $\sum_{n=1}^{\infty}\frac{(-1)^n}{n-\alpha}$

14. $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{\alpha+n^{-1}}}$

15. $\sum_{n=1}^{\infty}\ln\left(1+\frac{(-1)^{n-1}}{(n+1)^{\alpha}}\right)$

16. $\sum_{n=1}^{\infty}\frac{(-1)^n}{\left(2n+(-1)^n\right)^{\alpha}}$

17. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\left(\sqrt{n+1}+(-1)^n\right)^{\alpha}}$

18. $\sum_{n=1}^{\infty}\frac{(-1)^{[n]}}{n^{\alpha}}$

19. $\sum_{n=1}^{\infty}\frac{\cos n}{n^{\alpha}}$

20. $\sum_{n=1}^{\infty}\frac{\sin 2n\ln^2 n}{n^{\alpha}}$

21. $\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1\cdot3\cdot\ldots\cdot(2n-1)}{2\cdot4\cdot6\cdot\ldots\cdot(2n)}\right)^{\alpha}$

22. $\sum_{n=1}^{\infty}\frac{\sin nx}{n\ln^{\alpha}(n+1)}, 0<x<\pi$

23. $\sum_{n=1}^{\infty}\frac{(-1)^n\sin^{2n}\alpha}{n}$

24. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^n\cos^{2n}\alpha}{\sqrt{n}}$

25. $\sum_{n=1}^{\infty}\frac{\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-(n-1))}{n!}$

26. $1+\frac{1}{3^{\alpha}}-\frac{1}{2^{\alpha}}+\frac{1}{5^{\alpha}}+\frac{1}{7^{\alpha}}-\frac{1}{4^{\alpha}}+\ldots$

27. $1+\frac{1}{3^{\alpha}}-\frac{1}{1^{\alpha}}+\frac{1}{5^{\alpha}}+\frac{1}{7^{\alpha}}-\frac{1}{3^{\alpha}}+\frac{1}{9^{\alpha}}+$
$+\frac{1}{11^{\alpha}}-\frac{1}{5^{\alpha}}+\ldots$

28. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(1+\alpha)(2+\alpha)\ldots(n+\alpha)}{n^{\beta}n!}$

29. $\frac{1}{1^{\alpha}}-\frac{1}{2^{\beta}}+\frac{1}{3^{\alpha}}-\frac{1}{4^{\beta}}+\frac{1}{5^{\alpha}}-\frac{1}{6^{\beta}}+\ldots$

30. $1-\frac{2}{2^{\beta}}+\frac{1}{3^{\alpha}}+\frac{1}{4^{\alpha}}-\frac{2}{5^{\beta}}+\frac{1}{6^{\alpha}}+\frac{1}{7^{\alpha}}-\frac{2}{8^{\beta}}+\frac{1}{9^{\alpha}}+\ldots$

9.7.Ishorasi almashinuvchi qatorlar

Ishorasi almashinuvchi qatorlar.

Ta'rif: Hadlari ishoralari ham musbat, ham manfiy bo`lgan qatorlarga **ishorasi o`zgaruvchi qatorlar** deyiladi.

Ishorasi o`zgaruvchi qtor ishorasi o`zgarmas bo`lgan qatorga, ya'ni barcha hadlarining ishorasi bir xil bo`lgan qatorga qarama-qarshi qo`yiladi. Ishora o`zgaruvchi qatorning xususiy holiishorasi almashinuvchi qatordir.

Ta'rif: Hadlari navbat bilan musbat va manfiy bo`ldigan ishora o`zgaruvchi qatorga **ishorasi almashinuvchi qator** deyiladi va u quydagicha ifodalanadi:

$$a_1-a_2+a_3-a_4+\ldots+(-1)^{n-1}a_n+\ldots$$

Bunda $a_1,a_2,\ldots,a_n$ lar musbat sonlardir.

Leybnist alomati:

Teorema. Agar

$$a_1-a_2+a_3-a_4+\ldots+(-1)^{n-1}a_n+\ldots \quad (1)$$

ishorasi almashinuvchi qator hadlarining absolyut kattaliklari monoton kamayuvchi bo`lsa,

$$a_1\ge a_2\ge a_3\ge a_4\ge\ldots\ge a_n\ldots \quad (2)$$

hamda qatorning umumiy hadi a_n nolga intilsa,ya'ni

$$\lim_{n\to\infty}a_n=0 \quad (3)$$

bo`lsa,(1) qatorning yig`indisi uchun

$$0\le s\le a_1 \quad (4)$$

tengsizlik bajariladi hamdaberilgan qator yaqinlashuvchi bo`ladi.

Isbot: Berilgan qatorning juft va toq nomerli hadlarining alohida-alohida xususiy yig`indilarini topamiz. U holda, juft nomerli hadlari yig`indisi

$$S_{2n}=a_1-a_2+a_3-a_4+\ldots+a_{2n-1}-a_{2n}=(a_1-a_2)+(a_3-a_4)+\ldots+(a_{2n-1}-a_{2n}). \quad (5)$$

(1)ketma-ketlik manfiy bo`lmagani sababli, $S_{2n}\ge0$ dir.

Bundan tashqari,

$$S_{2n+2}=S_{2n}+(a_{2n+1}-a_{2n+2})\ge S_{2n} \quad (6)$$

bo`lgani uchun $n\to\infty$ da $S_{2n}$ kamayuvchi bo`lmaydi.

S_{2n} xususiy yig`indini quydagicha ifodalash mumkin:

$$S_{2n}=a_1-(a_2-a_3)-\ldots-(a_{2n-2}-a_{2n-1})-a_{2n} \quad (7)$$

Qavslar ichidagi ayirmalar va a_{2n} lar manfiy bo`lmagani sababli

$$s_{2n} \le a_1$$

Demak, juft nomerli hadlarning xususiy yig`indisi kamayuvchi bo`lganligi hamda yuqoridan chegaralanganligi uchun ulimitga ega,ya'ni:

$$\lim_{n\to\infty} S_{2n} = s \quad (\text{bunda} \quad s \ge 0) \tag{8}$$

Qatordagi toq nomerdagi hadlarning xususiy yig`indilari uchunquydagi o`rinlidir.

$$S_{2m+1} = S_{2n} + a_{2n+1}$$

Bundan,

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty}(S_{2n} + a_{2n+1}) = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} a_{2n+1} = S + 0 = S \tag{9}$$

U holda,quydagi ham o`rinli bo`ladi:

$$\lim_{n\to\infty} S_n = S \tag{10}$$

$S_{2n} \ge 0$ bo`lgani uchun $S \ge 0,\ n > 1$ da $S_{2n} \le a_1 - (a_2 - a_3) = b < a_1$

Bundan,

$$0 \le S = \lim_{n\to\infty} S_n \le b \le a_1$$

teorema isbot bo`ldi.

Misol. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots \frac{(-1)^{n+1}}{n} + \ldots$ qator yaqinlashishini Leybnits alomati yordamida tekshiring.

Yechilishi: Berilgan ishorasi almashinuvchi qator kamayuvchidir,ya'ni:

$$1 > \frac{1}{2} > \frac{1}{3} > \frac{1}{4} > \ldots > \frac{1}{n} > \ldots$$

$n \to \infty$ da a_n ning limiti nolga intiladi,ya'ni

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

Demak, Leybnits alomatidagi shartlar bajariladi.U holda berilgan qator yaqinlashuvchi bo`ladi.

Ishorasi almashuvchi qatorni yaqinlashishga tekshiring:

1. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \ln^2 n}{2^n}$
2. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[5]{n}} \arcsin \frac{\pi}{4n}$
3. $\sum_{n=1}^{\infty} \frac{(-n)^n}{(2n)!}$
4. $\sum_{n=1}^{\infty} \frac{(-1)^n (2n)!!}{(n+1)^n}$
5. $\sum_{n=1}^{\infty} \frac{(-1)^n \ln^2(n+1)}{n\sqrt{n+1}}$
6. $\sum_{n=1}^{\infty} (-1)^{\frac{n(n+1)}{2}} \frac{2^n + n^2}{3^n + n^3}$
7. $\sum_{n=1}^{\infty} \frac{(-1)^n \sin 3n}{n \ln(n+1) \cdot \ln^2(n+2)}$
8. $\sum_{n=1}^{\infty} (-1)^n \left(arctg \frac{1}{\sqrt{n}} - \arcsin \frac{1}{\sqrt{n}} \right)$
9. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n+1}}$
10. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \ln n}{\sqrt{n}}$
11. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \ln\ln(n+2)}{\ln(n+2)}$
12. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{(n+2)\sqrt[3]{n+1}}$
13. $\sum_{n=1}^{\infty} (-1)^n \left(1 - \cos \frac{\pi}{\sqrt{n}}\right)$
14. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt[n]{n^2+1}}$
15. $\sum_{n=1}^{\infty} (-1)^n \frac{n+2}{\sqrt{n+1}} arctg \frac{\pi}{\sqrt{n}}$
16. $\sum_{n=1}^{\infty} (-1)^{\frac{n(n-1)}{2}} \frac{1}{\sqrt{n}}$
17. $\sum_{n=1}^{\infty} (-1)^n \frac{\cos^2 2n}{\sqrt{n}}$
18. $\sum_{n=1}^{\infty} (-1)^n \frac{\sin^2\left(\frac{n}{2}\right)}{\sqrt[5]{n+1}}$
19. $\sum_{n=1}^{\infty} (-1)^{\frac{n(n-1)}{2}} \frac{1}{\sqrt{n+2}} \left(1 + \frac{2}{n}\right)^2$
20. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt[3]{n+1}}{\sqrt{n+2}}$
21. $\sum_{n=1}^{\infty} (-1)^n \frac{n^2 2^n}{3^n + 1}$
22. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n \ln(n+1)}$
23. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{(n+1)\sqrt{n+2}} tg \frac{1}{\sqrt{n}}$
24. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n \ln(n+1) \ln\ln(n+2)}$
25. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+4)}{\sqrt[3]{n^2+1}\left(2 + \sqrt{n^2+3}\right)}$
26. $\sum_{n=1}^{\infty} \frac{(-1)^n}{2\sqrt[3]{n} + (-1)^{n-1}}$
27. $\sum_{n=1}^{\infty} (-1)^n \frac{2 + (-1)^n}{n \ln(n+1)}$
28. $\sum_{n=1}^{\infty} (-1)^n n\, arctg \frac{n+1}{n^3}$
29. $\sum_{n=1}^{\infty} (-1)^n \sin\left(\sqrt{n^2-1} - n\right)$
30. $\sum_{n=1}^{\infty} (-1)^n sh \frac{n^2+1}{2n^3 - n^2}$

X BOB
FUNKSIONAL KETMA-KETLIK VA QATORLAR

10.1. Funksional ketma-ketlik va ularning yaqinlashuvchanligi.

$X \subset R$ to`plam berilgan bo`lib, unda

$$f_1(x), f_2(x), \dots, f_n(x), \dots$$

funksiyalar aniqlangan bo`lsin. Ana shu funksiyalardan tuzilgan ketma-ketlikka $X$ to`plamda berilgan **funksional ketma-ketlik** deyiladi va u $\{f_n(x)\}$ kabi belgilanadi:

$$\{f_n(x)\}: f_1(x), f_2(x), \dots, f_n(x), \dots \qquad (1)$$

$f_n(x)$ ga funksional ketma-ketlikning umumiy hadi deyiladi.

Ixtiyoriy $x_0 \in X$ nuta olib,ushbu

$$\{f_n(x_0)\}: f_1(x_0), f_2(x_0), \dots, f_n(x_0), \dots \qquad (2)$$

sonli ketma-ketlikni qaraymiz. Agar bu sonli ketma-ketlik **yaqinlashuvchi (uzoqlashuvchi)** bo`lsa, $\{f_n(x)\}$ funksional ketma-ketlik x_0 nuqtada **yaqinlashuvchi (uzoqlashuvchi)** deyiladi, x_0 nuqta esa funksional ketma-ketlikning **yaqinlashish (uzoqlashish) nuqtasi** deb ataladi.

$\{f_n(x)\}$ funksional ketma-ketlik barcha yaqinlashish nuqtalaridan iborat $M\ (M \subset R)$ to`plam $\{f_n(x)\}$ funksional ketma-ketlikning **yaqinlashish sohasi deyiladi.** $\Rightarrow \forall x \in M$ uchun ushbu $\lim\limits_{n\to\infty} f_n(x) - \exists$ bo`ladi.Agar $\forall x \in M$ uchun unga mos keluvchi $\lim\limits_{n\to\infty} f_n(x)$ ni mos qo`ysak,ya'ni

$$f: x \to \lim_{n\to\infty} f_n(x)$$

bo`lsa unda $M$ to`plamda aniqlangan $f(x)$ funksiya hosil bo`ladi.Bu $f(x)$ funksiya $\{f_n(x)\}$ ketma-ketlikning **limit funksiyasi** deyiladi. Demak,

$$\lim_{n\to\infty} f_n(x) = f(x) \quad (x \in M)$$

Tekis yaqinlashishga tekshiring.

1. $f_n(x) = \frac{x}{x+n}$, $E = [0;a]$, $0 < a < \infty$
2. $f_n(x) = \frac{nx^2}{1+2n+x}$, $E = [0;1]$
3. $f_n(x) = \frac{nx^2}{1+n^2x^4}$, $E = [1;\infty]$
4. $f_n(x) = arctg\frac{n}{x}$, $E = (0;\infty)$
5. $f_n(x) = n\left(\frac{x}{\sqrt{n}} - arctg\frac{x}{\sqrt{n}}\right)$, $E = [0;1]$
6. $f_n(x) = \frac{(n+x)^2}{x^2+n^2-nx}$, $E = [0;2)$
7. $f_n(x) = \frac{xn}{n+x^n}$, $E = (1;\infty)$
8. $f_n(x) = \frac{arctgn^2x}{x}$, $(0;1)$
9. $f_n(x) = narctg\frac{\ln x}{n}$ $E = (e;5)$
10. $f_n(x) = \sqrt[n]{x^2+nx+1}$, $E = (0;1)$
11. $f_n(x) = e^{-x^2-nx}$, $E = (1;\infty)$
12. $f_n(x) = \frac{x}{n}\ln\frac{x}{n}$, $E = (0;2)$
13. $f_n(x) = \frac{nx^2}{n^3+x^3}$, $E = [0;1]$
14. $f_n(x) = \frac{1}{x^3}\cos\frac{x}{n}$, $E = (1;\infty)$
15. $f_n(x) = \sqrt{n}\left(\sqrt{1+nx} - \sqrt{nx}\right)$, $E = (0;1)$
16. $f_n(x) = x^2\sqrt{1+\frac{1}{nx}}$, $E = (0;1)$
17. $f_n(x) = \frac{x^2\sqrt{n} - x\sqrt{n+1} - 1}{x\sqrt{n} - \sqrt{n+1}}$, $E\left(0;\frac{1}{2}\right)$
18. $f_n(x) = \frac{\sqrt{1+n^2x}}{nx}$, $E = (0;1)$
19. $f_n(x) = sh\left(ne^{-nx}\right)$, $E = (1;\infty)$
20. $f_n(x) = n^2\left(ch\left(\frac{x}{n}\right) - 1\right)$, $E = (0;1)$
21. $f_n(x) = nsh\frac{x}{n+x}$, $E = (0;1)$
22. $f_n(x) = ch\left(\frac{\ln n}{n} + \frac{x}{n}\right)$, $E = (0;1)$
23. $f_n(x) = n^2\left(nsh\frac{1}{nx} - \frac{1}{x}\right)$, $E = (1;\infty)$
24. $f_n(x) = \frac{th(1+nx)}{x}$, $E = (0;1)$
25. $f_n(x) = \frac{\sqrt{n} + arctgnx}{\sqrt{nx}}$, $E = (0;1)$
26. $f_n(x) = e^x\cos\left(\frac{1}{nx}\right)$, $E = (1;2)$
27. $f_n(x) = \frac{1}{n\sqrt{x}}\ln\frac{e^{nx}}{nx}$, $E = (0;4)$
28. $f_n(x) = narctgx^n$, $E = [0;1)$
29. $f_n(x) = e^{(x^2+nx+1)/n}$, $E = (0;1)$
30. $f_n(x) = n\left(1 - e^{\frac{x}{n}}\right)$, $E = (1;\infty)$

10.2. Funksional ketma-ketlikning tekis yaqinlashuvchanligi.

Funksional ketma-ketlikning tekis yaqinlashuvchanligi. Faraz qilaylik, $\{f_n(x)\}$:

$$f_1(x), f_2(x), \dots, f_n(x), \dots$$

funksional ketma-ketlik E_0 to`plamda yaqinlashuvchi (ya'ni yaqinlashish to`plami E_0) bo`lib, uning limit funksiyasi $f(x)$ bo`lsin:

$$\lim_{n\to\infty} f_n(x) = f(x).$$

Ma'lumki, bu munosabat

$$\forall \varepsilon > 0,\ \exists n_0 = n_0(\varepsilon, x) \in N,\ \forall n > n_0 : |f_n(x) - f(x)| < \varepsilon$$

bo`lishini anglatadi. Shuni ta'kidlash lozimki, yuqridagi natural $n_0$ son ixtiyoriy olingan $\varepsilon > 0$ son bilan birga qaralayotgan $x \in E_0$ nuqtaga ham bog`liq bo`ladi (chunki, $x \in E_0$ ning turli qiymatlarida ularga mos ketma-ketlik,umuman aytganda turlicha bo`ladi).

1-ta'rif. Agar $\forall \varepsilon > 0$ son olinghanda ham shu $\varepsilon > 0$ gagina bog`liq bo`lgan natural $n_0 = n_0(\varepsilon)$ son topilsaki, $\forall n > n_0$ va ixtiyoriy $x \in E_0$ da

$$|f_n(x) - f(x)| < \varepsilon$$

tengsizlik bajarilsa, ya'ni

$$\forall \varepsilon > 0,\ \exists n_0 = n(\varepsilon) \in N,\ \forall n > n_0,\ \forall x \in E : |f_n(x) - f(x)| < \varepsilon$$

bo`lsa, $\{f_n(x)\}$ funksional ketma-ketlik $E_0$ to`plamda $f(x)$ ga tekis yaqinlashadi (funksional ketma-ketlik E_0 to`plamda tekis yaqinlashuvchi) deyiladi.

Shunday qilib, $\{f_n(x)\}$ funksional ketma-ketlik E_0 to`plamda $f(x)$ limit funksiyaga ega bo`lsa,uning shu limit funksiyasiga yaqinlashish ikki xil bo`lar ekan:

1) $\forall \varepsilon > 0,\ \exists n_0 = n_0(\varepsilon, x) \in N,\ \forall n > n_0 : |f_n(x) - f(x)| < \varepsilon$

bo`lsa, $\{f_n(x)\}$ funksional ketma-ketlik E_0 da $f(x)$ ga yaqinlashadi (oddiy yaqinlashadi). Bu holda

$$f_n(x) \to f(x) \qquad (x \in E_0)$$

kabi belgilanadi.

2) $\forall \varepsilon > 0,\ \exists n_0 = n_0(\varepsilon) \in N,\ \forall n > n_0, \forall x \in E_0 : |f_n(x) - f(x)| < \varepsilon$

bo`lsa, $\{f_n(x)\}$ funksional ketma-ketlik E_0 da $f(x)$ ga tekis yaqinlashadi.Bu holda

$$f_n(x) \rightrightarrows f(x) \qquad (x \in E_0)\, f_n(x) \to f(x) \qquad (x \in E_0)$$

kabi belgilanadi.

Ravshanki, $\{f_n(x)\}$ funksional ketma-ketlik E_0 to`plamda $f(x)$ funksiyaga tekis yaqinlashsa u shu to`plamda $f(x)$ ga yaqinlashadi:

$$f_n(x) \rightrightarrows f(x) \quad \Rightarrow \quad f_n(x) \to f(x) \quad (x \in E_0).$$

Aytaylik,

$$f_n(x) \rightrightarrows f(x) \qquad (x \in E_0)$$

bo`lsin.Bu holda $\forall n > n_0$ va $\forall x \in E_0$ da

$$|f_n(x) - f(x)| < \varepsilon, \text{ ya'ni } f(x) - \varepsilon < f_n(x) < f(x) + \varepsilon$$

bo`ladi. Bu esa $\{f_n(x)\}$ funksional ketma-ketlikning biror hadidan boshlab, keyingi barcha hadlari $f(x)$ funksiyaning "$\varepsilon$ -oralig`i"da butunlay joylashishini bildiradi.

1-misol. Ushbu $f_n(x) = \dfrac{\sin nx}{n}$

funksional ketma-ketlikning R da tekis yaqinlashuvchiligi ko`rsatilsin.

Ravshanki,

$$\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{\sin nx}{n} = 0.$$

Demak,limit funksiya $f(x) = 0$.

Agar $\forall \varepsilon > 0$ son olinganda $n_0 = \left[\dfrac{1}{\varepsilon}\right]$ deyilsa, unda $\forall n > n_0$ va $\forall x \in R$ uchun

$$|f_n(x) - f(x)| = \left|\frac{\sin nx}{n} - 0\right| = \left|\frac{\sin nx}{n}\right| \le \frac{1}{n} \le \frac{1}{n_0 + 1} < \varepsilon$$

bo`lishini topamiz. Demak ta'rifga binoan

$$\frac{\sin nx}{n} \rightrightarrows 0$$

bo`ladi.

Faraz qilaylik, $\{f_n(x)\}$ funksional ketma-ketlik E_0 to`plamda $f(x)$ limit funksiyaga ega bo`lsin.

1-teorema. $\{f_n(x)\}$ funksional ketma-ketlik E_0 to`plamda $f(x)$ funksiyaga tekis yaqinlashishi uchun

$$\lim_{n \to \infty} \sup_{x \in E_0} |f_n(x) - f(x)| = 0$$

bo`lishi zarur va yetarli.

Isbot. Zarurligi. Aytaylik, $f_n(x) \rightrightarrows f(x) \quad (x \in E_0)$ bo`lsin. Ta'rifga binoan

$$\forall \varepsilon > 0,\ \exists n_0 = n_0(\varepsilon) \in N, \forall n > n_0, \forall x \in E_0 : |f_n(x) - f(x)| < \varepsilon$$

bo`ladi. Bu tengsizlikdan

$$\sup_{x \in E_0} |f_n(x) - f(x)| \le \varepsilon$$

bo`lib,unda

$$\lim_{n \to \infty} \sup_{x \in E_0} |f_n(x) - f(x)| = 0$$

bo`lishi kelib chiqadi.

Yetarliligi. Aytaylik, $\lim\limits_{n \to \infty} \sup\limits_{x \in E_0} |f_n(x) - f(x)| = 0$ bo`lsin.Limit ta'rifiga ko`ra

$$\forall \varepsilon > 0,\ \exists n_0 \in N\ \ \forall n > n_0, : \sup_{x \in E_0} |f_n(x) - f(x)| < \varepsilon$$

bo`ladi.Ravshanki

$$|f_n(x)-f(x)|\le\sup_{x\in E_0}|f_n(x)-f(x)|.$$

U holda $\forall x\in E_0$ uchun

$$|f_n(x)-f(x)|<\varepsilon$$

bo`ladi. Bundan

$$f_n(x)\rightrightarrows f(x)\quad (x\in E_0)$$

bo`lishi kelib chiqadi

2-misol. Ushbu $f_n(x)=\sqrt{x^2+\frac{1}{n^2}}$

funksional ketma-ketlikning $E_0=R$ da tekis yaqinlashuvchiligi ko`rsatilsin.

Berilgan funksional ketma-ketlikning limit funksiyasi

$$f(x)=\lim_{n\to\infty}f_n(x)=\lim_{n\to\infty}\sqrt{x^2+\frac{1}{n^2}}=|x|,\quad x\in R$$

bo`ladi.Endi

$$\sup_{x\in R}|f_n(x)-f(x)|$$

ni topamiz:

$$Sup_{x\in R}\left|\sqrt{x^2+\frac{1}{n^2}}-|x|\right|=Sup_{x\in R}\left|\frac{\frac{1}{n^2}}{\left(\sqrt{x^2+\frac{1}{n^2}}+|x|\right)}\right|=Sup_{x\in R}\frac{1}{n^2}\cdot\frac{1}{\left(\left(\sqrt{x^2+\frac{1}{n^2}}+|x|\right)\right)}=\frac{1}{n}.$$

Demak,

$$\lim_{n\to\infty}Sup_{x\in R}\left|\sqrt{x^2+\frac{1}{n^2}}-|x|\right|=\lim_{n\to\infty}\frac{1}{n}=0$$

bo`lib,

$$\sqrt{x^2+\frac{1}{n^2}}\rightrightarrows|x|\quad (x\in R)$$

bo`ladi.

Eslatma. Agar $\{f_n(x)\}$ funksional ketma-ketligi uchun $E\subset R$ to`plamda

$$\lim_{n\to\infty}\sup_{x\in E}|f_n(x)-f(x)|\ne 0$$

bo`lsa, $\{f_n(x)\}$ funksional ketma-ketlik E da tekis yaqinlashishi shart emas.

Endi funksional ketma-ketlikning limit funksiyasiga ega bo`lishi va unga tekis yaqinlashishini ifodalovchi teoremani keltiramiz:

2-teorema (Koshi teoremasi). $\{f_n(x)\}$ funksional ketma-ketlik E to`plamda limit funksiyaga ega bo`lishi va unga tekis yaqinlashishi uchun $\forall\varepsilon>0$ son olinganda ham shunday $n_0=n_0(\varepsilon)\in N$ topilib, $\forall n>n_0$, $\forall p\in N$ va $\forall x\in E$ da

$$|f_{n+p}(x)-f_n(x)|<\varepsilon,$$

ya'ni

$$\forall\varepsilon>0,\exists n_0=n_0(\varepsilon)\in N,\forall n>n_0,\forall p\in N \text{ va } \forall x\in E \text{ da}$$

$$|f_{n+p}(x)-f_n(x)|<\varepsilon \qquad (2)$$

bo`lishi zarur va yetarli.

Isbot. Zarurli. Aytaylik, E to`plamda $\{f_n(x)\}$ funksional ketma-ketlik limit funksiya $f(x)$ ga ega bo`lib, unga tekis yaqinlashsin:

$$f_n(x)\rightrightarrows f(x)\quad (x\in E_0)$$

Tekis yaqinlashish ta'rifiga ko`ra

$$\forall\varepsilon>0,\exists n_0=n_0(\varepsilon)\in N,\forall k>n_0,\forall x\in E:|f_k(x)-f(x)|<\frac{\varepsilon}{2}$$

bo`ladi.Xullas, $k=n$, $n>n_0$ va $k=n+p$, $p\in N$ da

$$|f_n(x)-f(x)|<\frac{\varepsilon}{2},\ |f_{n+p}(x)-f(x)|<\frac{\varepsilon}{2}$$

tengsizliklar bajarilib,ulardan

$$|f_{n+p}(x)-f_n(x)|=|f_{n+p}(x)-f(x)-(f_n(x)-f(x))|\le$$

$$\le|f_{n+p}(x)-f(x)|+|f_n(x)-f(x)|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

bo`lishi kelib chiqadi. Demak, (2) shart o`rinli.

Yetarliligi. $\{f_n(x)\}$ funksional ketma-ketlik uchun (2) shart bajarilsin.Uni quydagicha yozamiz:

$$\forall\varepsilon>0,\exists n_0=n_0(\varepsilon)\in N,\forall n>n_0,\forall p\in N,\ \forall x\in E \text{ da}$$

$$|f_{n+p}(x)-f_n(x)|<\frac{\varepsilon}{2} \qquad (3)$$

bo`ladi.

Ravshanki,tayin $x_0 \in E$ da $\{f_n(x_0)\}$ sonlar ketma-ketligi uchun (3) shartning bajarilishidan uning fundamental ketma-ketlik ekanligi kelib chiqadi.Koshi teoremasiga ko`ra $\{f_n(x_0)\}$ yaqinlashuvchi bo`ladi.Binobarin chekli

$$\lim_{n\to\infty} f_n(x_0) \qquad (4)$$

limit mavjud.

Modomiki, har bir $x \in E$ da (4) limit mavjud bo`lar ekan,undan avval aytganimizdek, $E$ to`plamda aniqlangan

$$x \to \lim_{n\to\infty} f_n(x) \quad (x \in E)$$

funksiya hosil bo`ladi.Uni $f(x)$ bilan belgilaymiz. Bu funksiya $\{f_n(x)\}$ funksional ketma-ketlikning limit funksiyasi bo`ladi:

$$f_n(x) \to f(x) \quad (x \in E).$$

Endi (3) tengsizlikda, n va x larni tanlab $(n > n_0, x \in E)$ $p \to \infty$ da limitga o`tamiz.Natijada

$$|f(x) - f_n(x)| \le \frac{\varepsilon}{2} < \varepsilon$$

hosil bo`ladi.Bu

$$f_n(x) \rightrightarrows f(x) \qquad (x \in E_0)$$

bo`lishini bildiradi.

3-misol. Ushbu $f_n(x) = \dfrac{\ln nx}{\sqrt{nx}}$

funksional ketma-ketlik $E = (0,1)$ to`palmda tekis yaqinlashuvchilikka tekshirilsin.

Agar ixtiyoriy $k \in N$ uchun

$$n = k,\ p = k = n,\ x^* = \frac{1}{k} = \frac{1}{n}$$

deyilsa,

$$\left|f_{n+p}(x) - f(x)\right| = \left|f_{2n}\left(\frac{1}{n}\right) - f_n\left(\frac{1}{n}\right)\right| = \left|\frac{\ln 2}{\sqrt{2}} - \ln 1\right| = \frac{\ln 2}{\sqrt{2}} = \varepsilon_0$$

bo`ladi. Demak,

$$\exists \varepsilon_0 = \frac{\ln 2}{\sqrt{2}} \quad \forall k \in N,\ \exists n \ge k, \exists p \in N,\ \exists x^* = \frac{1}{n} \in (0,1) : \left|f_{n+p}(x^*) - f_n(x^*)\right| \ge \varepsilon_0 .$$

Bu esa yuqoridagi teoremaning shartini bajarmasligini ko`rsatadi. Demak, berilgan funksional ketma-ketlik $E = (0,1)$ da tekis yaqinlashuvchi emas.

Aytaylik, $\{f_n(x)\}$ funksional ketma-ketlik E to`plamda yaqinlashuvchi bo`lib, $f(x)$ funksiya uning limit funksiyasi bo`lsin:

$$f_n(x) \to f(x) \quad (x \in E).$$

Agar

$$\exists \varepsilon_0 > 0\ \forall k \in N,\ \exists n > k,\ \exists x^* \in E : \left|f_{n+p}(x^*) - f_n(x^*)\right| \ge \varepsilon_0$$

bo`lsa, $\{f_n(x)\}$ funksional ketma-ketlikning $E$ to`plamda $f(x)$ funksiyaga notekis yaqinlashadi deyiladi.

4-misol. Ushbu $f_n(x) = n\sin\dfrac{1}{nx}$

funksional ketma-ketlik $E = (0,1)$ da tekis yaqinlashishga tekshirilsin.

Ravshanki,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} n\sin\frac{1}{nx} = \lim_{n\to\infty}\frac{\sin\dfrac{1}{nx}}{\dfrac{1}{nx}\cdot x} = \frac{1}{x}.$$

Demak, berilgan funksional ketma-ketlikning limit funksiyasi $f(x) = \dfrac{1}{x}$ bo`ladi.

Aytaylik, $x^* = \dfrac{1}{n}$ bo`lsin.Unda

$$\left|f_n(x^*) - f(x^*)\right| = \left|n\sin 1 - n\right| \ge 1 - \sin 1 = \varepsilon_0$$

munosabat ixtiyoriy $n \in N$ da o`rinli bo`ladi.

Demak, $f_n(x) = n\sin\dfrac{1}{nx}$ funksional ketma-ketlik limit funksiya $f(x) = \dfrac{1}{x}$ ga $E = (0,1)$ da tekis yaqinlashmaydi.

Tekis yaqinlashuvchi funksional ketma-ketlikning xossalari. Tekis yaqinlashuvchi funksional ketma-ketlikning qator xossalarga ega. Bu xossalarni keltiramiz.

Aytaylik, $\{f_n(x)\}$:

$$f_1(x), f_2(x), \ldots, f_n(x), \ldots$$

funksional ketma-ketlik $E \subset R$ to`plamda yaqinlashuvchi bo`lib, $f(x)$ uning limit funksiyasi bo`lsin:

$$f_n(x) \to f(x) \quad (x \in E).$$

1-xossa. Agar $\{f_n(x)\}$ funksional ketma-ketlikning har bir $f_n(x)$ $(n=1,2,3,...)$ hadi E to`plamda uzluksiz bo`lib,

$$f_n(x) \rightrightarrows f(x) \quad (x \in E_0)$$

bo`lsa, limit funksiya $f(x)$ shu $E$ to`plamda uzluksiz bo`ladi.

Demak,bu holda

$$\lim_{t\to x}\left(\lim_{n\to\infty} f_n(t)\right) = \lim_{n\to\infty}\left(\lim_{t\to x} f_n(t)\right)$$

munosabat o`rinli bo`ladi.

2-xossa. Agar $\{f_n(x)\}$ funksional ketma-ketlikning har bir $f_n(x)$ $(n=1,2,3,...)$ hadi $E=[a,b]$ da uzluksiz bo`lib,

$$f_n(x) \rightrightarrows f(x) \quad (x \in [a,b])$$

bo`lsa,

$$\lim_{n\to\infty}\int_a^b f_n(x)dx = \int_a^b f(x)\,dx$$

bo`ladi.

Demak,bu holda

$$\lim_{n\to\infty}\int_a^b f_n(x)dx = \int_a^b (\lim_{n\to\infty} f_n(x))dx$$

munosabat o`rinli bo`ladi.

3-xossa. Agar $\{f_n(x)\}$ funksional ketma-ketlikning har bir $f_n(x)$ $(n=1,2,3,...)$ hadi $E=[a,b]$ da uzluksiz $f_n'(x)$ $(n=1,2,3,$ hosilalarga ega bo`lib,

$$f_n'(x) \rightrightarrows \varphi(x) \quad (x \in [a,b])$$

bo`lsa,

$$\varphi(x) = f'(x)$$

bo`ladi.

Shu kabi xossalarga keyinroq o`rganiladigan tekis yaqinlashuvchi funksional qatorlar ham ega bo`ladi. Ayni paytda, ular bir mulohazaasosida isbotlandi. Mazkur xossalarning isbotini funksional qatorlarga nisbatan keltiramiz.

Tekis yaqinlashishga tekshiring.

1. $f_n(x)=\dfrac{\cos\sqrt{nx}}{\sqrt{n+2x}}$, $E=[0;\infty)$

16. $f_n(x)=\dfrac{n}{x}\ln\left(1+\dfrac{1}{x}\right)$, $(0;10)$

2. $f_n(x)=\sin\left(ne^{-nx}\right)$, $E=[1;\infty)$

3. $f_n(x)=\dfrac{\ln nx}{nx^2}$, $E=[1;\infty)$

4. $f_n(x)=\dfrac{4n\sqrt{nx}}{3+4n^2x}$, $E=[\delta;\infty), \delta>0$

5. $f_n(x)=xe^{-nx}\ln^2 n$, $E=[0;\infty)$

6. $f_n(x)=n^{3/2}\left(1-\dfrac{\sqrt[4]{x}}{n}\right)$, $E=[0;\infty)$

7. $f_n(x)=\dfrac{x+xn^3+x^3n^6}{1+x^2n^6}$, $E=[1;\infty)$

8. $f_n(x)=n\displaystyle\int_0^x \sin\frac{\pi t^n}{2}dt$, $E=[0;\alpha]$, $0<\alpha<1$

9. $f_n(x)=tg\left(\dfrac{n-1}{n}x\right)$, $E=\left(0;\dfrac{\pi}{4}\right)$

10. $f_n(x)=x^n-x^{n+2}$, $E=[0;1]$

11. $f_n(x)=\sqrt[4]{x^4+\dfrac{1}{n^\alpha}}$, $\alpha>0, E=R$

12. $f_n(x)=\sin\dfrac{1+nx}{2n}$, $E=R$

13. $f_n(x)=\sin\dfrac{x}{n^\alpha}$, $\alpha>0, E=R$

14. $f_n(x)=x^n+x^{2n}-2x^{3n}$, $E=[0;1]$

15. $f_n(x)=\sin^n x$, $E=\left(0;\dfrac{\pi}{2}\right)$

17. $f_n(x)=nx(1-x)^n$, $E=[0;1]$

18. $f_n(x)=n\left(x^{\frac{1}{n}}-1\right)$, $E=[1;a]$, $1<a<\infty$

19. $f_n(x)=n\ln\left(1+\dfrac{2^x}{n}\right)$, $E=(0;1)$

20. $f_n(x)=\dfrac{1+nx}{nx}$, $E=(1;\infty)$

21. $f_n(x)=\dfrac{1}{x+1}ch\dfrac{x+e^{-nx}}{n}$, $E=(0;1)$

22. $f_n(x)=n^2\left(\left(x+\dfrac{1}{n^2}\right)^4-x^4\right)$, $E=(0;1)$

23. $f_n(x)=\sin\left(\pi x 2^{-\frac{x}{n}}\right)$, $E=(0;1)$

24. $f_n(x)=e^{\frac{x^2+nx+1}{n}}$, $E=(1;\infty)$

25. $f_n(x)=\cos\left(\dfrac{\pi}{2}x^n\right)$, $E=(1;\infty)$

26. $f_n(x)=n^2\ln\left(1+\dfrac{x}{n^2}\right)$, $E=(0;1)$

27. $f_n(x)=\cos\left(ne^{-nx}\right)$, $E=(1;\infty)$

28. $f_n(x)=\dfrac{1}{x^2}\sqrt{1+\dfrac{x}{n}}$, $E=(0;1)$

29. $f_n(x)=\dfrac{1}{x^3}\cos\dfrac{x}{n}$, $E=(1;\infty)$

30. $f_n(x)=n^2\left(x-n\sin\dfrac{x}{n}\right)$, $E=(0;1)$

10.3. Funksional ketma-ketlik limit funksiyasi uzluksizligi.

Funksional ketma-ketlik va limit funksiya tushunchalari. Aytaylik, har bir natural n songa $E \subset R$ to`plamda aniqlangan bitta $f_n(x)$ funksiyani mos qo`yuvchi qoida berilgan bo`lsin. Bu qoidaga ko`ra

$$f_1(x), f_2(x), ..., f_n(x), ... \qquad (1)$$

to`plam hosil bo`ladi. Uni funksional ketma-ketlik deyiladi. E to`plam (1) funksional ketma-ketlikning aniqlanish to`plami deyiladi.

Odatda, (1) funksional ketma-ketlik, uning n-hadi yordamida $\{f_n(x)\}$ yoki $f_n(x)$ kabi belgilanadi. Masalan,

$$f_n(x) = \frac{n+1}{n+x^2} : \frac{2}{1+x^2}, \frac{3}{2+x^2}, ..., \frac{n+1}{n+x^2}, ...;$$

$$f_n(x) = \sin\frac{\sqrt{x}}{n} : \sin\frac{\sqrt{x}}{1}, \sin\frac{\sqrt{x}}{2}, ..., \sin\frac{\sqrt{x}}{n}, ...$$

lar funksional ketma-ketliklar bo`ladi va ularning aniqlanish to`plami mos ravishda

$$E = R \ , \ E = [0, +\infty)$$

bo`ladi. Ravshanki, $x$ o`zgaruvchining biror tayinlangan $x = x_0 \in E$ qiymatida ushbu

$$\{f_n(x_0)\} : f_1(x_0), f_2(x_0), ..., f_n(x_0), ...$$

sonlar ketma-ketligiga ega bo`lamiz.

1-ta'rif. Agar $\{f_n(x_0)\}$ sonli ketma-ketlik yaqinlashuvchi (uzoqlashuvchi) bo`lsa, $\{f_n(x)\}$ funksional ketma-ketlik $x = x_0$ nuqtada yaqinlashuvchi (uzoqlashuvchi) deyiladi. x_0 nuqta esa bu funksional ketma-ketlikning yaqinlashish (uzoqlashish) nuqtasi deyiladi.

2-ta'rif. $\{f_n(x)\}$ funksional ketma-ketlikning barcha yaqinlashish nuqtalarida iborat $E_0 \subset E$ to`plam, $\{f_n(x)\}$ funksional ketma-ketlikning yaqinlashish to`plami deyiladi.

Masalan, ushbu

$$f_n(x) = x^n : x, x^2, x^3, ..., x^n, ...$$

funksional ketma-ketlik aniqlanish to`plami $E = R$ bo`lib, u $\forall x \in (-1,1]$ nuqtada yaqinlashuvchi, $x \in R \setminus (-1,1]$ da uzoqlashuvchi bo`ladi.Demak, ketma-ketlikning yaqinlashish to`plami $E_0 = (-1,1]$ bo`ladi.

Faraz qilaylik, $\{f_n(x)\}$ funksional ketma-ketlikning yaqinlashish to`plami $E_0 (E_0 \subset R)$ bo`lsin. Ravshanki, bu holda har bir $x \in E_0$ da

$$f_1(x), f_2(x), ..., f_n(x), ...$$

ketma-ketlik yaqinlashuvchi, ya'ni

$$\lim_{n\to\infty} f_n(x)$$

mavjud bo`ladi. Endi har bir $x \in E$ ga $\lim_{n\to\infty} f_n(x)$ ni mos qo`ysak, ushbu

$$f : x \to \lim_{n\to\infty} f_n(x)$$

funksiya hosil bo`ladi. Bu $f(x)$ funksiya $\{f_n(x)\}$ funksional ketma-ketlikning limit funksiyasi deyiladi:

$$\lim_{n\to\infty} f_n(x) = f(x) \quad (x \in E_0).$$

Bu munosabat quydagini anglatadi: ixtiyoriy $\varepsilon > 0$ son va har bir $x \in E_0$ uchun shunday natural $n_0 = n_0(\varepsilon, x)$ son topiladiki,ixtiyoriy $n > n_0$ da

$$|f_n(x) - f(x)| < \varepsilon \ ,$$

ya'ni

$$\forall \varepsilon > 0, \exists n_0 = n_0(\varepsilon, x) \in N, \forall n > n_0 : |f_n(x) - f(x)| < \varepsilon$$

bo`ladi.

1-misol. Ushbu $f_n(x) = n \sin\frac{\sqrt{x}}{n}$

funksional ketma-ketlikning limit funksiyasi topilsin.

Berilgan funksional ketma-ketlik $E = [0, +\infty)$ da aniqlangan. Uning limit funksiyasi

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} n \sin\frac{\sqrt{x}}{n} = \lim_{n\to\infty} \frac{\sin\frac{\sqrt{x}}{n}}{\frac{\sqrt{x}}{n}} \cdot \sqrt{x} = \sqrt{x}$$

bo`ladi. Demak, funksional ketma-ketlik $E = [0, +\infty)$ da yaqinlashuvchi va

$$\lim_{n\to\infty} n \sin\frac{\sqrt{x}}{n} = \sqrt{x}.$$

2-misol. Ushbu $f_n(x) = x^n$

funksional ketma-ketlikning limit funksiyasi topilsin.

Bu funksional ketma-ketlik $E = R$ da aniqlangan. Ravshanki

$$\forall x \in (1,+\infty) \text{ da } \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = +\infty,$$

$$\forall x \in (-1,1) \text{ da } \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = 0,,$$

$$x = 1 \text{ da } \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} 1 = 1,,$$

$$\forall x \in (-\infty,-1) \text{ da } \lim_{n\to\infty} f_n(x) \text{ mavjud emas.}$$

Demak, berilgan funksional ketma-ketlik $E_0 = (-1,1]$ yaqinlashuvchi bo`lib, uning limit funksiyasi

$$f(x) = \lim_{n\to\infty} x^n = \begin{cases} 0 \text{ , agar } -1 < x < 1 \text{ bo`lsa,} \\ 1 \text{ , agar } x = 1 \text{ bo`lsa} \end{cases}$$

bo`ladi.

3-misol. Ushbu $f_n(x) = n^2\left(\sqrt[n]{x} - \sqrt[n+1]{x}\right)$ $(x > 0)$

funksional ketma-ketlikning limit funksiyasi topilsin.

Berilgan funksional ketma-ketlikning limit funksiyasi quydagicha topiladi:

$$f(x) = \lim_{n\to\infty}(f_n(x)) = \lim_{n\to\infty} n^2(\sqrt[n]{x} - \sqrt[n+1]{x}) = \lim_{n\to\infty} n^2\left(x^{\frac{1}{n}} - x^{\frac{1}{n+1}}\right) =$$

$$= \lim_{n\to\infty} n^2 x^{\frac{1}{n+1}}\left(x^{\frac{1}{n}-\frac{1}{n+1}} - 1\right) = \lim_{n\to\infty} \frac{n^2}{n^2+n} x^{\frac{1}{n+1}} \cdot \frac{x^{\frac{1}{n^2+n}} - 1}{\frac{1}{n^2+n}} = \ln x.$$

Berilgan funksional ketma-ketlikning limit funksiyasini toping.

1. $f_n(x) = x^n - 3x^{n+2} + 2x^{n+3}, E = [0;1]$
2. $f_n(x) = x^4 \cos\frac{1}{nx}, E = (0;\infty)$
3. $f_n(x) = \frac{nx^2}{x+3n+2}, E = [0;\infty)$
4. $f_n(x) = \sqrt{x^2 + \frac{1}{\sqrt{n}}}, E = R$
5. $f_n(x) = (x-1)arctg\, x^n, E = (0;\infty)$
6. $f_n(x) = \sqrt[n]{1+x^n}, E = [0;2]$
7. $f_n(x) = n^3x^2e^{-nx}, E = [0;\infty)$

16. $f_n(x) = \cos^n \frac{x}{\sqrt{n}}, E = R$
17. $f_n(x) = \frac{1+x^{2n}}{2+x^{4n}}, E = R$
18. $f_n(x) = n^2x(1-x^2)^n, E = [0;1]$
19. $f_n(x) = \sqrt[n]{\sin x}, E = [0;\pi]$
20. $f_n(x) = n^2x^n(1-x), E = R$
21. $f_n(x) = \left(1+\frac{x}{n}\right)^n, E = R$
22. $f_n(x) = n(\sqrt[n]{x} - 1), E = (0;\infty)$

8. $f_n(x) = n\left(\sqrt{x^2 + \frac{1}{n}} - x\right), E = (0;\infty)$
9. $f_n(x) = n\left(x^{\frac{1}{n}} - 1\right), E = [1;3]$
10. $f_n(x) = narcctgnx^2, E = (0;\infty)$
11. $f_n(x) = n\left(x^{\frac{1}{n}} - x^{\frac{1}{2n}}\right), E = (0;\infty)$
12. $f_n(x) = \sqrt{1 + x^n + \left(\frac{x^2}{2}\right)^n}, E = [0;\infty)$
13. $f_n(x) = \frac{1}{x^2+n}, E = R$
14. $f_n(x) = x^n - x^{2n}, E = [0;1]$
15. $f_n(x) = nx^2 \sin\frac{x}{n}, E = R$

23. $f_n(x) = e^{-nx^2}, [1;\infty)$
24. $f_n(x) = \frac{x^2}{x^2 + (1-nx)^2}, E = [0;1]$
25. $f_n(x) = \left(\frac{n+x}{n-x}\right)^n, E = R$
26. $f_n(x) = \left(\frac{\sqrt[n]{x}+1}{2}\right)^n, E = (0;\infty)$
27. $f_n(x) = n[\ln(x+n) - \ln n]$
28. $f_n(x) = \frac{x+e^{nx}}{1+e^{nx}}$
29. $f_n(x) = \ln\left(x^2 + \frac{1}{n}\right), [1;\infty)$
30. $f_n(x) = e^{-(x-n)^2}, [-1;1]$

10.4. Funksional ketma-ketlikni hadma-had limitga o`tish. Funksional ketma-ketlikni hadma-had integrallash. Funksional ketma-ketlikni hadma-had differensiallash.

$M(M \subset R)$ to`plamda $\{f_n(x)\}$:

$$f_1(x), f_2(x), \ldots, f_n(x), \ldots \qquad (1)$$

funksional ketma-ketlik berilgan bo`lib,uning limit funksiyasi $f(x)$ bo`lsin. x_0 nuqta esa M to`plamning limit nuqtasi.

Teorema. Agar $x \to x_0$ da $\{f_n(x)\}$ funksional ketma-ketlikning har bir $f_n(x)$ $(n = 1,2,\ldots)$ hadi chekli

$$\lim_{x\to x_0} f_n(x) = a_n \quad (n = 1,2,\ldots\ldots)$$

limitga ega bo`lib,bu ketma-ketlik $M$ da tekis yaqinlashuvchi bo`lsa,u holda $\{a_n\}$:

$$a_1, a_2, \ldots\ldots a_n, \ldots$$

ketma-ketlik ham yaqinlashuvchi,uning $a = \lim_{n\to\infty} a_n$ limiti esa $f(x)$ ning $x \to x_0$ dagi limitiga teng

$$\lim_{x\to x_0} f(x) = a$$

bo`ladi.

$[a,b]$ segmentda $\{f_n(x)\}$:

$$f_1(x), f_2(x), ...f_n(x).... \qquad (1)$$

funksional ketma-ketlik berilgan bo`lib,uning limit funksiyasi $f(x)$ bo`lsin.

Teorema. Agar $\{f_n(x)\}$ funksional ketma-ketikning har bir $f_n(x)$ $(n=1,2,...)$ hadi $[a,b]$ segmentda uzluksiz bo`lib,bu fuksional ketma-ketlik $[a,b]$ da tekis yaqinlashuvchi bo`lsa,u holda

$$\int_a^b f_1(x)dx, \int_a^b f_2(x)dx, ...\int_a^b f_n(x)dx,......$$

ketma-ketlik yaqinlashuvchi bo`ladi,uning limiti esa $\int_a^b f(x)dx$ ga teng bo`ladi,ya'ni

$$\lim_{n\to\infty}\int_a^b f_n(x)dx = \int_a^b f(x)dx. \qquad (2)$$

Bu teoremadagi (2) limit munosabatni quydagicha

$$\lim_{n\to\infty}\int_a^b f_n(x)dx = \int_a^b \left[\lim_{n\to\infty} f_n(x)\right]dx$$

ham yozish mumkin.

$[a,b]$ segmentda yaqinlashuvchi $\{f_n(x)\}$:

$$f_1(x), f_2(x), ...f_n(x).... \qquad (1)$$

funksional ketma-ketlik berilgan bo`lib,uning limit funksiyasi $f(x)$ bo`lsin.

Teorema. Agar $\{f_n(x)\}$ funksional ketma-ketlikning har bir hadi $f_n(x)$ $(n=1,2,....)$ $[a,b]$ segmentda uzluksiz $f_n'(x)$ $(n=1, 2, ...)$ hosilaga ega bo`lib,bu hosilalardan tuzilgan

$$f_1'(x), f_2'(x), ...f_n'(x),.....$$

funksional ketma-ketlik $[a,b]$ da tekis yaqinlashuvchi bo`lsa,u holda $f(x)$ limit funksiyasi shu $[a,b]$ da $f'(x)$ hosilaga ega bo`lib, $\{f_n'(x)\}$ ketma-ketlikning limiti $f'(x)$ ga teng bo`ladi.

Quydagilarni isbotlang:

1. $\{f_n(x)\} = \{nx(1-x)^n\}$ funksional ketma-ketlik $[0;1]$ segmentda $f(x)$ limit funksiyaga tekis yaqinlashsa ham,lekin

$$\lim_{n\to\infty}\int_0^1 f_n(x)dx = \int_0^1 \lim_{n\to\infty} f_n(x)dx$$

bo`lishini ko`rsating.

2. $\{f_n(x)\} = \{nxe^{-nx^2}\}$ funksional ketma-ketlik $[0;1]$ segmentda $f(x)$ limit funksiyaga tekis yaqinlashsa ham,lekin

$$\int_0^1 \left[\lim_{n\to\infty} f_n(x)\right]dx \neq \lim_{n\to\infty}\int_0^1 f_n(x)\, dx$$

bo`lishini ko`rsating.

3.Quydagi integral belgisiostidagi ifodani limitga o`tish mumkinmi?

$$\lim_{n\to\infty}\int_0^1 \frac{nx}{1+n^2x^4}dx = \int_0^1 \lim_{n\to\infty}\frac{nx}{1+n^2x^4}dx$$

4. $\{f_n(x)\} = \{x^n\}$ funksional ketma-ketlik $[0;1]$segmentda

$$f(x) = \begin{cases} 0, & 0 \le x < 1 \text{ bo`lganda} \\ 1, & x = 1 \text{ bo`lganda} \end{cases}$$

lmit funksiyaga notekis yaqinlashsa ham,lekin

$$\lim_{n\to\infty}\int_0^1 f_n(x)dx = \int_0^1 \lim_{n\to\infty} f_n(x)dx$$

bo`lishini ko`rsating.

5. $\{f_n(x)\} = \left\{x^2 + \frac{1}{n}\sin n\left(x + \frac{\pi}{2}\right)\right\}$ funksional ketma-ketlik $(-\infty;+\infty)$ da $f(x)$ limit funksiyaga tekis yaqinlashsa ham,lekin

$$\left(\lim_{n\to\infty} f_n(x)\right)' \neq \lim_{n\to\infty} f_n'(x)$$

bo`lishini ko`rsating.

10.5. Funksional qatorlar va ularning yaqinlashuvchanligi.

**Funksional qator va uning yig`indisi.** Faraz qilaylik, $E \subset R$ to`plamda aniqlangan

$$u_1(x), u_2(x), ..., u_n(x), ...$$

funksional ketma-ketlik berilgan bo`lsin.Bu ketma-ketlik hadlari yordamida tuzilgan quydagi

$$u_1(x) + u_2(x) + ... + u_n(x) + ...$$

ifoda funksional qator deyiladi va $\sum_{n=1}^{\infty} u_n(x)$ kabi belgilanadi:

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots \; . \qquad (1)$$

Bunda E funksional qatorning aniqlanish to`plami deyiladi.Masalan,

1) $\sum_{n=1}^{\infty} x^{n-1} = 1 + x + x^2 + \ldots + x^{n-1} + \ldots,$

2) $\sum_{n=1}^{\infty} ne^{nx} = e^x + 2e^{2x} + 3e^{3x} + \ldots + ne^{nx} + \ldots$

funksional qatorlar bo`lib, ularning aniqlanish to`plami $E = (-\infty, +\infty)$ bo`ladi. (1) funksional qator hadlaridan ushbu

$$S_1(x) = u_1(x),$$
$$S_2 = u_1(x) + u_2(x),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$S_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (2)$$

yig`indilarni tuzamiz.Ular (1) funksional qatorning qismiy yig`indilari deyiladi. Demak, (1) funksional qator berilgan holda har doim bu qatorning (2) qismiy yig`indilaridan iborat $\{S_n(x)\}$:

$$S_1(x), S_2(x), \ldots, S_n(x), \ldots$$

funksional ketma-ketlik hosil bo`ladi.Ravshanki, $x = x_0 \in E$ nuqtada $\{S_n(x_0)\}$ sonlar ketma-ketligi bo`ladi.

1-ta'rif. Agar $\{S_n(x_0)\}$ yaqinlashuvchi (uzoqlashuvchi) bo`lsa, $\sum_{n=1}^{\infty} u_n(x)$ funksional qator $x = x_0$ nuqtada yaqinlashuvchi (uzoqlashuvchi) deyiladi, x_0 nuqta funksional qatorning yaqinlashuvchi (uzoqlashuvchi) nuqtasi deyiladi.

2-ta'rif. $\sum_{n=1}^{\infty} u_n(x)$ funksional qatorning barcha yaqinlashish nuqtalaridan iborat $E_0 \subset E$ to`plam, $\sum_{n=1}^{\infty} u_n(x)$ funksional qatorning yaqinlashish to`plami

deyiladi.Bu holda $\sum_{n=1}^{\infty} u_n(x)$ funksional qator E_0 to`plamda yaqinlashuvchi ham deb yuritiladi.

Agar E_0 to`plamda ushbu

$$\sum_{n=1}^{\infty} |u_n(x)| = |u_1(x)| + |u_2(x)| + \ldots + |u_n(x)| + \ldots$$

qator yaqinlashuvchi bo`lsa, $\sum_{n=1}^{\infty} u_n(x)$ funksional qator E_0 da absolyut yaqinlashuvchi deyiladi.

3-ta'rif. $\sum_{n=1}^{\infty} u_n(x)$ funksional qatorning qismiy yig`indilaridan $\{S_n(x)\}$ iborat ketma-ketlikning limit funksiyasi $S(x)$:

$$S_n(x) \to S(x) \quad (x \in E_0)$$

$\sum_{n=1}^{\infty} u_n(x)$ funksional qator yig`indisi deyiladi va $\sum_{n=1}^{\infty} u_n(x) = S(x) \; (x \in E_0)$ kabi yoziladi.

1-misol. Ushbu $\sum_{n=1}^{\infty} x^{n-1} = 1 + x + x^2 + \ldots + x^{n-1} + \ldots$

funksional qatorning yaqinlashish to`plami va yig`indisi topilsin.

Berilgan funksional qatorning aniqlanish to`plami $E = R$ bo`ladi. Qatorning qismiy yig`indisini topamiz:

$$S_n(x) = 1 + x + x^2 + \cdots + x^{n-1} = \begin{cases} \dfrac{1-x^n}{1-x}, & \text{agar } x \neq 1 \\ n, & \text{agar } x = 1 \end{cases}$$

Ravshanki, $n \to \infty$ da $S_n(x)$ ning limiti x ga bog`liq bo`ladi:

a) $x \in (-1, 1)$ da

$$\lim_{n\to\infty} S_n(x) = \lim_{n\to\infty}\left(\frac{1}{1-x} - \frac{x^n}{1-x}\right);$$

b) $x \in [1, +\infty)$ da

$$\lim_{n\to\infty} S_n(x) = \infty;$$

c) $x \in (-\infty, -1]$ da $\lim_{n\to\infty} S_n(x)$ mavjud emas.

Demak,berilgan funksional qatorning yaqinlashish to`plami $E_0=(-1,1)$ bo`lib, yig`indisi

$$S(x)=\frac{1}{1-x}$$

bo`ladi.

2-misol. Ushbu $\sum_{n=1}^{\infty}\frac{x^n}{1+x^{2n}}$

funksional qatorning yaqinlashish to`plami topilsin.

Sonli qatorlar nazariyasidagi Dalamber alomatidan foydalanib topamiz:

$$\lim_{n\to\infty}\left|\frac{u_{n+1}(x)}{u_n(x)}\right|=\lim_{n\to\infty}\left|\frac{x^{n+1}}{1+x^{2n+2}}:\frac{x^n}{1+x^{2n}}\right|=\lim_{n\to\infty}\left|\frac{x\left(1+x^{2n}\right)}{1+x^{2n+2}}\right|;$$

a) $x\in(-1,1)$ da

$$\lim_{n\to\infty}\left|\frac{x(1+x^{2n})}{1+x^{2n+2}}\right|=|x|.$$

Bu holda berilgan funksional qator $(-1,1)$ da yaqinlashuvchi bo`ladi.

b) $x\in(-\infty,-1)\cup(1,+\infty)$ da

$$\lim_{x\to\infty}\left|\frac{x\left(1+x^{2n}\right)}{1+x^{2n+2}}\right|=\lim_{x\to\infty}\left|\frac{\frac{1}{x^{2n+1}}+\frac{1}{x}}{\frac{1}{x^{2n+1}}+1}\right|=\left|\frac{1}{x}\right|$$

bo`lib,funksional qator $x\in(-\infty,-1)\cup(1,+\infty)$ da yaqinlashuvchi bo`ladi.

c) $x=\pm1$ da berilgan funksional qator mos ravishda ushbu

$$\sum_{n=1}^{\infty}\frac{1}{2},\quad \sum_{n=1}^{\infty}\frac{(-1)^n}{2}$$

sonli qatorga aylanadi va ular uzoqlashuvchi bo`ladi.

Shunday qilib, qaralayotgan funksional qatorning yaqinlashish to`plami

$$E_0=R\setminus\{-1,1\}=(-\infty,-1)\cup(-1,1)\cup(1,+\infty)$$

bo`ladi

Qatorni berilgan oraliqda yaqinlashishga tekshiring:

1. $\sum_{n=1}^{\infty}\sin\left(\frac{\pi}{2}x^n\right), E=(0;1)$

2. $\sum_{n=1}^{\infty}e^{-narctgx}, E=(0;\infty)$

3. $\sum_{n=1}^{\infty}arctg\frac{x}{n^2}, E=[0;\alpha], \alpha>0$

4. $\sum_{n=1}^{\infty}\ln\left(1+\frac{x^2}{n^2}\right), E=[1;\infty)$

5. $\sum_{n=1}^{\infty}e^{-n\left(x^2+2\sin x\right)}, E=(0;1]$

6. $\sum_{n=1}^{\infty}\frac{1}{1+n^2x}, E=[1;\infty)$

7. $\sum_{n=1}^{\infty}\frac{nx^2}{1+n^{5/2}x^6}, E=(1;\infty)$

8. $\sum_{n=1}^{\infty}\frac{\sqrt{nx^3}}{x^2+n^2}, E=(0;1)$

9. $\sum_{n=1}^{\infty}\frac{n^2+x^2}{1+n^3x^3}, E=(0;1)$

10. $\sum_{n=1}^{\infty}\frac{xn+\sqrt{n}}{n+x}\ln\left(1+\frac{x}{n\sqrt{n}}\right), E=(1;\infty)$

11. $\sum_{n=1}^{\infty}sh\left(\frac{\sqrt{x}}{n\ln^3(n+2)}\right), E=(1;\infty)$

12. $\sum_{n=1}^{\infty}e^{-n\arcsin x}, E=\left(0;\frac{1}{2}\right)$

13. $\sum_{n=1}^{\infty}arctg\frac{1}{n^2x}, E=(0;1]$

14. $\sum_{n=1}^{\infty}\frac{1}{\sqrt{1+n^3x}}, E=(1;\infty)$

15. $\sum_{n=1}^{\infty}e^{-n\sin^2x}, E=\left(\frac{\pi}{4};\frac{\pi}{2}\right]$

16. $\sum_{n=1}^{\infty}\ln\left(1+\frac{1}{n^2x}\right), E=[1;2]$

17. $\sum_{n=1}^{\infty}xe^n\sin\frac{x}{5^n}, E=(0;1)$

18. $\sum_{n=1}^{\infty}\frac{sh\left(\frac{1}{xn}\right)\cos(xn)}{1+xn}, E=(1;\infty)$

19. $\sum_{n=1}^{\infty}\frac{nx^2}{n^2+x}\ln\left(1+\frac{x}{\sqrt{n}}\right), E=(0;1)$

20. $\sum_{n=1}^{\infty}\sqrt{\frac{nx}{1+x}}\left(e^{\frac{1}{n^2x^2}}-1\right), E=(1;\infty)$

21. $\sum_{n=1}^{\infty}\left(1-\cos\sqrt[3]{\frac{x}{n^2}}\right), E=(0;1)$

22. $\sum_{n=1}^{\infty}2^{-nx}arctg\left(n^2x\right), E=(0;\delta), \delta>0$

23. $\sum_{n=1}^{\infty}\frac{arctg\left(\frac{1}{nx}\right)\cos nx}{4+\ln^2 2nx}, E=(0;1)$

24. $\sum_{n=1}^{\infty}\frac{x}{n^2+\cos\left(\frac{n}{x+1}\right)}, E=(0;\delta), \delta>0$

25. $\sum_{n=1}^{\infty}2^n tg\frac{1}{3^nx+1}, E=(\delta;\infty), \delta>0$

26. $\sum_{n=1}^{\infty}\frac{1}{n+\sqrt{\frac{\ln x}{n}}}\sin\frac{1}{\sqrt{n}}, E=(e;\infty)$

27. $\sum_{n=1}^{\infty}\sqrt[n]{ch\left(x\sqrt{n}\right)}\left(\frac{x}{n}\right)^2, E=(0;1)$

28. $\sum_{n=1}^{\infty}\ln\left(1+\frac{e^x}{n}\right)\sin\frac{x^3}{\sqrt{n}}, E=(1;\infty)$

29. $\sum_{n=1}^{\infty}sh\frac{e^x}{n}\sin\frac{x}{\sqrt{n}}, E=(1;\infty)$

30. $\sum_{n=1}^{\infty}\sqrt{xn}\,arctg\frac{x}{(x-n)^2+nx}, E=(0;1)$

10.6. Funksional qatorlarni tekis yaqinlashuvchanligi.

Funksional qatorlarni tekis yaqinlashuvchanligi. Aytaylik,

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + ... + u_n(x) + ...$$

funksional qator E_0 to`plamda yaqinlashuvchi (ya'ni qatorning yaqinlashish to`plami E_0) bo`lib yig`indisi $S(x)$ bo`lsin:

$$S_n(x) \to S(x) \quad (x \in E_0) \qquad (1)$$

bunda, $S_n(x) = u_1(x) + u_2(x) + ... + u_n(x)$. (3) munosabat

$$\forall \varepsilon > 0,\ \forall x \in E_0, \exists n_0 = n_0(\varepsilon, x) \in N, \forall n > n_0 : |S_n(x) - S(x)| < \varepsilon$$

bo`lishini anglatadi.

1-tarif. Agar E_0 to`plamda

$$S_n(x) \rightrightarrows S(x), \quad (x \in E_0)$$

ya'ni

$$\forall \varepsilon > 0, \exists n_0 = n_0(\varepsilon) \in N, \forall n > n_0, \forall x \in E_0 : |S_n(x) - S(x)| < \varepsilon$$

bo`lsa, $\sum_{n=1}^{\infty} u_n(x)$ funksional qator $E_0$ to`plamda tekis ya qinlashuvchi deyiladi.

Agar

$$r_n(x) = S(x) - S_n(x),$$

deyilsa,funksional qatorning E_0 to`plamda tekis yaqinlashuvchiligini quydagicha

$$r_n(x) \rightrightarrows 0, \quad (x \in E_0),$$

ya'ni

$$\forall \varepsilon > 0, \exists n_0 = n_0(\varepsilon) \in N, \forall n > n_0, \forall x \in E_0 : |r_n(x)| < \varepsilon$$

ko`rinishda ta'riflash mumkin bo`ladi.

Shunday qilib

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + ... + u_n(x) + ...$$

funksional qator, uning qismiy yig`indisi

$$S_n(x) = u_1(x) + u_2(x) + ... + u_n(x)$$

va yig`indisi $S(x)$ uchun

$$S_n(x) \to S(x) \quad (x \in E_0)$$

bo`lsa, funksional qator E_0 da yaqinlashuvchi,

$$S_n(x) \rightrightarrows S(x), \quad (x \in E_0)$$

bo`lsa, funksional qator $E_0$ da tekis yaqinlashuvchi bo`ladi.

1-teorema. $\sum_{n=1}^{\infty} u_n(x)$ funksional qator E_0 da qator yig`indisi $S(x)$ funksiyaga tekis yaqinlashishi uchun

$$\lim_{n\to\infty} \sup_{x \in E_0} |S_n(x) - S(x)| = 0,$$

ya'ni

$$\lim_{n\to\infty} \sup_{x \in E_0} |r_n(x)| = 0$$

bo`lishi zarur va yetarli.

1-misol. Ushbu $\sum_{n=1}^{\infty} \frac{1}{(x+n)(x+n+1)}$

funksional qatorning $[0, +\infty)$ da tekis yaqinlashuvchi bo`lishi isbotlansin.

Berilgan funksional qator qismiy yig`indisini hisoblab, so`ng yig`indisini topamiz:

$$S_n(x) = \frac{1}{(x+1)(x+2)} + \frac{1}{(x+2)(x+3)} + ... + \frac{1}{(x+n)(x+n+1)} =$$

$$= \left(\frac{1}{x+1} - \frac{1}{x+2}\right) + \left(\frac{1}{x+2} - \frac{1}{x+3}\right) + ... + \left(\frac{1}{x+n} - \frac{1}{x+n+1}\right) =$$

$$= \frac{1}{x+1} - \frac{1}{x+n+1},$$

$$\lim_{n\to\infty} S_n(x) = \lim_{n\to\infty}\left(\frac{1}{x+1} - \frac{1}{x+n+1}\right) = \frac{1}{x+1}.$$

Demak,

$$S(x) = \frac{1}{x+1}.$$

Unda

$$S_n(x) - S(x) = \frac{1}{x+1} - \frac{1}{x+n+1} - \frac{1}{x+1} = -\frac{1}{x+n+1}$$

bo`lib,

$$\sup_{x \in [0,+\infty)} |S_n(x) - S(x)| = \frac{1}{n+1}$$

bo`ladi.Keyingi tenglikdan

$$\lim_{n\to\infty} \sup_{x \in [0,+\infty)} |S_n(x) - S(x)| = 0$$

bo`lishi kelib chiqadi. 1-teoremaga ko`ra berilgan funksional qator $[0,+\infty)$ da tekis yaqinlashuvchi

Eslatma. Agar

$$\lim_{n\to\infty}\sup_{x\in E_0}|S_n(x)-S(x)|\neq 0$$

bo`lsa, $\sum_{n=1}^{\infty}u_n(x)$ funksional qator $E_0$ da tekis yaqinlashuvchi bo`lishi shart emas: Masalan,

$$\sum_{n=1}^{\infty}x^{n-1}=1+x+x^2+...+x^{n-1}+...$$

funksional qatorning $(-1,1)$ da yaqinlashuvchi, yig`indisi

$$S(x)=\frac{1}{1-x}$$

bo`lishini ko`rgan edik.Bu funksional qator uchun

$$\lim_{n\to\infty}\sup_{-1<x<1}|S_n(x)-S(x)|=\lim_{n\to\infty}\sup_{-1<x<1}\left|\frac{x^n}{1-x}\right|=+\infty$$

bo`ladi. Demak, funksional qator $(-1,1)$ da tekis yaqinlashuvchi emas.

Faraz qilaylik,

$$\sum_{n=1}^{\infty}u_n(x)=u_1(x)+u_2(x)+...+u_n(x)+...$$

funksional qator $E\subset R$ to`plamda berilgan bo`lsin.

2-teorema (Koshi). $\sum_{n=1}^{\infty}u_n(x)$ funksional qator E to`plamda tekis yaqinlashuvchi bo`lishi uchun

$$\forall\varepsilon>0,\exists n_0=n_0(\varepsilon)\in N,\forall n>n_0,\forall p\in N,\ \forall x\in E \text{ da}$$

$$|S_{n+p}(x)-S_n(x)|=|u_{n+1}(x)+u_{n+2}(x)+...+u_{n+p}(x)|<\varepsilon$$

bo`lishi zarur va yetarli.

Funksional qatorning tekis yaqinlashuvchilik alomatlari.

a) Veyershtrass alomati. Aytaylik, E to`plamda

$$\sum_{n=1}^{\infty}u_n(x)=u_1(x)+u_2(x)+...+u_n(x)+... \qquad (2)$$

funksional qator berilgan bo`lib,

1) $\forall n\in N,\ \forall x\in E$ da $|u_n(x)|\le C_n$,

2) $\sum_{n=1}^{\infty}C_n=C_1+C_2+...+C_n+...$sonli qator yaqinlashuvchi bo`lsin.U holda (4) funksional qator $E$ to`plamda tekis yaqinlashuvchi bo`ladi.

1) – shartga ko`ra $\forall n>n_0,\forall p\in N$ va $\forall x\in E$ uchun

$$|u_{n+1}(x)+u_{n+2}(x)+...+u_{n+p}(x)|\le|u_{n+1}(x)|+|u_{n+2}(x)|+...+|u_{n+p}(x)|\le c_{n+1}+$$
$$+c_{n+2}+...+c_{n+p}$$

bo`lib,2) – shartda,ya'ni u $\sum_{n=1}^{\infty}c_n$ qatorning yaqinlashuvchiligidan Koshi teoremasiga binoan

$$\forall\varepsilon>0,\exists n_0=n_0(\varepsilon)\in N,\forall n>n_0,\forall p\in N \text{ da}$$

$$c_{n+1}+c_{n+2}+...+c_{n+p}<\varepsilon$$

bo`ladi.Demak

$$|u_{n+1}(x)+u_{n+2}(x)+...+u_{n+p}(x)|<\varepsilon.$$

Yuqoridagi 2- teoremaga ko`ra $\sum_{n=1}^{\infty}u_n(x)$ funksional qator $E$ to`plamda tekis yaqinlashuvchi bo`ladi.

2-misol. Ushbu $\sum_{n=1}^{\infty}\frac{x\sin x}{\sqrt{1+n^2}\left(1+nx^2\right)}$

funksional qator tekis yaqinlashishga tekshirilsin.

Berilgan qatorning aniqlanish to`plami $E=(-\infty,+\infty)$ bo`lib, uning umumiy hadi

$$u_n(x)=\frac{x\sin x}{\sqrt{1+n^2}\left(1+nx^2\right)} \qquad (n=1,2,3,....)$$

bo`ladi. Ravshanki,

$$u_n(x)=\left|\frac{x\sin x}{\sqrt{1+n^2}\left(1+nx^2\right)}\right|\le\frac{|x|}{\sqrt{1+n^2}\left(1+nx^2\right)}.$$

Endi $\forall x\in(-\infty,+\infty)$uchun

$$\frac{|x|}{1+nx^2}\le\frac{1}{2\sqrt{n}}$$

bo`lishini e'tiborga olib topamiz:

$$\frac{|x|}{\sqrt{1+n^2}\left(1+nx^2\right)}\le\frac{1}{2\sqrt{n\left(1+n^2\right)}}\le\frac{1}{2n^{3/2}}.$$

Demak,berilgan funksional qatorning hadlari uchun

$$|u_n(x)| \le \frac{1}{2n^{3/2}}$$

bo`ladi.Ma'lumki, $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ qator yaqinlashuvchi.Binobarin,Veyershtrass alomatiga ko`ra beilgan funksional qator $(-\infty,+\infty)$ da tekis yaqinlashuvchi bo`ladi.

Funksional qatorning tekis yaqinlashishini ifodalovchi keying alomatlarni isbotsiz keltiramiz.

b) Direxle alomati. Aytaylik, $E \subset R$ to`plamda aniqlangan $u_n(x)$ va $v_n(x)$ $(n=1,2,3,...)$ funksiyalar quydagi shartlarni bajarsin:

1) $\forall x \in E$ da $\{u_n(x)\}$ ketma-ketlik monoton;

2) $\{u_n(x)\}$ funksional ketma-ketlik E da 0 ga tekis yaqinlashuvchi:

$$u_n(x) \underset{\to}{\to} 0, \quad (x \in E_0);$$

1) shunday $C \in R$ mavjudki, $\forall n \in N$, $\forall x \in E$ da

$$|v_1(x)+v_2(x)+...+v_n(x)| = \left|\sum_{k=1}^{n} v_k(x)\right| \le C.$$

U holda

$$\sum_{n=1}^{\infty} u_n(x) \cdot v_n(x)$$

funksional qator E to`plamda tekis yaqinlashuvchi bo`ladi.

3-misol. Ushbu $\sum_{n=1}^{\infty} \frac{\sin x \cdot \sin nx}{\sqrt{n+x}}$ funksional qator $E=[0,+\infty)$ tekis yaqinlashuvchiligi isbotlansin.

Aytaylik,

$$u_n(x) = \frac{1}{\sqrt{n+x}}, \quad v_n(x) = \sin x \cdot \sin nx$$

bo`lsin. Bu funksiyalar uchun Direxle alomatidagi uchinchi shart bajariladi. Haqiqatdan ham,

1) $\forall x \in E$ da $u_n(x) = \frac{1}{\sqrt{n+x}}$ uchun

$$\frac{1}{\sqrt{n+x}} - \frac{1}{\sqrt{n+1+x}} = \frac{\sqrt{n+1+x}-\sqrt{n+x}}{\sqrt{n+1+x}\cdot\sqrt{n+x}} =$$

$$= \frac{1}{\sqrt{(n+x)(n+1+x)}\cdot\sqrt{n+1+x}+\sqrt{n+x}} > 0$$

bo`lganligidan uning kamayuvchiligi kelib chiqadi;

2) Ravshanki,

$$u_n(x) = \frac{1}{\sqrt{n+x}} \le \frac{1}{\sqrt{n}}, \; n \to \infty \text{ da } \frac{1}{\sqrt{n}} \to 0.$$

Demak,

$$u_n(x) \underset{\to}{\to} 0, \quad (x \in E_0);$$

3) bu holda

$$\left|\sum_{k=1}^{n} v_k(x)\right| = \left|\sum_{k=1}^{n} \sin x \sin kx\right| = 2\left|\cos\frac{x}{2}\right| \cdot \left|\sin\frac{nx}{2} \cdot \sin\frac{n+1}{2}x\right| \le 2$$

bo`ladi.

Dirixle alomatiga ko`ra berilgan funksional qator $E=[0,+\infty)$ da tekis yaqinlashuvchi

c) Abel alomati. Aytaylik, $E \subset R$ to`plamda aniqlangan $u_n(x)$ va $v_n(x)$ $(n=1,2,3,...)$ funksiyalar quydagi shartlarni bajarsin:

1) $\forall x \in E$ da $\{u_n(x)\}$ ketma-ketlik monoton;

2) shunday $C \in R$ topiladiki, $\forall n \in N$, $\forall x \in E$ da

$$|u_n(x)| \le C;$$

3) $\sum_{n=1}^{\infty} v_n(x)$ funksional qator E to`plamda tekis yaqinlashuvchi. U holda

$$\sum_{n=1}^{\infty} u_n(x) \cdot v_n(x)$$

funksional qator E to`plamda tekis yaqinlashuvchi bo`ladi.

4-misol. Ushbu $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$ funksional qatorning $E=[0,1]$ da tekis yaqinlashuvchi ekanligi isbotlansin.

Aytaylik,

$$u_n(x) = x^n, \quad v_n(x) = \frac{(-1)^{n+1}}{n} \quad (x \in [0,1])$$

bo`lsin. Bu funksiyalar uchun Abel alomatidaki uchta shart bajariladi (bu ravshan). Unda Abel alomatiga ko`ra berilgan funksional qator $[0,1]$ da tekis yaqinlashuvchi bo`ladi.

Tekis yaqinlashishga tekshiring:

1. $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\ln\frac{n+x}{n},\ E=(1;\infty)$

2. $\sum_{n=1}^{\infty}\frac{x^2n^2}{x^4+n^4}\sin\frac{n}{x},\ E=(0;1)$

3. $\sum_{n=1}^{\infty}\frac{n}{1+n^2x^2}tg\sqrt{\frac{x}{n}},\ E=\left[0;\frac{1}{2}\right]$

4. $\sum_{n=1}^{\infty}\frac{\sin\left(\frac{1}{nx}\right)}{1+(\ln nx)^2},\ E=(0;1)$

5. $\sum_{n=1}^{\infty}\frac{x}{x^2-nx+n^2},\ E=(0;1)$

6. $\sum_{n=1}^{\infty}\frac{x}{1+nx}\ln\left(1+\frac{x}{n}\right),\ E=(1;\infty)$

7. $\sum_{n=1}^{\infty}\frac{\sqrt{nx}}{x+n}\ln\left(1+\frac{x}{n}\right),\ E=(1;\infty)$

8. $\sum_{n=1}^{\infty}\frac{\ln(1+n^x)}{x+\ln n},\ E=(1;\infty)$

9. $\sum_{n=1}^{\infty}\frac{xn}{x+n^2}\ln\left(1+\sqrt{\frac{x}{n}}\right),\ E=(0;1)$

10. $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{2+x^3n^{3/2}}\sin\sqrt[3]{\frac{x}{n}},\ E=(0;1)$

11. $\sum_{n=1}^{\infty}\sqrt{n}\sin(2^n x)\sin\frac{1}{2^n x},\ E=(1;\infty)$

12. $\sum_{n=1}^{\infty}\left(e^{x/\sqrt{n}}-1\right)arctg\frac{x^2}{n+1},\ E=(1;\infty)$

13. $\sum_{n=1}^{\infty}\frac{\ln(nx)}{1+n^3\ln^2 x},\ E=\left(\frac{1}{2};1\right)$

14. $\sum_{n=1}^{\infty}\frac{x^2}{1+n^2x}\sin\frac{n}{x},\ E=(0;1)$

15. $\sum_{n=1}^{\infty}\frac{arctg(nx)}{1+n^2x^2},\ E=(0;1)$

16. $\sum_{n=1}^{\infty}\frac{1}{n^3\ln\left(\frac{n}{n+x}\right)},\ E=(1;\infty)$

17. $\sum_{n=1}^{\infty}(-1)^n\ln\frac{3+x\sqrt{n^5}}{2+x\sqrt{n^5}},\ E=(0;\infty)$

18. $\sum_{n=1}^{\infty}\frac{x^2}{x+n}\sin\frac{x}{\sqrt{n}}\sin\frac{\sqrt{n}}{x},\ E=(0;1]$

19. $\sum_{n=1}^{\infty}\frac{\cos\left(\frac{x}{n}\right)}{2^{n-x}+2^{\frac{n}{x}}},\ E=[10;\infty)$

20. $\sum_{n=1}^{\infty}\frac{n\sin\sqrt{n}}{1+n^3x^3},\ E=(0;\infty)$

21. $\sum_{n=1}^{\infty}\frac{(nx)^{\frac{3}{2}}}{1+n^3x^2},\ E=(0;1)$

22. $\sum_{n=1}^{\infty}\left(1-\cos\sqrt{\frac{x^5}{n}}\right)arctg\frac{e^x}{\sqrt{n}},\ E=(0;1)$

23. $\sum_{n=1}^{\infty}\frac{arctg(xn)}{1+x\sqrt{n}}\ln\left(1+\frac{1}{xn}\right),\ E=(1;\infty)$

24. $\sum_{n=1}^{\infty}n\,sh\frac{x}{n+x},\ E=(0;1)$

25. $\sum_{n=1}^{\infty}\frac{xn}{x^2+n^2}arctg\frac{x}{n},\ E=(0;1)$

26. $\sum_{n=1}^{\infty}\frac{\sqrt{nx}}{1+nx}\sin\frac{1}{nx},\ E=(1;\infty)$

27. $\sum_{n=1}^{\infty}\sqrt[3]{n}\left(\cos\frac{1}{xn}-1\right)\cos xn,$
$E=(0;1)$

28. $\sum_{n=1}^{\infty}\frac{\sqrt{1-\cos\frac{x}{n}}}{x^2\ln^2(1+nx)},\ E=(0;1)$

29. $\sum_{n=1}^{\infty}\frac{x^4}{n+x+1}arctg\frac{x^2}{\sqrt{n}},\ E=(1;\infty)$

30. $\sum_{n=1}^{\infty}\frac{1}{2+n^2x^2}\arcsin\sqrt{\frac{nx}{1+nx}},$
$E=(1;\infty)$

10.7. Funksional qator yig`indisining uzluksizligi.

**Funksional qator yig`indisining uzluksizligi.** Faraz qilaylik, $E\subset R$ to`plamda

$$\sum_{n=1}^{\infty}u_n(x)=u_1(x)+u_2(x)+...+u_n(x)+... \qquad (1)$$

funksional qator berilgan bo`lib, uning yig`indisi $S(x)$ bo`lsin.

1-teorema. Aytaylik, (1) qator ushbu shartlarni bajarsin:

1) qatorning har bir $u_n(x)$ $(n=1,2,3,...)$ hadi E to`plamda uzluksiz,

2) $\sum_{n=1}^{\infty}u_n(x)$ qator E da tekis yaqinlashuvchi. U holda funksional qator yig`indisi $S(x)$ funksiya $E$ to`plamda uzluksiz bo`ladi.

Isbot. Aytaylik, $x_0\in E$,

$$S_n(x)=u_1(x)+u_2(x)+...+u_n(x)$$

bo`lsin. Teoremaning 2) – shartiga ko`ra

$$S_n(x)\ \rightrightarrows S(x),\quad (x\in E_0)$$

bo`ladi. Ta'rifga binoan

$$\forall\varepsilon>0,\exists n_0=n_0(\varepsilon)\in N,\forall n>n_0 \text{ va } \forall x\in E \text{ da}$$

$$|S_n(x)-S(x)|<\frac{\varepsilon}{3} \qquad (2)$$

jumladan

$$|S_n(x_0)-S(x_0)|<\frac{\varepsilon}{3} \qquad (3)$$

tengsizliklar bajariladi.

Ravshanki, (2) va (3) tengsizliklar n ning n_0 dan katta biror tayin n_1 qiymatida ham o`rinli bo`ladi:

$$|S_{n_1}(x)-S(x)|<\frac{\varepsilon}{3}, \qquad (2')$$

$$|S_{n_1}(x_0)-S(x_0)|<\frac{\varepsilon}{3}. \qquad (3')$$

Teoremaning 1) shartidan va chekli sondagi funksiyalar yig`indisi yana uzluksiz bo`lishidan

$$S_{n_1}(x)=u_1(x)+u_2(x)+...+u_{n_1}(x)$$

funksiyaning E to`plamda uzluksiz ekanligi kelib chiqadi. Demak, $S_{n_1}(x)$ funksiya $x=x_0$ da uzluksiz. Unda, ta'rifga binoan

$\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0, |x - x_0| < \delta$ tengsizlikni qanotlatiruvchi barcha $x \in E$ da

$$|S_{n_1}(x) - S_{n_1}(x_0)| < \frac{\varepsilon}{3} \qquad (4)$$

bo`ladi.

Yuqoridagi ($2'$), ($3'$) va (4) tengsizliklardan foydalani topamiz:

$$|S(x) - S(x_0)| = |(S(x) - S_{n_1}(x)) + (S_{n_1}(x) - S_{n_1}(x_0)) + (S_{n_1}(x_0) - S(x_0))| \le |S(x) - S_{n_1}(x)| +$$

$$+|S_{n_1}(x) - S_{n_1}(x_0)| + |S_{n_1}(x_0) - S(x_0)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Bu esa $S(x)$ funksiyaning x_0 nuqtada uzluksiz bo`lishini bildiradi. Modomiki, $x_0$ nuqta $E$ to`plamning ixtiyoriy nuqtasi ekan, $S(x)$ funksiya $E$to`plamda uzluksiz bo`ladi.

Yuqorida keltirilgan teoremaning shartlari bajarilganda uning tasdig`ini quydagicha

$$\lim_{x \to x_0} \sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \left(\lim_{x \to x_0} u_n(x) \right)$$

ifodalash mumkin.

Tekis yaqinlashishga tekshiring:

1. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \ln \frac{n+x}{n}, E = (1;\infty)$

2. $\sum_{n=1}^{\infty} \frac{x^2 n^2}{x^4 + n^4} \sin \frac{n}{x}, E = (0;1)$

3. $\sum_{n=1}^{\infty} \frac{n}{1+n^2x^2} tg\sqrt{\frac{x}{n}}, E = \left[0; \frac{1}{2}\right]$

4. $\sum_{n=1}^{\infty} \frac{\sin\left(\frac{1}{nx}\right)}{1 + (\ln nx)^2}, E = (0;1)$

5. $\sum_{n=1}^{\infty} \frac{x}{x^2 - nx + n^2}, E = (0;1)$

6. $\sum_{n=1}^{\infty} \frac{x}{1+nx} \ln\left(1 + \frac{x}{n}\right), E = (1;\infty)$

7. $\sum_{n=1}^{\infty} \frac{\sqrt{nx}}{x+n} \ln\left(1 + \frac{x}{n}\right), E = (1;\infty)$

8. $\sum_{n=1}^{\infty} \frac{\ln(1+n^x)}{x + \ln n}, E = (1;\infty)$

16. $\sum_{n=1}^{\infty} \frac{1}{n^3 \ln\left(\frac{n}{n+x}\right)}, E = (1;\infty)$

17. $\sum_{n=1}^{\infty} (-1)^n \ln \frac{3 + x\sqrt{n^5}}{2 + x\sqrt{n^5}}, E = (0;\infty)$

18. $\sum_{n=1}^{\infty} \frac{x^2}{x+n} \sin \frac{x}{\sqrt{n}} \sin \frac{\sqrt{n}}{x}, E = (0;1]$

19. $\sum_{n=1}^{\infty} \frac{\cos\left(\frac{x}{n}\right)}{2^{n-x} + 2^{\frac{n}{x}}}, E = [10;\infty)$

20. $\sum_{n=1}^{\infty} \frac{n \sin \sqrt{n}}{1 + n^3 x^3}, E = (0;\infty)$

21. $\sum_{n=1}^{\infty} \frac{(nx)^{\frac{3}{2}}}{1 + n^3 x^2}, E = (0;1)$

22. $\sum_{n=1}^{\infty} \left(1 - \cos\sqrt{\frac{x^5}{n}}\right) arctg \frac{e^x}{\sqrt{n}}, E = (0;1)$

23. $\sum_{n=1}^{\infty} \frac{arctg(xn)}{1 + x\sqrt{n}} \ln\left(1 + \frac{1}{xn}\right), E = (1;\infty)$

9. $\sum_{n=1}^{\infty} \frac{xn}{x+n^2} \ln\left(1 + \sqrt{\frac{x}{n}}\right), E = (0;1)$

10. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{2 + x^3 n^{3/2}} \sin \sqrt[5]{\frac{x}{n}}, E = (0;1)$

11. $\sum_{n=1}^{\infty} \sqrt{n} \sin(2^n x) \sin \frac{1}{2^n x}, E = (1;\infty)$

12. $\sum_{n=1}^{\infty} \left(e^{x/\sqrt{n}} - 1\right) arctg \frac{x^2}{n+1}, E = (1;\infty)$

13. $\sum_{n=1}^{\infty} \frac{\ln(nx)}{1 + n^3 \ln^2 x}, E = \left(\frac{1}{2};1\right)$

14. $\sum_{n=1}^{\infty} \frac{x^2}{1+n^2x} \sin \frac{n}{x}, E = (0;1)$

15. $\sum_{n=1}^{\infty} \frac{arctg(nx)}{1+n^2x^2}, E = (0;1)$

24. $\sum_{n=1}^{\infty} n\, sh \frac{x}{n+x}, E = (0;1)$

25. $\sum_{n=1}^{\infty} \frac{xn}{x^2 + n^2} arctg \frac{x}{n}, E = (0;1)$

26. $\sum_{n=1}^{\infty} \frac{\sqrt{nx}}{1+nx} \sin \frac{1}{nx}, E = (1;\infty)$

27. $\sum_{n=1}^{\infty} \sqrt[3]{n} \left(\cos \frac{1}{xn} - 1\right) \cos xn,$ $E = (0;1)$

28. $\sum_{n=1}^{\infty} \frac{\sqrt{1 - \cos\frac{x}{n}}}{x^2 \ln^2(1+nx)}, E = (0;1)$

29. $\sum_{n=1}^{\infty} \frac{x^4}{n+x+1} arctg \frac{x^2}{\sqrt{n}}, E = (1;\infty)$

30. $\sum_{n=1}^{\infty} \frac{1}{2+n^2x^2} \arcsin\sqrt{\frac{nx}{1+nx}},$ $E = (1;\infty)$

10.8. Funksional qatorlarni hadma-had limitga o`tish.

$M (M \subset R)$ to`plamda yaqinlashuvchi

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + ... + u_n(x) + ... \qquad (1)$$

funksional qator berilgan bo`lib,uning yig`indisi $S(x)$ bo`lsin. $x_0$ nuqta esa $M$ to`plamning limit nuqtasi.

Teorema. Agar $x \to x_0$ da $\sum_{n=1}^{\infty} u_n(x)$ funksional qatorning har bir $u_n(x)$ $(n = 1,2,...)$ hadi chekli

$$\lim_{x \to x_0} u_n(x) = c_n \qquad (n = 1,2,...) \qquad (2)$$

limitga ega bo`lib,bu qator $M$ da tekis yaqinlashuvchi bo`lsa,u holda

$$\sum_{n=1}^{\infty} c_n = c_1 + c_2 + ... + c_n + ...$$

qator ham yaqinlashuvchi,uning yig`indisi C esa $S(x)$ ning $x \to x_0$ dagi limiti

$$\lim_{x \to x_0} S(x) = C$$

ga teng bo`ladi.

Isbot. Shartga ko`ra (1) funksional qator tekis yaqinlashuvchi.U holda $\forall\varepsilon>0$ olinganda ham shunday $n_0\in N$ topiladiki,barcha $n>n_0$, $m$ lar va $M$ to`plamning barcha x nuqtalari uchun

$$|u_{n+1}(x)+u_{n+2}(x)+...+u_m(x)|<\varepsilon \qquad (3)$$

tengsizlik bajariladi.(2) shartni e`tiborga olib,(3) tengsizlikda $x\to x_0$ da limitga o`tib quydagini topamiz:

$$|c_{n+1}+c_{n+2}+...+c_m|\le\varepsilon$$

Demak, $\forall\varepsilon>0$ olinganda ham,shunday $n_0\in N$ topiladiki,barcha $n>n_0$, $m>n$ lar uchun

$$|c_{n+1}+c_{n+2}+...+c_m|\le\varepsilon$$

tengsizlik bajarilar ekan.Qator yaqinlashuvchiligining yetarli va zaruriy shartini ifodalovchi teoremaga muvofiq

$$\sum_{n=1} c_n=c_1+c_2+...+c_n+...$$

qator yaqinlashuvchi bo`ladi.Demak,

$$\lim_{n\to\infty} C_n=C,$$

bunda

$$C_n=c_1+c_2+...+c_n \quad (n=1,2,...)$$

Endi $x\to x_0$ da (1) funksional qator yig`indisi $S(x)$ ning limiti C ga teng ya'ni

$$\lim_{x\to x_0} S(x)=C$$

bo`lishini ko`rsatamiz.Shu maqsadda ushbu

$$S(x)-C$$

ayirmani olib uni quydagicha yozamiz:

$$S(x)-C=[S(x)-S_n(x)]+[S_n(x)-C_n]+[C_n-C] \qquad (4)$$

bunda

$$S_n(x)=u_1(x)+u_2(x)+...+u_n(x).$$

Teorema shartoga ko`ra (1) qator tekis yaqinlashuvchi.Demak, $\forall\varepsilon>0$ olinganda ham, $\frac{\varepsilon}{3}$ ga ko`ra shunday $n_0\in N$ topiladiki,barcha $n>n_0$ va M to`plamning barcha x nuqtalari uchun

$$|S_n(x)-S(x)|<\frac{\varepsilon}{3} \qquad (5)$$

tengsizlik bajarildi.

(2) shartdan foydalanib quydagini topamiz.

$$\lim_{x\to x_0} S_n(x)=\lim_{x\to x_0}[u_1(x)+u_2(x)+...+u_n(x)]=c_1+c_2+...+c_n=C_n$$

Demak, $\forall\varepsilon>0$ olinganda ham, $\frac{\varepsilon}{3}$ ga ko`ra shunday $\delta>0$ topiladiki $|x-x_0|<\delta$ bo`lganda

$$|S_n-C_n|<\frac{\varepsilon}{3} \qquad (6)$$

tengsizlik bajariladi.

Yuqorida isbot etilganga ko`ra

$$\lim_{n\to\infty} C_n=C$$

Demak, $\forall\varepsilon>0$ olinganda ham, $\frac{\varepsilon}{3}$ ga ko`ra,shunday $n_0\in N$ topiladiki,barcha $n>n_0$ uchun

$$|C_n-C|<\frac{\varepsilon}{3} \qquad (7)$$

bo`ladi.Shuni ham aytish kerakki,agar $\overline{n_0}=\max\{n_0,\overline{n_0}\}$ deb olinsa,unda barcha $n>\overline{n_0}$ uchun (7) va (5) tengsizliklar bir vaqtda bajariladi.

Natijada (4) munosabatda (5),(6) va (7) tengsizliklarni e'tiborga olgan holda,quydagini topamiz.

$$|S(x)-C|\le|S(x)-S_n(x)|+|S_n(x)-C_n|+|C_n-C|<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon$$

Demak, $\forall\varepsilon>0$ olinganda ham,shunday $\delta>0$ topiladiki, $|x-x_0|<\delta$ uchun $(x\in M)$

$$|S(x)-C|<\varepsilon$$

tengsizlik bajariladi.Bu esa $\lim_{x\to x_0} S(x)=C$ ekanligini bildiradi.

Teorema isbot bo`ldi.

Yuqoridagi limit munosabatni quydagicha yozish mumkin:

$$\lim_{x\to x_0}\sum_{n=1}^{\infty} u_n(x)=\sum_{n=1}^{\infty}\left[\lim_{x\to x_0} u_n(x)\right]$$

Bu esa cheksiz qatorlarda ham hadlab limitga o`tish qoidasi o`rinli bo`lishini ko`rsatadi.

10.9. Funksional qatorlarni hadma-had integrallash.

Funksional qatorlarni hadma-had integrallash. Faraz qilaylik, $[a,b]$ segmentda

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + ... + u_n(x) + ... \quad (1)$$

funksional qator berilgan bo`lsin.

1-teorema. Aytaylik, (1) qator quydagi shartlarni bajarsin:

1) qatorning har bir $u_n(x)$ $(n=1,2,...)$ hadi $[a,b]$ segmenda uzluksiz,

2) $\sum_{n=1}^{\infty} u_n(x)$ qator $[a,b]$ segmentda tekis yaqinlashuvchi,

3) $\sum_{n=1}^{\infty} u_n(x) = S(x)$.

U holda

$$\sum_{n=1}^{\infty} \int_a^x u_n(t)\,dt = \int_a^x u_1(t)\,dt + \int_a^x u_2(t)\,dt + ...$$

qator $[a,b]$ da yaqinlashuvchi va

$$\sum_{n=1}^{\infty} \int_a^x u_n(t)\,dt = \int_a^x S(t)\,dt \quad (x \in [a,b])$$

bo`ladi.

Isbot. Berilgan funksional qatorning qismiy yig`indisi

$$S_n(x) = u_1(x) + u_2(x) + ... + u_n(x)$$

ni olamiz. Unda teoremaning 2) – va 3) – shartlariga ko`ra

$$S_n(x) \overset{\to}{\underset{\to}{}} S(x) \qquad (x \in [a,b])$$

bo`ladi. Tekis yaqinlashish ta'rifga

$$\forall \varepsilon > 0, \exists n_0 = n_0(\varepsilon) \in N, \forall n > n_0 \text{ va } \forall t \in [a,b] \text{ da}$$

$$|S_n(t) - S(t)| < \frac{\varepsilon}{b-a}$$

tengsizlik bajariladi.

Teoremaning 1) – shartidan hamda yuqorida isbot etilgan 1-teoremadan foydalanib

$$\int_a^x u_n(t)\,dt \qquad (n=1,2,3,...), \qquad \int_a^x S(t)\,dt$$

integrallarning mavjudligini topamiz.

Ushbu

$$\sum_{n=1}^{\infty} \int_a^x u_n(t)\,dt = \int_a^x u_1(t)\,dt + \int_a^x u_2(t)\,dt + ... + \int_a^x u_n(t)\,dt + ...$$

funksional qatorni qaraymiz. Bu qatorning qismiy yig`indisi

$$\sigma_n(x) = \sum_{k=1}^{n} \int_a^x u_k(t)\,dt \qquad (x \in [a,b])$$

bo`lsin. Ravshanki,

$$\sum_{k=1}^{n} \int_a^x u_k(t)\,dt = \int_a^x \left(\sum_{k=1}^{n} u_k(t) \right) dt.$$

Demak,

$$\sigma_n(x) = \int_a^x S_n(t)\,dt.$$

Endi

$$\sum_{n=1}^{\infty} \int_a^x u_n(t)\,dt$$

funksional qatorning $[a,b]$ da tekis yaqinlashuvchiligini ko`rsatamiz. Quydagi

$$\left| \sigma_n(x) - \int_a^x S(t)\,dt \right|$$

ayirma uchun

$$\left| \sigma_n(x) - \int_a^x S(t)\,dt \right| = \left| \int_a^x S_n(t)\,dt - \int_a^x S(t)\,dt \right| \le \int_a^x |S_n(t) - S(t)|\,dt < \frac{\varepsilon}{b-a} \int_a^x dt = \frac{\varepsilon}{b-a} \cdot (x-a) < \varepsilon$$

bo`ladi. Demak,

$$\sigma_n(x) \overset{\to}{\underset{\to}{}} \int_a^x S(t)\,dt \qquad (x \in [a,b]).$$

Bu esa

$$\sum_{n=1}^{\infty} \int_a^x u_n(t)\,dt$$

funksional qatorni $[a,b]$ da tekis yaqinlashuvchiligi va

$$\sum_{n=1}^{\infty} \int_a^x u_n(t)\,dt = \int_a^x S(t)\,dt$$

bo`lishini bildiradi

Keltirilgan teoremaning shartlari bajarilganda teoremaning tasdig`ini quydagicha

$$\sum_{k=1}^{\infty}\int_a^x u_k(t)\,dt=\int_a^x\left(\sum_{k=1}^{\infty}u_k(t)\right)dt$$

ifodalash mumkin.

10.10. Funksional qatorlarni hadma-had differensiallash.

Funksional qatorlarni hadma-had differensiallash. Faraz qilaylik, $[a,b]$ segmentda

$$\sum_{n=1}^{\infty}u_n(x)=u_1(x)+u_2(x)+...+u_n(x)+... \qquad (1)$$

funksional qator berilgan bo`lsin.

1-teorema. Aytaylik, (1) funksional qator quydagi shartlarni bajarsin:

1) qatorning har biri hadi $[a,b]$ segmentda uzluksiz $u_n'(x)$ $(n=1,2,...)$ hosilaga ega,

2) Ushbu

$$\sum_{n=1}^{\infty}u_n'(x)=u_1'(x)+u_2'(x)+...+u_n'(x)+...$$

funksional qator $[a,b]$ da tekis yaqinlashuvchi,

3) $x_0\in[a,b]$ nuqta mavjudki,

$$\sum_{n=1}^{\infty}u_n(x_0)=u_1(x_0)+u_2(x_0)+...+u_n(x_0)+...$$

qator yaqinlashuvchi. U holda

a) $\sum_{n=1}^{\infty}u_n(x)$ funksional qator $[a,b]$ da tekis yaqinlashuvchi,

b) bu qator yig`indisi

$$S(x)=\sum_{n=1}^{\infty}u_n(x)$$

$[a,b]$ da uzluksiz $S'(x)$ hosilaga ega,

c) $S'(x)=\sum_{n=1}^{\infty}u_n'(x)$

bo`ladi.

Ushbu $\sum_{n=1}^{\infty}u_n'(x)$

qatorning yig`indisini $\sigma(x)$ bilan belgilaylik:

$$\sigma(x)=\sum_{n=1}^{\infty}u_n'(x). \qquad (2)$$

Bu qator tekis yaqinlashuvchi va har bir hadi $[a,b]$ da uzluksiz. Yuqorida keltirilgan teoremaga ko`ra (2) ni hadlab integrallash mumkin:

$$\int_{x_0}^{x}\sigma(x)\,dx=\sum_{n=1}^{\infty}u_n'(x)\,dx,$$

bunda $x_0\in[a,b]$, $x\in[a,b]$. Ayni paytda,

$\sum_{n=1}^{\infty}\int_{x_0}^{x}u_n'(x)$ funksional qator $[a,b]$ da tekis yaqinlashuvchi.

Ravshanki,

$$\int_{x_0}^{x}u_n'(x)\,dx=u_n(x)-u_n(x_0).$$

Demak, $\sum_{n=1}^{\infty}(u_n(x)-u_n(x_0))$ qator $[a,b]$ tekis yaqinlashuvchi.

Shartga ko`ra

$$\sum_{n=1}^{\infty}u_n(x_0)$$

qator yaqinlashuvchi (uni $[a,b]$ da tekis yaqinlashuvchi deb qarash mumkin).

Shunday qilib

$$\sum_{n=1}^{\infty}(u_n(x)-u_n(x_0)),\ \sum_{n=1}^{\infty}u_n(x_0)$$

qatorlar $[a,b]$ da tekis yaqinlashuvchi bo`ladi. Bundan esa bu qatorlarning yig`indisi bo`lgan

$$\sum_{n=1}^{\infty}u_n(x)$$

funksional qatorning $[a,b]$ da tekis yaqinlashuvchiligi kelib chiqadi. Shuni e'tiborga olib topamiz:

$$\int_{x_0}^{x}\sigma(x)\,dx=\sum_{n=1}^{\infty}(u_n(x)-u(x_0))=\sum_{n=1}^{\infty}u_n(x)-\sum_{n=1}^{\infty}u(x_0)=S(x)-S(x_0).$$

$\sigma(x)$ funksiya,har bir hadi uzluksiz,o`zi tekis yaqinlashuvchi

$$\sum_{n=1}^{\infty}u_n'(x)$$

qatorning yig`indisi bo`lgani uchun, $[a,b]$ da uzluksiz bo`ladi.

Unda keying tenglikdan

$$\sigma(x) = (S(x) - S(x_0)) = S'(x)$$

bo`lishi kelibchiqadi.

Demak,

$$\sum_{n=1}^{\infty} u_n(x)$$

qator yig`indisi uzluksiz $S'(x)$ hosilaga ega va

$$S'(x) = \sum_{n=1}^{\infty} u_n'(x)$$

bo`ladi.

Bu keltirilgan teoremaning shartlari bajarilganda uning tasdiqini quydagicha yozish mumkin.

$$\frac{d}{dx}\left(\sum_{n=1}^{\infty} u_n(x)\right) = \sum_{n=1}^{\infty}\left(\frac{d}{dx} u_n(x)\right).$$

1-misol. Ushbu $\sum_{n=1}^{\infty} \ln \frac{(n+1)(n+x)}{n(n+1+x)} \quad (0 \le x < +\infty)$

funksional qator yig`indisi topilsin.

Ma'lumki,

$$\sum_{n=1}^{\infty} \frac{1}{(n+x)(n+1+x)}$$

funksional qator $[0,+\infty)$ da tekis yaqinlashuvchi bo`lib,uning yig`indisi

$$S(x) = \frac{1}{1+x}$$

ga teng:

$$\frac{1}{1+x} = \sum_{n=1}^{\infty} \frac{1}{(n+x)(n+x+1)}.$$

Ravshanki, bu qatorning har bir hadi $[0,+\infty)$ da uzluksiz. Demak,uni 2 – teoremaga ko`ra hadlab integrallash mumkin:

$$\int_0^x \frac{dt}{1+t} = \sum_{n=1}^{\infty} \int_0^x \frac{dt}{(n+t)(n+1+t)}.$$

Aniq integrallarni hisoblaymiz:

$$\int_0^x \frac{dt}{1+t} = \ln(1+t)\Big|_0^x = \ln(1+x),$$

$$\int \frac{1}{(n+t)(n+1+t)} dt = \int\left(\frac{1}{n+t} - \frac{1}{n+1+t}\right) dt = \ln(1+t)\Big|_0^x - \ln(n+1+t)\Big|_0^x = \ln\frac{(n+1)(n+x)}{n(n+1+x)}$$

Demak, $\sum_{n=1}^{\infty} \ln \frac{(n+1)(n+x)}{n(n+1+x)} = \ln(1+x)$.

Qatorning yig`indisini toping:

1. $\sum_{n=2}^{\infty} (n+1)x^{n-2}$
2. $\sum_{n=3}^{\infty} (n+4)x^{x-3}$
3. $\sum_{n=0}^{\infty} n(2n+1)x^{n+2}$
4. $\sum_{n=0}^{\infty} (2n^2 - n - 2)x^{n+1}$
5. $\sum_{n=3}^{\infty} (n+3)x^{n-2}$
6. $\sum_{n=2}^{\infty} (n+5)x^{n-1}$
7. $\sum_{n=0}^{\infty} (2n^2 + 5n + 3)x^{n+1}$
8. $\sum_{n=0}^{\infty} (2n^2 - n - 1)x^{n}$
9. $\sum_{n=1}^{\infty} (n+3)x^{n-1}$
10. $\sum_{n=2}^{\infty} (n+4)x^{x-2}$
11. $\sum_{n=0}^{\infty} (n^2 + 5n + 3)x^{n}$
12. $\sum_{n=0}^{\infty} (n^2 - 2n - 1)x^{n+2}$

16. $\sum_{n=0}^{\infty} (n^2 + 2n + 2)x^{x+2}$
17. $\sum_{n=2}^{\infty} (n+5)x^{x-2}$
18. $\sum_{n=0}^{\infty} (2n^2 + 7n + 5)x^{n+1}$
19. $\sum_{n=1}^{\infty} (n+4)x^{n-1}$
20. $\sum (n^2 - 2n - 2)x^{n+1}$
21. $\sum_{n=0}^{\infty} (n^2 + 7n + 4)x^{n}$
22. $\sum_{n=0}^{\infty} (n+2)x^{n-1}$
23. $\sum_{n=0}^{\infty} \left(n^2 + 6n + 5\right)x^{n+1}$
24. $\sum_{n=0}^{\infty} \left(n^2 - n + 1\right)x^{n}$
25. $\sum_{n=1}^{\infty} (n+6)x^{n-1}$
26. $\sum_{n=0}^{\infty} \left(2n^2 - 2n + 1\right)x^{n}$
27. $\sum_{n=0}^{\infty} \left(n^2 + 2n - 1\right)x^{n+1}$

13. $\sum_{n=3}^{\infty}(n+1)x^{n-3}$

14. $\sum_{n=0}^{\infty}(n^2+n+1)x^{n+3}$

15. $\sum_{n=3}^{\infty}(n+2)x^{n-3}$

28. $\sum_{n=2}^{\infty}nx^{n-2}$

29. $\sum_{n=2}^{\infty}(n+2)x^{n-2}$

30. $\sum_{n=0}^{\infty}(2n^2+n+1)x^{n+1}$

10.11. Darajali qator, uning yaqinlashish radiusi va yaqinlashish intervali.

Darajali qator, uning yaqinlashish radiusi va yaqinlashish intervali. Faraz qilaylik,

$$\sum_{n=0}^{\infty}a_nx^n = a_0+a_1x+a_2x^2+...+a_nx^n+...$$

darajali qator berilgan bo'lsin. Bu qatorning yaqinlashish yoki uzoqlashish nuqtalari haqida quydagi uch hol bo'lishi mumkin:

1) barcha musbat sonlar qatorning yaqinlashish nuqtalari bo'ladi;
2) barcha musbat sonlar qatorning uzoqlashish nuqtalari bo'ladi;
3) shunday musbat sonlar borki, ular qatorning yaqinlashish nuqtalari bo'ladi, shunday musbat sonlar borki, ular qatorning uzoqlashish nuqtalari bo'ladi.

Birinchi holda, Abel teoremasiga ko'ra darajali qator barcha $x\in R$ da yaqinlashuvchi bo'lib, darajali qatorning yaqinlashish to'plami $E=(-\infty,+\infty)$ bo'ladi. Bunday qatorga ushbu

$$\sum_{n=0}^{\infty}\frac{1}{n!}x^n = 1+x+\frac{1}{2!}x^2+...+\frac{1}{n!}x^n+...$$

darajali qator misol bo'ladi

Ikkinchi holda, Abel teoremasi natijasiga ko'ra darajali qator barcha $x\in R\setminus\{0\}$ da uzoqlashuvchi bo'lib, uning yaqinlashish to'plami $E=\{0\}$ bo'ladi. Bunday qatorga ushbu

$$\sum_{n=1}^{\infty}n!x^n = x+2!x^2+3!x^3+...+n!x^n+...$$

darajali qator misol bo'la oladi.

Endi uchinchi holga qaraymiz. Bu halda ushbu

$$\sum_{n=0}^{\infty}x^n = 1+x+x^2+x^3+...+x^n+...$$

darajali qator misol bo'ladi. Bu darajali qator barcha $x\in(0,1)$ da yaqinlashuvchi va demak, Abel teoremasiga ko'ra qator $(-1,1)$ da yaqinlashadi, barcha $x\in[1,+\infty)$ da qator uzoqlashuvchi va demak, Abel teoremasining natijasiga ko'ra qator $(-\infty,-1]\cup[1,+\infty)$ da uzoqlashadi. Demak, darajali qatorning yaqinlashish to'plami $E=(-1,1)$ bo'ladi.

Aytaylik,

$$\sum_{n=0}^{\infty}a_nx^n = a_0+a_1x+a_2x^2+...+a_nx^n+...$$

darajali qator r_1 nuqtada $(r_1>0)$ yaqinlashuvchi, R_1 nuqtada $(R_1>0)$ esa uzoqlashushchi bo'lsin. Ravshanki,

$$r_1<R_1$$

bo'ladi.

Agar $\sum_{n=0}^{\infty}a_nx^n$ darajali qator

$$\frac{r_1+R_1}{2}$$

nuqtada yaqinlashuvchi bo'lsa,

$$r_2=\frac{r_1+R_1}{2},\qquad R_2=R_1$$

deb, uzoqlashuvchi bo'lsa,

$$r_2=r_1,\qquad R_2=\frac{r_1+R_1}{2}$$

deb r_2 va R_2 nuqtalarni olamiz. Ravshanki,

$$r_1\le r_2,\ R_1\ge R_2 \text{ va } R_2-r_2=\frac{R_1-r_1}{2}$$

bo'ladi. Bu munosabatdagi r_2 va R_2 sonlarga ko'ra r_3 va R_3 sonlarni yuqoridagiga o'xshash aniqlaymiz:

Agar $\sum_{n=0}^{\infty}a_nx^n$ darajali qator

$$\frac{r_1+R_1}{2}$$

nuqtada yaqinlashuvchi bo'lsa,

$$r_3=\frac{r_2+R_2}{2},\quad R_3=R_2$$

deb, uzoqlashuvchi bo'lsa,

$$r_3 = r_2, \quad R_3 = \frac{r_2 + R_2}{2}$$

deb r_3 va R_3 nuqtalarni olamiz. Bunda

$$r_2 \le r_3,\ R_2 \ge R_3 \ \text{ va } \ R_3 - r_3 = \frac{R_2 - r_2}{2}$$

bo`ladi.

Bu jarayonni davom ettiraborish natijasida $\sum_{n=0}^{\infty} a_n x^n$ darajali qatorning yaqinlashish nuqtalaridan iborat $\{r_n\}$, uzoqlashish nuqtalaridan iborat $\{R_n\}$ ketma-ketliklar hosil bo`ladi.Bunda

$$r_1 \le r_2 \le \ldots \le r_n \le \ldots, \quad R_1 \ge R_2 \ge \ldots. \ge R_n \ge \ldots.$$

va $n \to \infty$ da

$$R_n - r_n = \frac{R_1 - r_1}{2^{n-1}} \to 0$$

bo`ladi.Unda yuqorida keltirilgan teoremaga ko`ra $\lim_{n\to\infty} r_n$ va $\lim_{n\to\infty} R_n$ limitlar mavjud va

$$\lim_{n\to\infty} r_n = \lim_{n\to\infty} R_n$$

bo`ladi. Uni r bilan belgilaymiz:

$$\lim_{n\to\infty} r_n = \lim_{n\to\infty} R_n = r.$$

Endi x o`zgaruvchining $|x| < r$ tengsizlikni qanotlantiruvchi ixtiyoriy qiymatini olaylik.Unda

$$\lim_{n\to\infty} r_n = r$$

bo`lishidan, shunday $n_0 \in N$ topiladiki,

$$|x| < r_{n_0} < r$$

bo`ladi. Binobarin, berilgan darajali qator $r_{n_0}$ nuqtada, demak qaralyotgan $x$ nuqtada yaqinlashuvchi bo`ladi.

x o`zgaruvchining $|x| > r$ tenglikni qanotlantiruvchi ixtiyoriy qimatini olaylik. Unda

$$\lim_{n\to\infty} R_n = r$$

bo`lishidan,shunday $n_1 \in N$ topiladiki,

$$|x| > R_{n_1} > r$$

bo`ladi. Binobarin, berilgan darajali qator $R_{n_1}$ nuqtada, demak qaralyotgan $x$ nuqtada uzoqlashuvchi bo`ladi.

Demak, 3)-holda

$\sum_{n=0}^{\infty} a_n x^n$ darajali qator uchun shunday musbat r soni mavjud bo`ladiki, $|x| < r$, ya'ni $\forall x \in (-r, r)$ da qator yaqinlashuvchi, $|x| > r$, ya'ni $\forall x \in (-\infty, -r) \cup (r, +\infty)$ da qator uzoqlashuvchi bo`ladi. $x = \pm r$ nuqtalarda $\sum_{n=0}^{\infty} a_n x^n$ darajali qator yaqinlashuvchi ham bo`lishi mumkin, uzoqlashuvchi ham bo`lishi mumkin.

1-ta'rif. Yuqorida keltirilgan r son $\sum_{n=0}^{\infty} a_n x^n$ darajali qatorning yaqinlashish radiusi, $(-r, r)$ interval esa darajali qatorning yaqinlashish intervali deyiladi.

Eslatma. 1)-holda darajali qatorning yaqinlashish radiusi $r = +\infty$ deb, 2)-holda darajali qatorning yaqinlashish radiusi $r = 0$ deb olinadi.

3^0. Darajali qatorning yaqinlashish radiusini topish. Biror

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$$

darajali qatorni qaraylik. Bu qator koeffisentlaridan tuzilgan $\{a_n\}$ $(n = 0, 1, 2, \ldots)$ ketma-ketlik uchun

1) $\forall n \ge 0$ da $a_n \ne 0$,

2) $\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$ mavjud bo`lsin. U holda $\sum_{n=0}^{\infty} a_n x^n$ darajali qatorning yaqinlashish radiusi

$$r = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

bo`ladi.

Aytaylik, darajali qator uchun

$$\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = L \quad (a_n \ne 0, n = 0, 1, 2, 3 \ldots)$$

bo`lsin. Qaralayotgan $\sum_{n=0}^{\infty} a_n x^n$ darajali qatorda $x$ ni parametr hisoblab,Dalamber alomatiga ko`ra uni yaqinlashishga tekshiramiz:

$$\lim_{n\to\infty} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \cdot |x| = |x| \lim_{n\to\infty} \left| \frac{1}{\frac{a_n}{a_{n+1}}} \right| = |x| \cdot \frac{1}{L}$$

Demak,

$$\frac{|x|}{L}<1, \text{ ya'ni } |x|<L$$

bo`lganda qator yaqinlashuvchi bo`ladi,

$$\frac{|x|}{L}>1 \quad \text{ya'ni} \quad |x|>L$$

bo`lganda darajali qator uzoqlashuvchi bo`ladi.

Bundan $\sum_{n=0}^{\infty} a_n x^n$ darajali qatorning yaqinlashish radiusi

$$r=L=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| \qquad (4)$$

bo`lishi kelib chiqadi.

1-misol. Ushbu $\sum_{n=0}^{\infty}\frac{n^n}{e^n n!}x^n \quad (0!=1)$

darajali qatorning yaqinlashish radiusi topilsin.

Bu qator uchun

$$a_n=\frac{n^n}{e^n n!}, \quad a_{n+1}=\frac{(n+1)^{n+1}}{e^{n+1}(n+1)!}$$

bo`ladi. Ravshanki,

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=\lim_{n\to\infty}\left|\frac{n^n}{e^n\cdot n!}\cdot\frac{e^{n+1}(n+1)!}{(n+1)^{n+1}}\right|=\lim_{n\to\infty}\frac{e}{\left(1+\frac{1}{n}\right)^n}=1.$$

Demak, berigan darajali qatorning yaqinlashish radiusi $r=1$ bo`ladi.

Ixtiyoriy darajali qatorning yaqinlashish radiusini aniqlab beradigan teoremani isbotsiz keltiramiz.

2-teorema (Koshi-Adamar). Ushbu

$$\sum_{n=0}^{\infty}a_n x^n=a_0+a_1x+a_2x^2+\ldots+a_nx^n+\ldots$$

darajali qatorning yaqinlashish radiusi

$$r=\frac{1}{\overline{\lim_{n\to\infty}}\sqrt[n]{|a_n|}} \qquad (5)$$

bo`ladi.

Eslatma. Agar

$$\overline{\lim_{n\to\infty}}\sqrt[n]{|a_n|}=+\infty$$

bo`lsa, $\sum_{n=0}^{\infty}a_nx^n$ darajali qatorning yaqinlashish radiusi $r=0$ deb,

$$\overline{\lim_{n\to\infty}}\sqrt[n]{|a_n|}=0$$

bo`lsa, $\sum_{n=0}^{\infty}a_nx^n$ darajali qatorning yaqinlashish radiusi $r=+\infty$ deb olinadi.

2-misol. Ushbu

$$\sum_{n=0}^{\infty}2^n x^{5n}$$

darajali qatorning yaqinlashish radiusi topilsin.

Avvalo

$$2x^5=t$$

deb olamiz. Natijada berilgan qator quydagi

$$\sum_{n=0}^{\infty}t^n=1+t+t^2+\ldots+t^n$$

ko`rinishga keladi. Bu qatorning yaqinlashish radiusi (5) formulaga ko`ra

$$r=\frac{1}{\overline{\lim_{n\to\infty}}\sqrt[n]{|a_n|}}=\frac{1}{\overline{\lim_{n\to\infty}}\sqrt[n]{1}}=1$$

bo`ladi. Demak, $|t|<1$ da qator yaqinlashuvchi, $|t|>1$ da uzoqlashuvchi. Unda

$|2x^5|<1$, ya'ni $|x|<\frac{1}{\sqrt[5]{2}}$ da berilgan qator yaqinlashuvchi,

$|2x^5|>1$, ya'ni $|x|>\frac{1}{\sqrt[5]{2}}$ da uzoqlashuvchi bo`ladi. Berilgan darajali qatorning yaqinlashish radiusi $r=\frac{1}{\sqrt[5]{2}}$ bo`ladi.

3-misol. Ushbu $\sum_{n=0}^{\infty}\frac{(-1)^n}{3^{n-1}\sqrt{n}}x^n$

darajali qatorning yaqinlashish to`plami topilsin.

Ravshanki, $a_n=\frac{(-1)^n}{3^{n-1}\sqrt{n}}$, $a_{n+1}=\frac{(-1)^{n+1}}{3^n\sqrt{n+1}}$.

Berilgan darajali qatorning yaqinlashish radiusini (4) formulaga ko`ra topamiz:

$$r=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=\lim_{n\to\infty}\left|\frac{(-1)^n}{3^{n-1}\sqrt{n}}\cdot\frac{3^n\sqrt{n+1}}{(-1)^{n+1}}\right|=\lim_{n\to\infty}3\sqrt{\frac{n+1}{n}}=3.$$

Darajali qator $x=-3$ nuqtada ushbu $\sum_{n=1}^{\infty}\frac{3}{\sqrt{n}}$ sonli qatorga aylanadi va bu sonli qator uzoqlashuvchi bo`ladi. $x=3$ nuqtada esa quydagi $\sum_{n=1}^{\infty}3\frac{(-1)^n}{\sqrt{n}}$ sonli qator hosil bo`ladi va bu qator Leybnist teoremasiga ko`ra yaqinlashuvchi bo`ladi. Demak, berilgan darajali qatorning yaqinlashish to`plami $E=(-3,3]$ dan iborat.

Darajali qatorning yaqinlashish radiusini toping:

1. $\sum_{n=1}^{\infty}\frac{(x-1)^n}{n\sqrt{n}}$

2. $\sum_{n=1}^{\infty}\left(\frac{2n-1}{3n+2}\right)^n(x+2)^n$

3. $\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{2n+1}$

4. $\sum_{n=0}^{\infty}\frac{(-1)^{n-1}(2n+3)}{3n^2+4}x^{2n+1}$

5. $\sum_{n=1}^{\infty}\frac{1}{\sqrt[n]{3}}\left(\frac{x-1}{3}\right)^n$

6. $\sum_{n=1}^{\infty}\sqrt{\frac{n^4+3}{n^3+4n}}(x+2)^n$

7. $\sum_{n=1}^{\infty}\frac{5^n+(-3)^n}{n+1}x^n$

8. $\sum_{n=1}^{\infty}\frac{(x+1)^n}{\sqrt{n+1}}\ln\frac{3n-2}{3n+2}$

9. $\sum_{n=1}^{\infty}\frac{\sqrt[3]{2n+1}-\sqrt[3]{2n-1}}{\sqrt{n}}(x+3)^n$

10. $\sum_{n=1}^{\infty}(\sqrt[n]{a}-1)x^n,\ a>0$

11. $\sum_{n=0}^{\infty}3^n(n^3+2)(x-1)^{2n}$

12. $\sum_{n=1}^{\infty}4^{n^2}(x+1)^{n^2}$

16. $\sum_{n=1}^{\infty}\left(1+\frac{1}{2}+...+\frac{1}{n}\right)(x-1)^n$

17. $\sum_{n=1}^{\infty}\frac{(2+(-1)^n)^n}{n}(x+1)^n$

18. $\sum_{n=1}^{\infty}\frac{\left(1+2\cos\left(\frac{\pi n}{4}\right)\right)^n x^n}{\ln^2(n+1)}$

19. $\sum_{n=1}^{\infty}\left(\frac{a^n}{n}+\frac{b^n}{n^2}\right)x^n,\ a>0,b>0$

20. $\sum_{n=1}^{\infty}\frac{x^n}{a^n+b^n},\ a>0,b>0$

21. $\sum_{n=1}^{\infty}\frac{a(a-1)...(a-(n+1))}{n!}x^n$

22. $\sum_{n=1}^{\infty}\frac{3^{-\sqrt{n}}}{\sqrt{n^2+n+1}}(x-1)^n$

23. $\sum_{n=1}^{\infty}\frac{(-1)^{[\sqrt{n}]}}{n}x^n$

24. $\sum_{n=1}^{\infty}\left(\frac{x}{\sin n}\right)^n$

25. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}\left(\frac{n}{e}\right)^n(x-1)^n$

26. $\sum_{n=1}^{\infty}2^n x^{n^2}$

27. $\sum_{n=1}^{\infty}\frac{(x-1)^n}{a^{\sqrt{n}}},a>0$

13. $\sum_{n=1}^{\infty}\frac{(2n-1)!!}{n!}(x+2)^n$

14. $\sum_{n=1}^{\infty}\frac{(x+3)^{n^2}}{n^n}$

15. $\sum_{n=1}^{\infty}\left(1-\frac{1}{n}\right)^{n^2}x^n$

28. $\sum_{n=1}^{\infty}\frac{n!}{a^{n^2}}x^n,a>1$

29. $\sum_{n=1}^{\infty}2^n\cos^n x$

30. $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{-n^2}e^{-\pi x}$

10.12.Darajali qatorlarning xossalari.

Darajali qator tushunchasi. Abel teoremasi. Har bir hadi

$$u_n(t)=a_n(t-t_0)^n\qquad (t_0\in R;\ n=0,1,2...)$$

funksiyadan iborat bo`lgan ushbu

$$\sum_{n=0}^{\infty}a_n(t-t_0)^n=a_0+a_1(t-t_0)+a_2(t-t_0)^2+...\qquad(1)$$

funksional qator darajali qator deyiladi, bunda

$$a_0,a_1,...,a_n,...$$

haqiqiy sonlar darajali qatorning koeffisentlari deyiladi.

(1) da $t-t_0=x$ deyilsa, u quydagi

$$\sum_{n=0}^{\infty}a_nx^n=a_0+a_1x+a_2x^2+...+a_nx^n+...\ (x\in R)\qquad(2)$$

ko`rinishga keladi va biz shu ko`rinishdagi darajali qatorni o`rganamiz.

Ravshanki, (2) qatorning qismiy yig`indisi

$$S_n(x)=a_0+a_1x+a_2x^2+...+a_nx^n$$

ko`phaddan iborat.Ayni paytda, $x=0$ da $S_n(0)=a_0$ bo`ladi. Demak,har qanday (2) ko`rinishdagi darajali qator $x=0$ nuqtada yaqinlashuvchi bo`ladi.

1-teorema (Abel). Agar

$$\sum_{n=0}^{\infty}a_nx^n=a_0+a_1x+a_2x^2+...+a_nx^n+...$$

darajali qator $x=x_0\neq 0$ nuqtada yaqinlashuvchi, ushbu

$$|x|<|x_0|$$

tengsizlikni qanotlantiruvchi barcha x larda darajali qator yaqinlashuvchi (absolyut yaqinlashuvchi) bo`ladi.

Isbot. Faraz qilaylik, $x=x_0\neq 0$ da

$$\sum_{n=0}^{\infty} a_n x_0^n$$

qator yaqinlashuvchi bo`lsin. Qator yaqinlashishining zaruriy shartiga ko`ra

$$\lim_{n\to\infty} a_n x_0^n = 0$$

bo`ladi. Demak, $\{a_n x_0^n\}$ ketma-ketlik chegaralangan:

$$\exists M > 0,\ \forall n \in N \text{ da } |a_n x_0^n| \le M .$$

Ravshanki,

$$|a_n x^n| = |a_n x_0^n| \left|\frac{x}{x_0}\right|^n \le M \cdot \left|\frac{x}{x_0}\right|^n \qquad (3)$$

va $|x| < |x_0|$ da $\left|\frac{x}{x_0}\right| = q < 1$ bo`ladi.Demak $\sum_{n=0}^{\infty}\left|\frac{x}{x_0}\right|^n = \sum_{n=0}^{\infty} q^n$ geometrik qator yaqinlashuvchi. Unda ushbu

$$\sum_{n=0}^{\infty} M\left|\frac{x}{x_0}\right|^n$$

qator ham yaqinlashuvchi bo`ladi. (3) munosabatni e'tiborga olib, so`ng solishtirish teoremasidan foydalanib

$$\sum_{n=0}^{\infty} a_n x^n$$

darajali qatorning yaqinlashishini (absolyut yaqinlashishini) topamiz.

Natija. Agar $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$

darajali qator $x = x_1$ nuqtada uzoqlashuvchi (ushbu

$$\sum_{n=0}^{\infty} a_n x_1^n = a_0 + a_1 x_1 + a_2 x_1^2 + \ldots + a_n x_1^n + \ldots$$

sonli qator uzoqlashuvchi) bo`lsa, quydagi

$$|x| > |x_1|$$

tengsizlikni qanotlantiruvchi barcha x larda $\sum_{n=0}^{\infty} a_n x^n$ qator uzoqlashuvchi bo`ladi.

Isbot. Teskarisini faraz qilaylik, $\sum_{n=0}^{\infty} a_n x^n$ qator $|x| > |x_1|$ tengsizlikni qanotlantiruvchi biror $x = x^*$ nuqtada $(|x^*| > |x_1|)$ yaqinlashuvchi bo`lsin. U holda Abel teoremasiga ko`ra $|x| < |x^*|$ tengsizlikning qanotlantiruvchi barcha x larda yaqinlashuvchi, jumladan x_1 nuqtada ham yaqinlashuvchi bo`lib qoladi. Bu esa shartga ziddir.

Abel teoremasi va uning natijasi darajali qatorlarning yaqinlashish (uzoqlashish) to`plamining strukturasini (tuzilishini) aniqlab beradi.

Darajali qatorning yaqinlashish radiusini toping:

1. $\sum_{n=0}^{\infty} n^2 z^n$
2. $\sum_{n=0}^{\infty} 3^n (z+1)^n$
3. $\sum_{n=0}^{\infty} \left(\frac{n+1}{2n+3}\right)^n z^n$
4. $\sum_{n=0}^{\infty} \frac{(n+1)(z-2)^n}{4^{n+2}}$
5. $\sum_{n=0}^{\infty} n^n z^n$
6. $\sum_{n=0}^{\infty} \left(\frac{n+2}{n+5}\right)^{n^2} z^n$
7. $\sum_{n=1}^{\infty} 2^{\ln^2 n} (z-3)^n$
8. $\sum_{n=1}^{\infty} \left(\frac{z}{\sqrt{n}}\right)^n$
9. $\sum_{n=1}^{\infty} n!\left(\frac{z}{n}\right)^n$
10. $\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n!} z^n$
11. $\sum_{n=1}^{\infty} \frac{2^n z^{4n}}{n^2}$

16. $\sum_{n=1}^{\infty} \frac{(kn)!}{n!(n+1)!\ldots(n+(k+1))!},\ k \in N$
17. $\sum_{n=1}^{\infty} n! z^{n!}$
18. $\sum_{n=1}^{\infty} (2-\sqrt{e})(2-\sqrt[3]{e})\ldots.(2-\sqrt[n]{e}) z^n$
19. $\sum_{n=1}^{\infty} n\left(\cos\frac{1}{n}\right)^{2n^2} z^n$
20. $\sum_{n=1}^{\infty} \frac{2^n \cos\left(\frac{3\pi n}{4}\right)}{n} z^n$
21. $\sum_{n=1}^{\infty} \left(\left(1-\frac{1}{n}\right)^n arctge^{-n}\right)(z+1)^n$
22. $\sum_{n=1}^{\infty} \left(\frac{n^2+3}{n^2+5}\right)(z-1)^n$
23. $\sum_{n=1}^{\infty} \left(1 + arctg\frac{n+1}{n^2}\right)(z+2)^n$
24. $\sum_{n=1}^{\infty} \left(\ln\cos\frac{1}{3^n}\right) z^n$
25. $\sum_{n=1}^{\infty} \frac{2^n (1+i)^{3n}}{(n+1)(n+2)}$
26. $\sum_{n=1}^{\infty} \left(\frac{1-i}{2}\right) z^{2n}$

12. $\sum_{n=1}^{\infty}\frac{(n+1)^2}{3^n}z^{5n}$

13. $\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}z^n$

14. $\sum_{n=1}^{\infty}\frac{z^n}{(n!)^{\alpha}},\ \alpha>0$

15. $\sum_{n=1}^{\infty}n!e^{-n^{\alpha}}z^n,\ \alpha>1$

27. $\sum_{n=1}^{\infty}\frac{(2+i\sqrt{5})^n}{(3-i\sqrt{7})^{2n}}(z-i)^{3n}$

28. $\sum_{n=1}^{\infty}\frac{n!}{(1+i)(1+2i)...(1+ni)}z^n$

29. $\sum_{n=1}^{\infty}\left(\sin\sqrt{n+1}-\sqrt{n}\right)(x+1)^n$

30. $\sum_{n=1}^{\infty}\frac{2^n n}{n^n}(x-1)^{2n}$

10.13. Teylor qatori. Elementr funksiyalarni Teylor qatoriga yoyish.

Funksiyaning Teylor qatori. Aytaylik, $f(x)$ funksiya $x_0\in R$ nuqtaning biror

$$U_{\delta}(x_0)=\{x\in R: x_0-\delta<x<x_0+\delta;\delta>0\}$$

atrofida istalgan tartibli hosilaga ega bo`lsin. Bu hol $f(x)$ funksiyaning Teylor formulasini yozishga imkon beradi.

$$f(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+...+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+r_n(x),$$

bunda $r_n(x)$-qoldiq had.

Modomiki, $f(x)$ funksiya $U_{\delta}(x_0)$ da istalgan tartibli hosilaga ega ekan,unda

$$f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+...+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+... \quad (1)$$

darajali qatorni qarash mumkin.

(1) darajali qatorning koeffisentlari sonlar bo`lib,ular $f(x)$ funksiya va uning hosilalarning x_0 nuqtadagi qiymatlari orqali ifodalangan.

(1) darajali qator $f(x)$ funksiyaning Teylor qatori deyiladi.

Xususan, $x_0=0$ bo`lganda (1) darajali qator ushbu

$$f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+...+\frac{f^{(n)}(0)}{n!}x^n+...=\sum_{n=1}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$$

ko`rinishga keladi.

Faraz qilaylik, 0 $f(x)$ funksiya biror $(-r,r)$ da $(r>0)$ istalgan tartibdagi hosilaga ega bo`lib, uning $x_0=0$ nuqtadagi Teylor qatori

$$f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+...+\frac{f^{(n)}(0)}{n!}x^n+... \quad (2)$$

bo`lsin. Bu qatorning qoldiq hadini $r_n(x)$ deylik:

$$f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+...+\frac{f^{(n)}(0)}{n!}x^n+r_n(x).$$

1-teorema. (2) qarajali qator $(-r,r)$ da $f(x)$ ga yaqinlashishi uchun ushbu

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+...+\frac{f^{(n)}(0)}{n!}x^n+r_n(x)$$

Teylor formulasida, $\forall x\in(-r,r)$ uchun

$$\lim_{n\to\infty}r_n(x)=0$$

bo`lishi zarur va yetarli.

Isbot. Zarurligi. Aytaylik, (2) darajali qator $(-r,r)$ da yig`indisi, $f(x)$ bo`lsin. Ta'rifga binoan

$$\lim_{n\to\infty}S_n(x)=f(x),\ \ (x\in(-r,r))$$

bo`ladi,bunda

$$S_n(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+...+\frac{f^{(n)}(0)}{n!}x^n.$$

Ravshanki, $\forall x\in(-r,r)$ da $\lim_{n\to\infty}S_n(x)=f(x)$ bo`lishidan

$$\lim_{n\to\infty}[f(x)-S_n(x)]=\lim_{n\to\infty}r_n(x)=0$$

bo`lishi kelib chiqadi.

Yetarliligi. Aytaylik, $\forall x\in(-r,r)$ da $\lim_{n\to\infty}r_n(x)=0$ bo`lsin.U holda

$$\lim_{n\to\infty}[f(x)-S_n(x)]=\lim_{n\to\infty}r_n(x)=0$$

bo`lib,unda

$$\lim_{n\to\infty}S_n(x)=f(x)$$

bo`lishi kelib chiqadi.Demak

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+...+\frac{f^{(n)}(0)}{n!}x^n+...$$

bo`ladi.

Odatda, bu munosabat o`rinli bo`lsa, $f(x)$ funksiya Teylor qatoriga yoyilgan deyiladi.

2^0. Funksiyani Teylor qatoriga yoyish. Faraz qilaylik, $f(x)$ funksiya biror $(-r,r)$ da istalgan tartibli hosilalarga ega bo`lsin.

2-teorema. Agar $\exists M>0,\ \forall x\in(-r,r),\ \forall n\geq 0$ da

$$|f^{(n)}(x)| \le M$$

bo`lsa, $f(x)$ funksiya $(-r,r)$ da Teylor qatoriga yoyiladi:

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+...+\frac{f^{(n)}(0)}{n!}x^n+... \qquad (3)$$

Ma'lumki, $f(x)$ funksiyaning Lagranj ko`rinishdagi qoldiq hadi Teylor formulasi quydagicha bo`ladi:

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+...+\frac{f^{(n)}(0)}{n!}x^n+r_n(x),$$

bunda,

$$r_n(x)=\frac{f^{(n)}(\theta x)}{(n+1)!}x^{n+1}. \quad (0<\theta<1)..$$

Teoremanig shartidan foydalanib topamiz:

$$|r_n(x)|=\left|\frac{f^{(n)}(\theta x)}{(n+1)!}x^{n+1}\right|\le M\cdot\frac{r^{n+1}}{(n+1)!}. \quad (x\in(-r,r)).$$

Ravshanki,

$$\lim_{n\to+\infty}\frac{r^{n+1}}{(n+1)!}=0.$$

Demak, $\forall x\in(-r,r)$ da

$$\lim_{n\to\infty}r_n(x)=0$$

bo`lib, unda qaralayotgan $f(x)$ funksiyaning Teylor qatoriga yoyilishi kelib chiqadi.

3^0. Elementar funksiyalarni Teylor qatoriga yoyish.

a) Ko`rsatgichli va giperbolik funsiyalarni Teylor qatorlarini topamiz.Aytaylik,

$$f(x)=e^x$$

bo`lsin. Ravshanki, $f(0)=1, f^{(n)}(0)=1 \quad (n\in N)$ bo`lib, $\forall\in(-\alpha,\alpha)$ da $(\alpha>0)$

$$0<f(x)<e^{\alpha},\ 0<f^{(n)}(x)<e^{\alpha}$$

bo`ladi. Binobarin, 2-teoremaga ko`ra $f(x)=e^x$ funksiya $(-\alpha,\alpha)$ da Teylor qatoriga yoyiladi va (3) formuladan foydalanib topamiz:

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+\frac{x}{1!}+\frac{x^2}{2!}+...+\frac{x^n}{n!}+... \quad 0!=1. \qquad (4)$$

$\alpha>0$ ixtiyoriy musbat son. Demak, (4) darajali qatorning yaqinlashish radiusi $r=+\infty$ bo`ladi.

(4) munosabatda x ni $-x$ ga almashtirib topamiz:

$$e^{-x}=\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}=1-\frac{x}{1!}+\frac{x^2}{2!}-...+(-1)^n\frac{x^n}{n!}+...$$

Ma'lumki giperbolik sinus giperbolik kosinus funksiyalari quydagicha

$$shx=\frac{e^x-e^{-x}}{2},\quad chx=\frac{e^x+e^{-x}}{2}$$

ta'riflardan endi.

Yuqoridagi

$$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+...+\frac{x^n}{n!}+...,$$

$$e^{-x}=1-\frac{x}{1!}+\frac{x^2}{2!}-...+(-1)^n\frac{x^n}{n!}+...$$

formulalardan foydalanib topamiz:

$$shx=\frac{x}{1!}+\frac{x^3}{3!}+...+\frac{x^{2n+1}}{(2n+1)!}+...=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!},$$

$$chx=1+\frac{x^2}{2!}+\frac{x^4}{4!}+...+\frac{x^{2n}}{(2n)!}+...=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}.$$

Bu shx, chx funksiyalarning Teylor qatorlari bo`lib, ular ifodalangan darajali qatorning yaqinlashish radiuslari $r=+\infty$ bo`ladi.

b) Trigonometrik funksiyalarning Teylor qatorlarini topamiz. Aytaylik, $f(x)=\sin x$ bo`lsin. Ravshanki, $\forall x\in R,\ \forall n\in N$ da

$$|f(x)|\le 1,\quad |f^{(n)}(x)|\le 1$$

bo`lib, $f(0),\ f'(0)=1,\ f^{(2n)}(0)=0, f^{(2n+1)}(0)=(-1)^n \quad (n\in N)$ bo`ladi. Demak, 2-teoremaga ko`ra $f(x)=\sin x$ funksiya Teylor qatoriga yoyiladi va (3) formulaga binoan

$$\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}=x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-... \qquad (5)$$

bo`ladi.

Aytaylik,

$$f(x)=\cos x$$

bo`lsin. Bu funksiya uchun $\forall x\in R,\ \forall n\in N$ da

$$|f(x)|\le 1,\quad |f^{(n)}(x)|\le 1$$

bo`lib,

$$f(0)=1,\ f'(0)=0,\ f^{(2n)}(0)=(-1)^n, f^{(2n+1)}(0)=0 \quad (n\in N)$$

bo`ladi. Unda 2–teoremaga ko`ra $f(x)=\cos x$ funksiya Teylor qatoriga yoyiladi va (3) formulaga binoan

$$\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}=1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-... \qquad (6)$$

bo`ladi.

(5) va (6) darajali qatorning yaqinlashish radiusi $r=+\infty$ bo`ladi.

d) Logarifmik funksiyaning Teylor qatorini topamiz.Aytaylik,

$$f(x)=\ln(1+x)$$

bo`lsin.Ma`lumki,

$$f^{(n)}(x)=\frac{(-1)^{n-1}(n-1)!}{(1+x)^n} \quad (n\in N)$$

bo`lib,

$$\frac{f^{(n)}(0)}{n!}=\frac{(-1)^{n-1}}{n}$$

bo`ladi. Bu funksiyaning Teylor formulasi

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+...+(-1)^{n-1}\frac{x^n}{n}+r_n(x) \qquad (7)$$

ko`rinishga ega.

$f(x)=\ln(1+x)$ funksiyani Teylor qatoriga yoyishda 1-teoremadan foydalanamiz. Buning uchun (7) formulada $r_n(x)$ ning 0 ga intilishini ko`rsatish yetarli bo`ladi.

Faraz qilaylik, $x\in[0,1]$ bo`lsin. Bu holda Lagranj ko`rinishida yozilgan

$$r_n(x)=\frac{(-1)^n x^{n+1}}{(n+1)(1+\theta x)^{n+1}} \quad (0<\theta<1)$$

qoldiq had uchun

$$|r_n(x)|\le\frac{1}{n+1}$$

bo`ladi va

$$\lim_{n\to\infty}r_n(x)=0$$

tenglik bajariladi.

Faraz qilaylik, $x\in[-\alpha,0]$ bo`lsin, bunda $0<\alpha<1$.

Bu holda Koshi ko`rinishda yozilgan

$$r_n(x)=\frac{(-1)^n(1-\theta_1)^n\cdot x^{n+1}}{(1+\theta_1 x)^{n+1}} \quad (0<\theta_1<1)$$

qoldiq had uchun

$$|r_n(x)|\le\frac{\alpha^{n+1}}{1-\alpha}$$

bo`lib,

$$\lim_{n\to\infty}r_n(x)=0$$

bo`ladi.

Demak, $\forall\in(-1,1]$

$$\lim_{n\to\infty}r_n(x)=0.$$

Unda 1-teoremaga ko`ra

$$\ln(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n=x-\frac{x^2}{2}+\frac{x^3}{3}-...+(-1)^{n-1}\frac{x^n}{n}+... \qquad (8)$$

bo`ladi.

(8) darajali qatorning yaqinlashish radiusi $r=1$ ga teng.

Agar yuqoridagi $\ln(1+x)$ ning yoyilmasida x ni $-x$ ga almashtirilsa, unda

$$\ln(1-x)=-\sum_{n=1}^{\infty}\frac{x^n}{n}=-x-\frac{x^2}{2}-\frac{x^3}{3}-...-\frac{x^n}{n}-...$$

formula kelib chiqadi.

e) Darajali funksiyaning Teylor qatorini topamiz.

Faraz qilaylik,

$$f(x)=(1+x)^\alpha \quad (\alpha\in R)$$

bo`lsin. Ma'lumki,

$$f^{(n)}(x)=\alpha(\alpha-1)(\alpha-2)...(\alpha-n+1)(1+x)^{\alpha-n} \quad (n\in N)$$

bo`lib,

$$f^{(n)}(0)=\alpha(\alpha-1)(\alpha-2)...(\alpha-n+1)$$

bo`ladi. Bu funksiyaning Teylor formulasi ushbu

$$(1+x)^\alpha=1+\frac{\alpha}{1!}x+\frac{\alpha(\alpha-1)}{2!}x^2+...+\frac{\alpha(\alpha+1)...(\alpha-n+1)}{n!}x^n+r_n(x)$$

ko`rinishga ega.

Endi $n\to\infty$ da $r_n(x)\to 0$ bo`lishini ko`rsatamiz.

Ma'lumki, Teylor formulasidagi qoldiq handing Koshi ko`rinishi quydagicha

$$r_n(x)=\frac{(\alpha-1)(\alpha-2)...[(\alpha-1)-(n-1)]}{n!}x^n\alpha\cdot x(1+\theta x)^{\alpha-1}\left(\frac{1-\theta}{1+\theta x}\right)^n$$

$(0<\theta<1)$ bo`lar edi.

Aytaylik, $x\in(-1,1)$ bo`lsin. Bu holda:

1) $\lim\limits_{n\to\infty}\frac{1}{n!}(\alpha-1)(\alpha-2)...[(\alpha-1)-(n-1)]x^n=0$ bo`ladi,

chunki, limit ishorasi ostidagi ifoda yaqinlashuvchi ushbu

$$1+\sum_{n=1}^{\infty}\frac{\alpha(\alpha-1)...(\alpha-n+1)}{n!}x^n$$

qatorning umumiy hadi;

2) $|\alpha\cdot x|(1-|x|)^{\alpha-1}<\alpha\cdot x(1+\theta x)^{\alpha-1}<|\alpha\cdot x|(1+|x|)^{\alpha-1}$;

3) $\left|\frac{1-\theta}{1+\theta x}\right|^n\le\left|\frac{1-\theta}{1+\theta x}\right|<1$

bo`ladi. Bu munosabatlardan foydalanib, $\forall x\in(-1,1)$ da

$$\lim_{n\to\infty}r_n(x)=0$$

bo`lishini topamiz. 1-teoremaga ko`ra

$$(1+x)^\alpha=1+\frac{\alpha}{1!}x+\frac{\alpha(\alpha-1)}{2!}x^2+...+\frac{\alpha(\alpha+1)...(\alpha-n+1)}{n!}x^n+... \qquad (9)$$

bo`ladi.

Bu darajali qatorning yaqinlashish radiusi $\alpha\ne0,\alpha\notin N$ bo`lganda 1 ga teng: $r=1$.

(9) munosabatda $\alpha=-1$ deb olinsa, unda ushbu

$$\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n=1-x+x^2-x^3+x^4-...+(-1)^n x^n+...$$

formula hosil bo`ladi. Bu formulada x ni $-x$ ga almashtirib topamiz:

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}(-1)^n x^n=1+x+x^2+...+x^n+...$$

1-misol. Ushbu $f(x)=\ln\frac{1+x}{1-x}$

funksiya Teylor qatoriga yoyilsin.

Ma'lumki,

$$\ln\frac{1+x}{1-x}=\ln(1+x)-\ln(1-x)$$

bo`ladi.

Biz yuqorida

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-...+(-1)^{n-1}\frac{x^n}{n}+...$$

$$\ln(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-...-\frac{x^n}{n}-...$$

bo`lishini ko`rgan edik. Bu munosabatlardan foydalanib topamiz:

$$\ln(1+x)-\ln(1-x)=x-\frac{x^2}{2}+\frac{x^3}{3}-...+(-1)^{n-1}\frac{x^n}{n}+...-$$

$$-\left(-x-\frac{x^2}{2}-\frac{x^3}{3}-...-\frac{x^n}{n}-...\right)=2x+\frac{2x^3}{3}+\frac{2x^5}{5}+...+\frac{2x^{2n-1}}{2n-1}+...$$

Demak,

$$\ln\frac{1+x}{1-x}=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+...+\frac{x^{2n-1}}{2n-1}+...\right). \qquad (10)$$

(10) darajali qatorning yaqinlashish radiusi $r=1$ bo`lib, yaqinlashish to`plami $(-1,1)$ bo`ladi.

2-misol. Ushbu $f(x)=\int\limits_0^x\frac{\sin t}{t}dt$

funksiya Teylor qatoriga yoyilsin.

Ma'lumki,

$$\sin t=t-\frac{t^3}{3!}+\frac{t^5}{5!}-...+(-1)^{n-1}\frac{t^{2n-1}}{(2n-1)!}+....$$

Unda

$$\sin t=1-\frac{t^2}{3!}+\frac{t^4}{5!}-...+(-1)^{n-1}\frac{t^{2n-2}}{(2n-1)!}+...$$

bo`ladi. Bu darajali qatorni hadlab integrallab topqmiz:

$$\int_0^x\frac{\sin t}{t}dt=\int\left(1-\frac{t^2}{3!}+\frac{t^4}{5!}-...+(-1)^{n-1}\frac{t^{2n-2}}{(2n-1)!}+...\right)dt=$$

$$=x-\frac{x^3}{3!\cdot3}+\frac{x^5}{5!\cdot5}-...+(-1)^n\frac{x^{2n-1}}{(2n-1)!\cdot(2n-1)}+...$$

Keyingi darajali qatorning yaqinlashish radiusi $r=+\infty$ bo`ladi.

Berilgan funksiyalarni ko`rsatilgan nuqtada Teylor qatoriga yoying.

1. $y=\ln x \quad x_0=1$

2. $y=\sqrt{x^3} \quad x_0=1$

3. $y=\frac{1}{x} \quad x_0=3$

16. $y=e^{-x^2} \quad x_0=0$

17. $y=\begin{cases}\frac{e^x-1}{x}, & \text{bo'lsa } x\ne0\\ 1, & \text{bo'lsa } x=0\end{cases}$

18. $y=\sin\frac{x}{2} \quad x_0=0$

4. $y=\sin\frac{\pi x}{4}$ $x_0=2$

5. $y=chx$ $x_0=0$

6. $y=e^x\sin x$ $x_0=0$

7. $y=x^2e^x$ $x_0=0$

8. $y=\cos(x+\alpha)$ $x_0=0$

9. $y=\cos x chx$ $x_0=0$

10. $y=\ln(1+e^x)$ $x_0=0$

11. $y=\frac{1+x}{(1-x)^3}$ $x_0=3$

12. $y=e^{\cos x}$ $x_0=2$

13. $y=\cos^n x$ $x_0=1$

14. $y=-\ln\cos x$ $x_0=0$

15. $y=e^{2x}$ $x_0=0$

19. $y=\cos^2 x$ $x_0=0$

20. $y=(x-tgx)\cos x$ $x_0=0$

21. $y=\ln(10+x)$ $x_0=0$

22. $y=\sqrt{1+x^2}$ $x_0=0$

23. $y=\frac{1}{\sqrt[3]{1+x^3}}$ $x_0=0$

24. $y=x\ln(1+x)$ $x_0=0$

25. $y=\sqrt[3]{8-x^3}$ $x_0=0$

26. $y=\frac{x^2}{\sqrt{1-x^2}}$ $x_0=0$

27. $y=\begin{cases}\frac{e^{x^2}-e^{-x^2}}{x}, & \text{bo'lsa } x\neq 0\\ 1, & \text{bo'lsa } x=0\end{cases}$

28. $y=\begin{cases}\frac{\sin x}{x}, & \text{bo'lsa } x\neq 0\\ 1, & \text{bo'lsa } x=0\end{cases}$

29. $y=\ln\ln x$ $x_0=0$

30. $y=e^{\sin x}$ $x_0=2$

10.14. Fure qatori.

Fure qatori ta'rifi. Har bir hadi

$$u_n(x)=a_n\cos nx+b_n\sin nx \qquad (n=0,1,2,\ldots)$$

garmonikadan iborat ushbu

$$a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx) \qquad (1)$$

funksional qator trigonometrik qator deyiladi.Bunda

$$a_0,a_1,b_1,a_2,b_2,\ldots,a_n,b_n,\ldots$$

sonlar trigonometrik qatorning koeffisentlari deyiladi.

Odatda, (1) trigonometrik qatorning qismiy yig`indisi

$$T_n(x)=a_0+\sum_{k=1}^{n}(a_k\cos kx+b_k\sin kx)$$

trigonometrik ko`phad deyiladi.

$f(x)$ funksiya $[-\pi,\pi]$ da berilgan bo`lib, u shu oraliqda integrallanuvchi bo`lsin. Ravshanki,

$$f(x)\cos nx \quad , \quad f(x)\sin nx \qquad (n=1,2,3,\ldots)$$

funksiyalar ham integrallanuvchi bo`ladi.Yuqorida keltirilgan funksiyalarning integrallarini quydagicha belgilaymiz:

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx,$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx, \qquad (n=1,2,\ldots) \qquad (2)$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx. \qquad (n=1,2,\ldots)$$

So`ng ushbu

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx) \qquad (3)$$

trigonometrik qatorni tuzamiz.

Ravshanki,(3) trigonometrik qator (2) munosabatlardan topiladigan

$$a_0,a_1,b_1,a_2,b_2,\ldots,a_n,b_n,\ldots$$

sonlar bilan to`la aniqlanadi.

1-ta'rif. Koeffisentlari (2) munosabatlar bilan aniqlangan (3) trigonometrik qator $f(x)$ funksiyaning Fure qatori deyiadi Bunda

$$a_0,a_1,b_1,a_2,b_2,\ldots,a_n,b_n,\ldots$$

sonlar $f(x)$ funksiyaning Fure koeffisentlari deyiladi.

Demak, $f(x)$ funksiyaning Fure qatori shunday trigonometrik qatorki, uning koeffisentlari (5) formulalar yordamida aniqlanadi.Shuni e'tiborga olib, $f(x)$ funksiyaning Fure qatori quydagicha yoziladi:

$$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx).$$

1-misol. Ushbu $f(x)=e^{\alpha x}$ $\quad(-\pi\le x\le\pi,\alpha\neq 0)$

funksiyaning Fure qatori topilsin.

(2) formulalardan foydalanib,berilgan funksiyaning Fure koeffisentlarini hisoblaymiz:

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}e^{\alpha x}dx=\frac{1}{\alpha\pi}\left(e^{\alpha\pi}-e^{-\alpha\pi}\right)=\frac{2}{\alpha\pi}sh\alpha\pi,$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}e^{\alpha x}\cos nxdx=\frac{1}{\pi}\frac{\alpha\cos nx+n\sin nx}{\alpha^2+n^2}e^{\alpha x}\Bigg|_{-\pi}^{\pi}=$$

$$=(-1)^n\frac{1}{\pi}\cdot\frac{2\alpha}{\alpha^2+n^2}sh\alpha\pi \qquad (n=1,2,\ldots),$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}e^{\alpha x}\sin nxdx=\frac{1}{\pi}\frac{\alpha\sin nx-n\cos nx}{\alpha^2+n^2}e^{\alpha x}\Bigg|_{-\pi}^{\pi}=$$

$$=(-1)^{n-1}\frac{1}{\pi}\cdot\frac{2n}{\alpha^2+n^2}sh\alpha\pi \qquad (n=1,2,\ldots).$$

Demak,

$$f(x)=e^{\alpha x}$$

funksiyaning Fure qatori

$$f(x)=e^{\alpha x}\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=$$

$$=\frac{2sh\alpha\pi}{\pi}\left[\frac{1}{2\alpha}+\sum_{n=1}^{\infty}\frac{(-1)^n}{\alpha^2+n^2}(\alpha\cos nx-n\sin nx)\right]$$

bo`ladi.

Aytaylik,ushbu shartlar bajarilsin:

1) quydagi

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx) \tag{4}$$

trigonometrik qator $[-\pi,\pi]$ da yaqinlashuvchi va uning yig`indisi $f(x)$ ga teng:

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx), \tag{5}$$

2) (5) ni hamda uni $\cos kx$ va $\sin kx$ larga $(k=0,1,2,\ldots)$ ko`paytirishdan hosil bo`lgan

$$f(x)\cos kx=\frac{a_0}{2}\cos kx+\sum_{n=1}^{\infty}(a_n\cos nx\cos kx+b_n\sin nx\cos kx),$$

$$f(x)\sin kx=\frac{a_0}{2}\sin kx+\sum_{n=1}^{\infty}(a_n\cos nx\sin kx+b_n\sin nx\sin kx),$$

qatorlar $[-\pi,\pi]$ da hadlab integrallansin.U holda

$$a_0,a_1,b_1,a_2,b_2,\ldots,a_n,b_n,\ldots$$

sonlar $f(x)$ funksiyaning Fure koeffisentlari bo`ladi,(4) trigonometrik qator esa $f(x)$ funksiyaning Fure qatori bo`ladi.

Bu tasdiqning isboti quydagi

$$\int_{-\pi}^{\pi}f(x)dx \quad , \quad \int_{-\pi}^{\pi}f(x)\cos kxdx \quad , \quad \int_{-\pi}^{\pi}f(x)\sin kxdx$$

integrallarni hisoblashdan kelib chiqadi.

4[0]. Juft va toq funksiyalarning Fure qatori. Faraz qilaylik, $f(x)$ funksiya $[-\pi,\pi]$ da berilgan juft funksiya bo`lib,u shu oraliqda integrallanuvchi bo`lsin.Bu funksiyaning Fure koeffisentlarini topamiz:

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=\frac{1}{\pi}\left[\int_{-\pi}^{0}f(x)\cos nxdx+\int_{0}^{\pi}f(x)\cos nxdx\right]=$$

$$=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nxdx \qquad (n=0,1,2,\ldots),$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx=\frac{1}{\pi}\left[\int_{-\pi}^{0}f(x)\sin nxdx+\int_{0}^{\pi}f(x)\sin nxdx\right]=$$

$$=\frac{1}{\pi}\left[-\int_{0}^{\pi}f(x)\sin nxdx+\int_{0}^{\pi}f(x)\sin nxdx\right]=0 \qquad (n=1,2,\ldots).$$

Demak,juft $f(x)$ funksiyaning Fure koeffisentlari

$$a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nxdx \qquad (n=0,1,2,\ldots)$$

$$b_n=0 \qquad (n=1,2,\ldots)$$

bo`lib,Fure qatori

$$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$$

bo`ladi.

Aytaylik, $f(x)$ funksiya $[-\pi,\pi]$ da berilgan toq funksiya bo`lib,u shu oraliqda integrallanuvchi bo`lsin.Bu funksiyaning Fure koeffisentlarini topamiz:

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx=\frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nxdx+\int_{0}^{\pi} f(x)\cos nxdx\right]=$$

$$=\frac{1}{\pi}\left[-\int_{0}^{\pi} f(x)\cos nxdx-f(x)\cos nxdx\right]=0 \qquad (n=0,1,2,\ldots),$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx=\frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nxdx+\int_{0}^{\pi} f(x)\sin nxdx\right]=$$

$$=\frac{2}{\pi}\left[\int_{0}^{\pi} f(x)\sin nxdx\right] \qquad (n=1,2,\ldots).$$

Demak,toq $f(x)$ funksiyaning Fure koeffisentlari

$$a_n=0\ , \quad (n=0,1,2,\ldots),$$

$$b_n=\frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nxdx\ , \qquad (n=1,2,\ldots)$$

bo`lib,Fure qatori

$$f(x)\sim\sum_{n=1}^{\infty} b_n\sin nx$$

bo`ladi

2-misol. Ushbu $f(x)=x^2 \quad (-\pi\le x\le\pi)$ juft funksiyaning Fure qatori topilsin.

Avvalo berilgan funksiyaning Fure koeffisisentlarini topamiz:

$$a_0=\frac{2}{\pi}\int_{0}^{\pi} x^2dx=\frac{2}{3}\pi^2 \quad ,$$

$$a_n=\frac{2}{\pi}\int_{0}^{\pi} x^2\cos nxdx=\frac{2}{\pi}x^2\frac{\sin nx}{n}\Big|_0^{\pi}-\frac{4}{n\pi}\int_{0}^{\pi} x\sin nxdx=$$

$$=\frac{4}{\pi n}\left(\frac{x\cos nx}{n}\Big|_0^{\pi}-\frac{1}{n}\int_{0}^{\pi}\cos nxdx\right)=(-1)^n\cdot\frac{4}{n^2}. \qquad (n=1,2,\ldots)$$

Demak, $f(x)=x^2$ funksiyaning Fure qatori

$$f(x)=x^2\sim\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}$$

bo`ladi

3-misol. Ushbu $f(x)=x \qquad (-\pi\le x\le\pi)$

toq funksiyaning Fure qatori topilsin.

Berilgan funksiyaning Fure koeffisientlarini hisoblaymiz:

$$b_n=\frac{2}{\pi}\int_{0}^{\pi} x\sin nxdx=\frac{2}{\pi}\left(-\frac{x\cos nx}{n}\Big|_0^{\pi}+\frac{1}{n}\int_{0}^{\pi}\cos nxdx\right)=\frac{2(-1)^{n-1}}{n}.$$

Demak, $f(x)=x$ funksiyaning Fure qatori

$$f(x)\sim\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2}{n}\sin nx$$

bo`ladi

5^0. $[-l,l]$ oraliqdagi berilgan funksiyaning Fure qatori. Faraz qilaylik, $f(x)$ funksiya $[-l,l]$ oraliqda $(l>0)$ berilgan bo`lib,u shu oraliqda integrallanuvchi bo`lsin.

Ravshanki ,ushbu

$$t=\frac{\pi}{l}x$$

almashtirish natijasida $[-l,l]$ oraliq $[-\pi,\pi]$ oraliqqa o`tadi. Agar

$$f(x)=f\left(\frac{1}{\pi}t\right)=\varphi(t).$$

deyilsa, $\varphi(t)$ funksiya $[-\pi,\pi]$ oraliqda berilgan va shu oraliqda integrallanuvchi funksiya bo`ladi. Uning Fure qatori

$$\varphi(t)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin nt)$$

bo`lib,

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\varphi(t)\cos ntdt \quad , \qquad (n=0,1,2,\ldots)$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\varphi(t)\sin ntdt \qquad (n=1,2,\ldots)$$

bo`ladi, Endi

$$t=\frac{\pi}{l}x$$

bo`lishini e'tiborga olib topamiz:

$$\varphi\left(\frac{\pi}{l}x\right)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos n\frac{\pi}{l}x+b_n\sin n\frac{\pi}{l}x\right),$$

$$a_n=\frac{1}{l}\int_{-l}^{l}\varphi\left(\frac{\pi}{l}x\right)\cos n\frac{\pi}{l}xdx \quad , \ (n=0,1,2\ldots)$$

$$b_n=\frac{1}{l}\int_{-l}^{l}\varphi\left(\frac{\pi}{l}x\right)\sin n\frac{\pi}{l}xdx \quad . \ (n=1,2\ldots)$$

Natijada berilgan $f(x)$ funksiyaning Fure qatori quydagicha

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

bo`lib, bunda

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx \qquad (n = 0,1,2\ldots)$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi}{l}xdx \qquad (n = 1,2\ldots)$$

bo`ladi.

4-misol: Ushbu $f(x) = e^x \quad (-1 \le x \le 1)$ funksiyaning Fure qatori topilsin.

Yuqoridagi formulalardan foydalanib, $f(x) = e^x$ funksiyaning Fure koeffisentilarini topamiz:

$$a_0 = \int_{-1}^{1} e^x dx = e - e^{-1},$$

$$a_n = \int_{-1}^{1} e^x \cos n\pi x dx = \frac{n\pi \sin n\pi x - \cos n\pi x}{1 + n^2\pi^2} e^x \Big|_{-1}^{1} =$$

$$= \frac{1}{1 + n^2\pi^2}\left(e\cos n\pi - e^{-1}\cos n\pi\right) = (-1)^n \frac{e - e^{-1}}{1 + n^2\pi^2} \qquad (n = 1,2,\ldots),$$

$$b_n = \int_{-1}^{1} e^x \cos n\pi x dx = \frac{\sin n\pi x - n\pi \cos n\pi x}{1 + n^2\pi^2} e^x \Big|_{-1}^{1} =$$

$$= \frac{1}{1 + n^2\pi^2}\left(en\pi\cos n\pi + n\pi e^{-1}\cos n\pi\right) = \frac{n\pi(-1)^n}{1 + n^2\pi^2}\left(e^{-1} - e\right) =$$

$$= (-1)^{n+1}\frac{e - e^{-1}}{1 + n^2\pi^2} \qquad (n = 1,2,\ldots).$$

Demak,

$$f(x) = e^x \qquad (-1 \le x \le 1)$$

funksiyaning Fure qatori

$$e^x \sim \frac{e - e^{-1}}{2} + \left(e - e^{-1}\right)\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{1 + n^2\pi^2}\cos n\pi + \frac{(-1)^{n+1}}{1 + n^2\pi^2} n\pi \sin n\pi x\right]$$

bo`ladi.

Berilgan funksiyalarni Fure qatoriga yoying:

1. $f(x) = \mathrm{sign}(\cos x)$
2. $f(x) = \arcsin(\sin x)$
3. $f(x) = \arcsin(\cos x)$
4. $f(x) = x - [x]$
5. $f(x) = |\cos x|$
6. $f(x) = |\sin x|$
7. $f(x) = \left|\cos\left(\frac{x}{2}\right)\right|$
8. $f(x) = e^{ax},\ 0 < x < \pi$
9. $f(x) = x\sin x,\ 0 \le x \le \pi$
10. $f(x) = e^x,\ (0;\ln 2)$
11. $f(x) = \sin ax,\ 0 \le x \le \pi$
12. $f(x) = x + \mathrm{sign}x,\ (-\pi;\pi)$
13. $f(x) = \pi^2 - x^2,\ (-\pi;\pi)$
14. $f(x) = x^3,\ (-\pi;\pi)$
15. $f(x) = e^{2|x|},\ (-\pi;\pi)$
16. $f(x) = |x|,\ [-1;1]$
17. $f(x) = \ln\left|\sin\frac{x}{2}\right|$
18. $f(x) = \ln\left|\cos\frac{x}{2}\right|$
19. $f(x) = \ln\left|tg\frac{x}{2}\right|$
20. $f(x) = \frac{q\sin x}{1 - 2q\cos x + q^2},\ (|q| < 1)$
21. $f(x) = \sec x,\ -\frac{\pi}{4} < x < \frac{\pi}{4}$
22. $f(x) = \sum_{n=0}^{\infty}\alpha^n\frac{\sin nx}{\sin x},\ (\alpha < 1)$
23. $f(x) = x,\ -\pi \le x \le \pi,\ x_0 = \pi$
24. $f(x) = \begin{cases}1, & 0 \le x \le \pi \\ 0, & -\pi \le x < 0\end{cases} \quad x_0 = 0$
25. $f(x) = |x|,\ -\pi \le x \le \pi,\ x_0 = \pi$
26. $f(x) = \pi + x,\ -\pi \le x \le \pi,\ x_0 = \pi$
27. $f(x) = \begin{cases}-2x & -\pi < x \le 0 \\ 3x & 0 < x \le \pi\end{cases} \quad x_0 = \pi$
28. $f(x) = \begin{cases}A, & 0 < x < l \\ \frac{A}{2}, & x = l \\ 0, & l < x < 2l\end{cases} \quad (0;2l)$
29. $f(x) = \begin{cases}ax, & -\pi < x < 0 \\ bx, & 0 \le x < \pi\end{cases} \quad (-\pi;\pi)$
30. $f(x) = \begin{cases}a, & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ b, & \frac{\pi}{2} \le x < \frac{3\pi}{2}\end{cases} \quad \left(-\frac{\pi}{2};\frac{3\pi}{2}\right)$

FOYDALANILGAN ADABIYOTLAR RO'YXATI:

1. T.Azlarov, H.Mansurov Matematik analiz Toshkent "O'qituvchi" 1994.
2. A.Gaziyev, I.Isroilov, M.Yaxshiboyev Matematik analizdan misol masalalar Toshkent-2021.
3. A.Sa'dullayev,H.Mansurov,G.Xudoyberganov,A.Vorisov,R.G'ulomov. Matematik analiz kursidan misol va masalalar to'plami.Toshkent "O'zbekiston" 1993.
4. B.A.Shoimqulov,T.T.Tuychiyev,D.H.Djumaboyev Matematik analizdan mustaqil ishlar "O`zbekiston faylasuflari milliy jamiyati",Toshkent-2008.
5. Б.П.Демидович.Сборник задач и упражнений по математическому анализу:Москва "Наука" главная редакция физико-математической литературы 1990.
6. Г.Н.Берман.Сборник задач по курсу математического анализа:Москва "Наука" главная редакция физико-математической литературы 1985.
7. Кудрявцев.Л.Д., Кутасов.А.Д., Чехлов.В.И., Шабунин.М.И., Сборник задач по математическому анализу.Том 2. Интегралы.ФИЗМАТЛИТ,2003.
8. Sh.R.Xurramov. Oliy matematika (maslalar to`plami,nazorat topshiriqlari). Oliy ta'lim muassasalari uchun o`quv qo`llanma. 1-qism. –T.: "Fan va texnalogiya",2015.
9. A.Sa'dullayev,H.Mansurov,G.Xudoyberganov,A.Vorisov,R.G'ulomov. Matematik analiz kursidan misol va masalalar to'plami.Toshkent "O'zbekiston" 1993.
10. M.Xushvaqtov Matematik analiz Toshkent:"Yangiyul poligraph service",2008.
11. Sh.O.Alimov,R.R.Ashurov Matematik analiz Toshkent "MUNTOZ SO`Z",2018.

F.S.Aktamov, E.M.Mahkamov, G.B.Quzmanova

MATEMATIK ANALIZ

O'QUV QO'LLANMA
I QISM

Muharrir:	X. Taxirov
Tehnik muharrir:	S. Melikuziva
Musahhih:	M. Yunusova
Sahifalovchi:	A.Ziyamuhamedov

Nashriyot litsenziya № 2044, 25.08.2020 й

Bichimi 60x84 1/16. "Times new roman" garniturasi, kegli 16.

Offset bosma usulida bosildi. Shartli bosma tabog'i 22,8. Adadi 100 dona.

Buyurtma № 1668234

Yangi chirchiq prints MCHJda chop etildi.

9789910995019