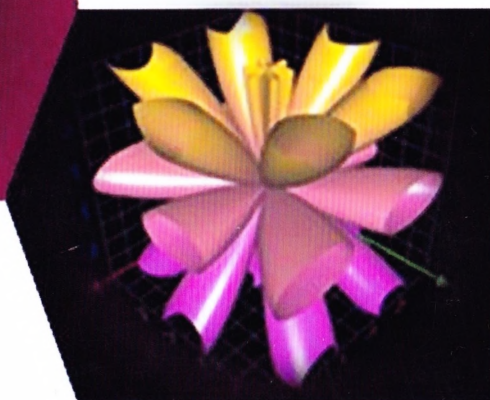
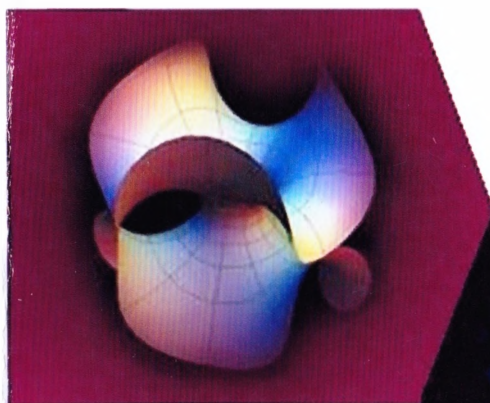


A. Gaziyeu, M.U. Yaxshiboyev

MATEMATIK ANALIZ



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SAMARQAND DAVLAT UNIVERSITETI

A. Gaziyeu, M.U. Yaxshiboyev

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Mazkur darslik "Matematik analiz" fanining ko'p o'zgaruvchi funksiyalar, funksional ketma-ketliklar va qatorlar, parametrga bog'liq xos va xosmas integrallar, karrali integrallar, egri chiziqli integrallar, sirt integrallari va maydonlar nazariyasi elementlari hamda Furiy qatorlari kabi bo'limlari bo'yicha materiallarni qamrab olgan.

Darslikda keltirilayotgan mavzularning avvalo nazariyasini sodda ravon tilda bayon qilishga, so'ngra berilgan tushunchalar, tasdiqlar va teoremlarni mustahkamlash uchun amaliy, nazariy, mexanik, fizika xarakterdagi misollarni yechib ko'rsatishga ko'proq e'tibor qaratigan.

Darslik bakalavriyatning "matematika", "mexanika", "fizika" yo'nalishi talabalari uchun mo'ljallangan bo'lib, undan bakalavriyatning "amaliy matematika va informatika" hamda "oliy matematika" chuqur o'rganiladigan texnika oliy o'quv yurtlarida ta'lim oluvchi talabalar hamda o'qituvchilar ham foydalanishlari ko'zda tutilgan.

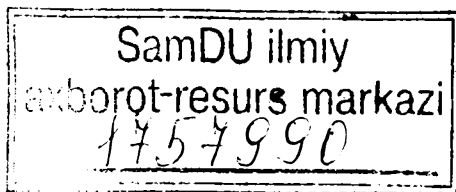
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«Matematik analiz» fanining o'quv dasturiga kirgan ko'p o'zgaruvchili funksiyalar, funksional ketma-ketliklar va qatorlar parametrga bog'liq xos va xosmas integrallar, ikki va uch karra integrallar, birinchi va ikkinchi tur egri chiziqli integrallar, birinchi va ikkinchi tur sirt integrallari, Stoks va Ostroradskiy formulalari, maydonlar nazariyasi elementlari hamda Furyc qatorlari bo'limlarini qisqa va soddabayon qilish imkonini beradi.

Darslikda keltirilayotgan mavzularning avvalo nazariyasini soddaravon tilda bayon qilishga, so'ngra berilgan tushunchalar, tasdiqlar va teoremlarni mustahkamlash uchun amaliy, nazariy, mexanik, fizik xarakterdagi misollarni yechib ko'rsatishga ko'proq e'tibor qaratilgan.

Mualliflar o'zining ko'p yillik mehnati davomida Samarqand Davlat universitetining mexanika-matematika, fizika fakultetlarida matematik analiz fanidan o'qigan ma'ruzalari, olib borgan amaliy mashg'ulotlar davrida orttirgan tajribasi hamda shu davrlarda talabalarning davlat tilida yozilgan o'quv adabiyotlariga (jumladan, matematik analiz fani bo'yicha) bo'lgan ehtiyojini sezgan holda, uni imkon darajasida qondirish maqsadida mualliflar ushbu darslikni yozishga jazm qildi.

Darslik 5130100 – matematika yo'nalishida bakalavr tayyorlash o'quv rejasiga moslashtirib yozilgan bo'lsada, undan universitetlarning 5140200-fizika, 5130200- amaliy matematika va informatika, 5140300-mexanika va "Oliy matematika" chuqur o'rganiladigan texnika oliy o'quv yurtlarida ta'lim oluvchi talabalar uchun ham mo'ljallangan.

Kitobda matematik belgilardan keng foydalanish bilan birga tasdiqlar, teoremlar isbotining tugaganligi esa, ■ belgi orqali belgilangan. Ushbu kitobning boshidan oxirigacha mavzular, ta'riflar, teoremlar kerakli formulalar bandlarga ajratilgan va ular tartib bilan nomerlangan.

Kitobning qo'l yozmasini sinchiklab o'qib chiqib, uning ilmiy va uslubiy jihatdan yaxshilanishiga o'z hissalarini qo'shgan profesorlar A.Jalilov, A.Xasanov va hamda dotsent A. Arziqulovlarga mualliflar o'z minnodorchiligini bildiradi.

Kitobxonlardan ushbu darslik haqidagi fikr-mulohazalarini mamnuniyat bilan qabul qilamiz.

42-§. R^m fazo va uning muhim to'plamlari. R^m fazoda ketma-ketlik va uning limiti

10.1. Ko'p o'zgaruvchi miqdorlar orasidagi funksional bog'lanish. Tabiatdagi ko'pgina masalalarni yechish jarayonida o'zgaruvchi miqdorlar orasida, bir o'zgaruvchi miqdorning qiymati boshqalarning qiymatlari orqali aniqlanadigan bog'lanishlar uchraydi.

Uning biror xarakteristikalarini o'rganayotganligimizda jismning bir nuqtasidan boshqa nuqtasiga o'tganda bu xarakteristikalarining o'zgarishini hisobga olish kerak bo'ladi. Masalan, jismning bir nuqtasidagi zichligi ρ yoki temperaturasi T , ikkinchi nuqtasiga o'tganda, zichlikning yoki temperaturaning o'zgarishini hisobga olish lozim bo'ladi. Jismning har bir nuqtasi x, y, z dekart koordinatalari bilan aniqlanadi uchun qaralayotgan xarakteristikalar (zichlik va temperatura) bu o'zgaruvchi x, y, z larning qiymatlari bilan aniqlanadi. O'zgaruvchi har qanday jarayonni qarayotganda fizik xarakteristikalar qiymati to'rtta o'zgaruvchi: nuqtaning koordinatalari x, y, z va t - vaqtning qiymati bilan aniqlanadi. Masalan, gazda tovush tebranishlarining tarqalishini o'rganayotganda, gazning zichligi ρ va bosimi p to'rtta o'zgaruvchi x, y, z va t ning qiymati bilan aniqlanadi. Tabiatdagi shunga o'xshash bog'lanishlarni o'rganish uchun ko'p o'zgaruvchili funksiya tushunchasini kiritish va ularning xususiyatlarini matematik nutai nazardan o'rganish zaruriyati to'g'riladi.

10.2. m o'lchovli Evklid fazosi. m ta $x_1, x_2, \dots, x_m$ haqiqiy sonlarning barcha mumkin bo'lgan tartiblangan $(x_1, x_2, \dots, x_m)$ nuqtalar to'plamiga m -o'lchovli *koordinat fazosi* deyiladi va u

$$R^m = \{ (x_1, x_2, \dots, x_m) : x_k \in R, k = 1, 2, \dots, m \}$$

tabii belgilanadi. R^m koordinat fazosining elementini qisqacha bitta M harfi bilan belgilaymiz. $x_1, x_2, \dots, x_m$ sonlar M nuqtaning *koordinatalari*

deyiladi. R^m koordinat fazosining ixtiyoriy ikki $M'(x'_1, x'_2, \dots, x'_m)$ va $M''(x''_1, x''_2, \dots, x''_m)$ nuqtalari orasidagi $\rho(M', M'')$ masofa ushbu

$$\rho(M', M'') = \sqrt{(x'_1 - x''_1)^2 + (x'_2 - x''_2)^2 + \dots + (x'_m - x''_m)^2} \quad (10.2.1)$$

formula bilan aniqlangan bo'lsa, u holda R^m ga m -o'lchovli evklid fazo deyiladi.

(10.2.1) masofa $\rho(M', M'')$ quyidagi xossalarga ega:

1^o. $\rho(M', M'') \geq 0 \quad \forall M', M'' \in R^m; \quad \rho(M', M'') = 0 \Leftrightarrow M' = M''$

2^o. $\rho(M', M'') = \rho(M'', M') \quad \forall M', M'' \in R^m;$

3^o. $\rho(M', M''') \leq \rho(M', M'') + \rho(M'', M''') \quad \forall M', M'', M''' \in R^m.$

Bu xossalarni isbotsiz keltiramiz.

10.2.2. Eslatma. Agar R^m koordinat fazosini koordinatalar $(x_1, x_2, \dots, x_m)$ bo'lgan x vektorlar to'plamidan iborat bo'lsin de qarasa, unda $x = (x_1, x_2, \dots, x_m)$, $y = (y_1, y_2, \dots, y_m)$ vektorlar yig'indisi ayirmasi

$$x \pm y = (x_1 \pm y_1, x_2 \pm y_2, \dots, x_m \pm y_m), \quad \forall x, y \in R^m,$$

vektorni o'zgarmas songa ko'paytmasi esa, $\lambda x = (\lambda x_1, \lambda x_2, \dots, \lambda x_m)$, $\lambda \in R$ ko'rinishda aniqlangan bo'lsa, u holda R^m ni *chiziqli (vektorli) fazo* deb ataladi.

R^m fazoda nol element sifatida $0 = (0, 0, \dots, 0)$ vektori olinadi.

x vektorning moduli: $|x| = \sqrt{x_1^2 + x_2^2 + \dots + x_m^2}$, u holda $\rho(x, y) = |x - y|$.

10.3. R^m - evklid fazosidagi nuqtalar to'plami. $M_0 \in R^m$ $M_0(x_1^0, x_2^0, \dots, x_m^0)$, r - biror musbat son berilgan bo'lsin.

1. $\{M : \rho(M, M_0) \leq r\}$ - markazi M_0 nuqtada, radiusi r bo'lgan m -o'lchovli shar deyiladi.

2. $\{M : \rho(M, M_0) < r\}$ - markazi M_0 nuqtada, radiusi r bo'lgan *ochiq sha* deyiladi.

3. $\{M : \rho(M, M_0) = r\}$ - markazi M_0 nuqtada, radiusi r bo'lgan *sfer* deyiladi.

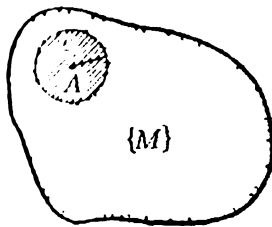
$\{M \mid x_1 = x_1^0, |x_2 - x_2^0| \leq d_2, \dots, |x_m - x_m^0| \leq d_m\}$, ($d_1, d_2, \dots, d_m$ - lar o'ltiribsonlar) *m-o'ltiribsonli parallelepiped* deyiladi. $m=2$ da bu plamga *K* - evklid fazosidagi to'rtburchak deyiladi.

$\{M \mid x_1 = x_1^0, |x_2 - x_2^0| < d_2, \dots, |x_m - x_m^0| < d_m\}$, ($d_1, d_2, \dots, d_m$ - lar o'ltiribsonlar) *m-o'ltiribsonli ochiq parallelepiped* deyiladi.

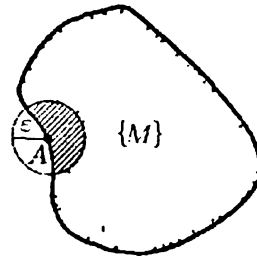
$\{M\} \subset R^m$, $M_0 \in R^m$ bo'lsin. Markazi M_0 nuqtada, radiusi ε ga teng ochiq to'rtburchak sharga, M_0 nuqtaning ε atrofi deyiladi.

$\{M\} \subset R^m$, $A \in \{M\}$ bo'lsin.

10.1.1. Ta'rif. Agar A nuqtaning shunday $\exists U_\varepsilon(A)$ bo'lib, $U_\varepsilon(A) \subset \{M\}$ bo'lsa, u holda A nuqta $\{M\}$ to'rtlamning *ichki nuqtasi* deyiladi (10.1-chizma).



10.1-chizma.



10.2-chizma.

10.3.2. Ta'rif. Agarda A nuqtaning ixtiyoriy ε atrofi $U_\varepsilon(A)$ da $\{M\}$ to'rtlamga qarashli va qarashli bo'lmagan nuqtalar mavjud bo'lsa, u holda A nuqta $\{M\}$ to'rtlamning *chegaraviy nuqtasi* deyiladi (10.2-chizma). Takidlaymizki, chegara nuqtalari to'rtlamga qarashli va qarashli bo'lmashligi ham mumkin.

10.3.3. Ta'rif. Agar $\{M\}$ to'rtlamning hamma nuqtalari uning ichki nuqtalari bo'lsa, u holda $\{M\}$ to'rtlamga *ochiq to'rtlam* deyiladi.

10.3.4. Ta'rif. Agar $\{M\}$ to'rtlam, o'zining hamma chegara nuqtalarini o'zida saqlasa, u holda $\{M\}$ to'rtlamga *yopiq to'rtlam* deyiladi.

10.3.6. Ta'rif. Agar $\{M\}$ to'plamning hamma nuqtalari biror sharni ichida joylashsa, u holda $\{M\}$ to'plam chegaralangan deyiladi.

(bunda $\varphi_i(t), i=1,2,\dots,m$, funksiyalar $[\alpha, \beta]$ da uzluksiz) to'plamga fazodagi uzluksiz chiziq deyiladi. $A(\varphi_1(\alpha), \varphi_2(\alpha), \dots, \varphi_m(\alpha))$ $B(\varphi_1(\beta), \varphi_2(\beta), \dots, \varphi_m(\beta))$ nuqtalar L chiziqning uchlari deyiladi.

10.3.7. Ta'rif. Agar $\{M\} \subset R^n$ to'plamning ixtiyoriy ikki nuqtasin shu to'plamda butunlay yotuvchi uzluksiz chiziq bilan tutashtiris mumkin bo'lsa, u holda $\{M\}$ to'plamga *bog'lamli to'plam* deyiladi.

10.4. R^m fazoda sonli ketma-ktlik va uning limiti. Natural sonlar to'plami N va R^m evklid fazosi berilgan bo'lsin. Har bir $n (n \in N)$ ga R^m ning biror muayyan $M_n = M_n(x_1^{(n)}, x_2^{(n)}, \dots, x_m^{(n)})$ nuqtasi mos qo'yilgan bo'lsin; ya'ni

$$\begin{aligned} 1 &\rightarrow M_1(x_1^{(1)}, x_2^{(1)}, \dots, x_m^{(1)}) , \\ 2 &\rightarrow M_2(x_1^{(2)}, x_2^{(2)}, \dots, x_m^{(2)}) , \\ &\vdots \\ n &\rightarrow M_n(x_1^{(n)}, x_2^{(n)}, \dots, x_m^{(n)}) , \end{aligned}$$

natijada hosil bo'lgan $M_1, M_2, \dots, M_n, \dots$ tartiblangan nuqtalaar majmuasi R^m da ketma-ketlik deb ataladi va u qisqacha $\{M_n\} (M_n \subset R^m)$ bilan belgilanadi.

$$1) M_n = M_n\left(\frac{1}{n}, \frac{1}{n}\right): M_1(1,1), M_2\left(\frac{1}{2}, \frac{1}{2}\right), \dots, M_n\left(\frac{1}{n}, \frac{1}{n}\right), \dots$$

$$2) M_n = M_n\left(0, \frac{1}{n}\right): M_1(0,1), M_2\left(0, \frac{1}{2}\right), \dots, M_n\left(0, \frac{1}{n}\right), \dots$$

$$3) M_n = M_n\left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right): M_1(1,1,\dots,1), M_2\left(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}\right), \dots, M_n\left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right), \dots$$

bu ketma-ketliklarning birinchisi va ikkinchisi R^2 evklid fazosining nuqtalaridan, uchinchisi esa, R^m evklid fazosining nuqtalaridan tashkil topgan ketma-ketliklardir. R^m da

$$M_1, M_2, \dots, M_n, \dots \quad (10.4.1)$$

nuqtalar ketma-ketligi va $A = A(a_1, a_2, \dots, a_n) \in R^m$ nuqta berilgan bo'lsin.

10.4.2. Ta'rif. Agar $\forall \varepsilon > 0$ son olinganda ham $\exists n_0 \in \mathbb{N}$ topilsaki, barcha $n > n_0(\varepsilon)$ lar uchun

$$\rho(M_n, A) < \varepsilon$$

qarashlik, ya'ni $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n > n_0(\varepsilon): \rho(M_n, A) < \varepsilon$ bajarilsa, A nuqta $\{M_n\} (M_n \subset R^m)$ ketma-ketlikning limiti deyiladi va $\lim_{n \rightarrow \infty} M_n = A$ yoki $n \rightarrow \infty$ da $M_n \rightarrow A$ kabi belgilanadi.

Agar (10.4.1) ketma-ketlik chekli limitga ega bo'lsa, u yaqinlashuvchi ketma-ketlik deb ataladi. Limit ta'rifidagi shartni qanoatlantiruvchi A mavjud bo'lmasa, $\{M_n\}$ ketma-ketlik *uzoqlashuvchi* deyiladi.

$\forall n > n_0$ da $\rho(M_n, A) < \varepsilon$ tengsizlikning bajarilishi, (10.4.1) ketma-ketlikning n_0 dan katta nomerli barcha hadlari $U_\varepsilon(A)$ atrofiga tegishli bo'lishini anglatadi. Bu hol (10.4.1) ketma-ketlik limitini quyidagicha ta'riflash imkonini beradi.

10.4.3. Ta'rif. Agar $A \in R^m$ nuqtaning $\forall U_\varepsilon(A)$ atrofi olinganda ham, $\{M_n\}$ ketma-ketlikning biror hadidan keying barcha hadlari shu atrofga qarashli bo'lsa, A nuqta $\{M_n\}$ ketma-ketlikning *limiti* deyiladi.

10.4.4. Misol. R^m fazoda ushbu $\{M_n\} = \{M_n(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})\}$ ketma-ketlikning limiti $A = A(0, 0, \dots, 0)$ bo'lishi ekanini ko'rsating.

Yechilishi. $\forall \varepsilon > 0$ sonni olaylik. Shu songa ko'ra $n_0 = [\frac{\sqrt{m}}{\varepsilon}] + 1$ dolsak, natijada barcha $n > n_0(\varepsilon)$ uchun

$$\rho(M_n, A) = \frac{\sqrt{m}}{n} < \frac{\sqrt{m}}{n_0} = \frac{\sqrt{m}}{[\frac{\sqrt{m}}{\varepsilon}] + 1} < \varepsilon.$$

Demak, **10.4.2**-ta'rifga ko'ra,

$$\lim_{n \rightarrow \infty} M_n = \lim_{n \rightarrow \infty} (M_n(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})) = A(0, 0, \dots, 0)$$

bo'ladi.

10.4.5. Misol. Ushbu

$$\{M_n\} = \{M_n((-1)^{n+1}, (-1)^{n+1})\}$$

ketma-ketlikning limitga ega emasligini ko'rsating.

Yechilishi. Teskarisini faraz qiamiz, ya'ni berilgan ketma-ketli limitga ega va u $A = A(a_1, a_2)$ bo'lsin. $\varepsilon = 1$ bo'lsin deylik, bu songa ko'r $\exists n_0 \in \mathbb{N}$ topiladiki, barcha $n > n_0$ lar uchun

$$\rho((1, 1), (a_1, a_2)) < \varepsilon, \quad \rho((-1, -1), (a_1, a_2)) < \varepsilon,$$

tengsizliklar bajariladi. Bu esa,

$2\sqrt{2} = \rho((-1, -1), (1, 1)) \leq \rho((-1, -1), (a_1, a_2)) + \rho((a_1, a_2), (1, 1)) < \varepsilon + \varepsilon = 2\varepsilon = 2$ ziddiyatga olib keladi. Bunga sabab qaralayotgan ketma-ketliknin limitga ega deyilishidir. Shunday qilib, ketma-ketlik limitga ega emas.

$\{M_n\} \subset R^m$ va $A \in R^m$ berilgan bo'lsin.

10.4.6. Teorema. Agar $\{M_n\}$ ketma-ketlik R^m fazoda $A = A(a_1, a_2, \dots, a_m)$ limitga ega bo'lsa, yani $\lim_{n \rightarrow \infty} M_n = A$, u holda

$$\lim_{n \rightarrow \infty} x_1^{(n)} = a_1, \lim_{n \rightarrow \infty} x_2^{(n)} = a_2, \dots, \lim_{n \rightarrow \infty} x_m^{(n)} = a_m$$

bo'ladi va aksincha agar M_n nuqtaning koordinatalari $(x_1^{(n)}, x_2^{(n)}, \dots, x_m^{(n)})$ mos ravishda A nuqtaning koordinatalari $a_1, a_2, \dots, a_m$ ga yaqinlashsa, holda $\{M_n\}$ ketma-ketlik $A(a_1, a_2, \dots, a_m)$ nuqtaga yaqinlashadi.

Isbot. a) $\{M_n\}$ ketma-ketlik A nuqtaga yaqinlashuvchi bo'lsin. U holda $n \rightarrow \infty$ limitiga ega: $\forall \varepsilon > 0$ son olinganda ham shunday $\exists n_0 \in \mathbb{N}$, $\forall n > n_0$ bo'lib $\rho(M_n, A) < \varepsilon$, ya'ni

$$\sqrt{(x_1^{(n)} - a_1)^2 + (x_2^{(n)} - a_2)^2 + \dots + (x_m^{(n)} - a_m)^2} < \varepsilon \quad (10.4.7)$$

tingizlik o'rinli bo'ladi. Bundan $\forall n > n_0$ lar uchun

$$|x_1^{(n)} - a_1| < \varepsilon, |x_2^{(n)} - a_2| < \varepsilon, \dots, |x_m^{(n)} - a_m| < \varepsilon,$$

tingizliklar bajariladi, ya'ni M_n ning koordinatalari $(x_1^{(n)}, x_2^{(n)}, \dots, x_m^{(n)})$ mavishda A nuqtaning koordinatalari $a_1, a_2, \dots, a_m$ ga yaqinlashadi.

b) M_n ning koordinatalari $(x_1^{(n)}, x_2^{(n)}, \dots, x_m^{(n)})$ mos ravishda A nuqtaning koordinatalari $a_1, a_2, \dots, a_m$ ga yaqinlashsin. U holda $\forall \varepsilon$ son olinganda ham $n_1^0, n_2^0, \dots, n_m^0$ nomerlar mavjud bo'lib, $n > n_1^0, n > n_2^0, \dots, n > n_m^0$ lardan boshlab mavishda

$$|x_1^{(n)} - a_1| < \frac{\varepsilon}{\sqrt{m}}, |x_2^{(n)} - a_2| < \frac{\varepsilon}{\sqrt{m}}, \dots, |x_m^{(n)} - a_m| < \frac{\varepsilon}{\sqrt{m}}$$

tingizliklar bajariladi. Bundan, $n > n_0 = \max\{n_1^0, n_2^0, \dots, n_m^0\}$ dan boshlab (10.4.7) tengsizlikning bajarilishi kelib chiqadi, ya'ni $\forall n > n_0$ lar uchun $\rho(M_n, A) < \varepsilon$ tengsizlik o'rinli bo'ladi, ya'ni $\lim_{n \rightarrow \infty} M_n = A$

Shunday qilib,

$$\lim_{n \rightarrow \infty} M_n = A \Leftrightarrow \begin{cases} \lim_{n \rightarrow \infty} x_1^{(n)} = a_1, \\ \lim_{n \rightarrow \infty} x_2^{(n)} = a_2, \\ \dots\dots\dots \\ \lim_{n \rightarrow \infty} x_m^{(n)} = a_m \end{cases}$$

kelib chiqadi. Demak, R^m fazodagi yaqinlashish, koordinatlari bo'yicha yaqinlashish bo'lar ekan. Shunday qilib, bu teorema R^m fazodagi ketma-ketlikning limitini o'rganishni R dagi sonli ketma-ketlikning limitini o'rganishga keltirilar ekan. Ma'lumki, biz bundan avval ([9], 2-bob, 2.1-2.7 bandlarga q.) sonlar ketma-ketligi va limitini o'rgangan edik. Shuni e'tiborga olgan holda, biz quyida R^m fazoda ketma-ketliklar nazariyasi bayonida asosiy faktlarni keltirish va ularning ba'zilarining isbotini berish bilan chegaralanamiz.

$\{M_n\} \subset R^m$ berilgan bo'lsin.

1⁰. $\{M_n\}$ ketma-ketlik yaqinlashuvchi bo'lsa, uning limiti yagona bo'ladi.

Keyingi xossalari keltirishda avval $\{M_n\}$ ketma-ketlikning chegaralanganlik tushunchasini keltirib o'tamiz.

10.4.8. Ta'rif. Agar $\{M_n\}$ ketma-ketlik hadlari qiymatlaridan tuzilgan to'plam chegaralangan bo'lsa, u holda $\{M_n\}$ ketma-ketlikga *chegaralangan* ketma-ketlik deyiladi.

40.4.9. Teorema. R^m fazoda $\{M_n\}$ ketma-ketlikning chegaralangan bo'lishi uchun uning koordinatalari $\{x_1^{(n)}\}, \{x_2^{(n)}\}, \dots, \{x_m^{(n)}\}$ sonli ketma-ketliklarning har birining chegaralangan bo'lishi zarur va yetarli.

2⁰. $\{M_n\}$ ketma-ketlik yaqinlashuvchi bo'lsa, u holda u chegaralangan bo'ladi.

$A \in R^m, B \in R^m, A = A(a_1, a_2, \dots, a_m), B = B(b_1, b_2, \dots, b_m)$ berilgan bo'lsa R^m fazoning koordinatasi $(a_1 + b_1, a_2 + b_2, \dots, a_m + b_m)$ bo'lgan nuqtaga va B nuqtalarning yig'indisi deyiladi va $A + B$ kabi belgilanadi.

R^m fazoning koordinatasi $(\lambda a_1, \lambda a_2, \dots, \lambda a_m)$ bo'lgan nuqtaga λ haqiqiy son bilan $A \in R^m$ nuqta ko'paytmasi deyiladi va λA kabi belgilanadi.

R^m fazoda A va B nuqtalarning ayirmasi $A + (-1)B$ kabi aniqlanadi, $A - B$ kabi belgilanadi.

3⁰. Agar $\{M_n\}$ ketma-ketlik yaqinlashuvchi bo'lib, uning limiti bo'lsa, u holda $\{kM_n\}$ ($k \in R$) ketma-ketlik ham yaqinlashuvchi va

$$\lim_{n \rightarrow \infty} kM_n = k \lim_{n \rightarrow \infty} M_n = kA$$

bo'ladi.

4⁰. Agar $\{M_n\}$ va $\{N_n\}$ ketma-ketliklar yaqinlashuvchi bo'lib, ularning limitlari mos ravishda A va B bo'lsa, u holda $\{M_n \pm N_n\}$ ketma-ketlik ham yaqinlashuvchi bo'lib, uning limiti $A \pm B$ ga teng bo'ladi, ya'ni

$$\lim_{n \rightarrow \infty} (M_n \pm N_n) = \lim_{n \rightarrow \infty} M_n \pm \lim_{n \rightarrow \infty} N_n = A \pm B.$$

$\{M_n\} \subset R^m$ berilgan bo'lsin.

10.4.10. Ta'rif. Agar $\forall \varepsilon > 0$ son olinganda ham $\exists n_0 \in \mathbb{N}$ topilsaki, har $n > n_0$ va $p > n_0$ lar uchun

$$\rho(M_n, M_p) < \varepsilon$$

tekis o'rinni bo'lsa, $\{M_n\}$ ketma-ketlik R^m da *fundamental ketma-ketlik* ataladi.

10.4.11. Misol. $\{M_n(\frac{1}{n}, \frac{1}{n})\}$ ketma-ketlikning R^2 da fundamental ekanligini ko'rsating.

Yechilishi. Haqiqatdan ham, $\forall \varepsilon > 0$ ga ko'ra $n_0 = \left\lceil \frac{2\sqrt{2}}{\varepsilon} \right\rceil + 1$ deb olinsa,

$$\begin{aligned} \rho(M_p(\frac{1}{p}, \frac{1}{p}), M_n(\frac{1}{n}, \frac{1}{n})) &= \sqrt{(\frac{1}{p} - \frac{1}{n})^2 + (\frac{1}{p} - \frac{1}{n})^2} = \sqrt{2} \left| \frac{1}{p} - \frac{1}{n} \right| = \\ &= \sqrt{2} \left(\frac{1}{p} - \frac{1}{n} \right) \leq \sqrt{2} \left(\frac{1}{n_0} + \frac{1}{n_0} \right) \leq \frac{2\sqrt{2}}{2\sqrt{2}} \varepsilon = \varepsilon \end{aligned}$$

Demak, **10.4.3**-ta'rifga asosan, berilgan ketma-ketlik fundamental ketma-ketlik bo'ladi.

10.4.12. Misol. $\{M_n(x_n, 0) : x_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}\}$ ketma-ketlikning R^2 da fundamental emasligini ko'rsating.

Yechilishi. Haqiqatdan ham, $n > p$ deb olinsa,

$$\rho(M_p(x_p, 0), M_n(x_n, 0)) = \sqrt{(x_n - x_p)^2} = |x_n - x_p| = \frac{1}{p+1} + \frac{1}{p+2} + \dots + \frac{1}{n} > \frac{n-p}{n}$$

Hunda $n - 2p$ bo'lganda esa,

$$\rho(M_p(x_p, 0), M_n(x_n, 0)) > \frac{1}{2}$$

bo'ladi. Bundan berilgan ketma-ketlikning fundamental emasligi kelib chiqadi.

Ravshanki, agar $\{M_n\} (M_n \subset R^m)$ ketma-ketlik fundamental bo'lsa, u holda M_n ning koordinatalari $\{x_1^{(n)}\}, \{x_2^{(n)}\}, \dots, \{x_m^{(n)}\}$ ketma-ketliklarning har biri ham fundamental ketma-ketlik bo'ladi va aksincha, $\{x_1^{(n)}\}, \{x_2^{(n)}\}, \dots, \{x_m^{(n)}\}$ larining har biri fundamental bo'lsa, u holda $\{M_n\}$ ketma-ketlik ham fundamental bo'ladi, ya'ni quyidagi teorema o'rinni bo'ladi.

10.4.13. Teorema. R^m fazoda $\{M_n\} = \{M_n(x_1^{(n)}, x_2^{(n)}, \dots, x_m^{(n)})\}$ ketma-ketlikning fundamental bo'lishi uchun uning koordinatalaridan iborat $\{x_1^{(n)}\}, \{x_2^{(n)}\}, \dots, \{x_m^{(n)}\}$ ketma-ketlikning har birining fundamental bo'lishi zarur va yetarli.

10.4.14. Teorema (Koshi prinsipi). $\{M_n\} (\{M_n\} \subset R^m)$ ketma-ketlikning yaqinlashuvchi bo'lishi uchun uning fundamental bo'lishi zarur va yetarli.

10.5. Qisman ketma-ketliklar. Bolsano-Veyshstrass teorema $\{M_n\} \subset R^m$ berilgan bo'lsin. Bu ketma-ketlikning $n_1, n_2, \dots, n_k$, ($n_1 < n_2 < \dots < n_k < \dots; n_k \in \mathbb{N}, k = 1, 2, \dots$) nomerli hadlaridan tashkil topgan ushbu

$$M_{n_1}, M_{n_2}, \dots, M_{n_k}, \dots$$

ketma-ketlikga berilgan ketma-ketlikning *qisman ketma-ketligi* deyiladi va $\{M_{n_k}\}$ kabi belgilanadi.

Masalan, R^2 fazoda ushbu

$$(1, 1), \left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{n}, \frac{1}{n}\right), \dots, \left(\frac{1}{2n}, \frac{1}{2n}\right), \dots,$$

$$(1, 1), \left(\frac{1}{2^2}, \frac{1}{2^2}\right), \left(\frac{1}{3^2}, \frac{1}{3^2}\right), \dots, \left(\frac{1}{n^2}, \frac{1}{n^2}\right), \dots$$

ketma-ketliklar, quyidagi

$$(1, 1), \left(\frac{1}{2}, \frac{1}{2}\right), \dots, \left(\frac{1}{n}, \frac{1}{n}\right), \dots$$

ketma-ketlikning qisman ketma-ketliklari bo'ladi.

10.5.1. Teorema. Agar $\{M_n\}$ ketma-ketlik yaqinlashuvchi bo'lib, uning limiti $A (A \in R^m)$ bo'lsa, u holda bu ketma-ketlikning har qanda $\{M_{n_k}\}$ qisman ketma-ketligi ham yaqinlashuvchi bo'lib, uning limiti har A ga teng bo'ladi.

10.5.2. Eslatma. Ketma-ketlikning, qisman ketma-ketliklarining mavjud bo'lishidan berilgan ketma-ketlikning limiti mavjud bo'lishi haqida doim ham kelib chiqavermaydi. Masalan, $(1, 1), (-1, -1), \dots, ((-1)^{n+1}, (-1)^{n+1}), \dots$ ketma-ketlik berilgan bo'lsin. Bu ketma-ketlik limitga ega emas, lekin uning qisman ketma-ketliklari:

$$(1, 1), (1, 1), \dots; \quad (-1, -1), (-1, -1), \dots$$

joylarida $(1, 1)$ va $(-1, -1)$ limitlarga ega.

Shunday qilib, $\{M_n\}$ ketma-ketlik limitga ega bo'lmasa ham, uning har bir ketma-ketliklari limitga ega bo'lishi mumkin ekan.

10.5.3. Teorema (Bolsano-Veyshstrass). Har qanday chegaralangan ketma-ketlikdan yaqinlashuvchi qisman ketma-ketliklar ajratish mumkin.

11 §. Ko'p o'zgaruvchili funksiya va uning limiti tushunchalari

10.6. Ko'p o'zgaruvchili funksiya tushunchasi. $\{M\} \subset R^m$ to'plam berilgan bo'lsin.

10.6.1. Ta'rif. $\{M\}$ to'plamning har bir M nuqtasiga biror qonun yoki qonunda yordamida biror $u(u \in R)$ son mos qo'yilgan bo'lsa, $\{M\}$ to'plamga $u = u(M)$ yoki $u = f(M)$ *funksiya aniqlangan* deyiladi. Bunda $\{M\}$ to'plam funksiyaning *aniqlanish sohasi*, $\{u\}$ to'plam esa funksiyaning *qiymatlar sohasi* deyiladi.

Missollar. 1) $u = \sqrt{9 - x^2 - y^2}$ funksiyaning aniqlanish sohasi $D(f)$: markazi koordinatalar boshida joylashgan radiusi 3 ga teng bo'lgan donadan iborat, qiymatlar to'plami esa $E(f)$: $0 \leq u \leq 3$ kesmadan iborat.

2) $u = \ln xy$ funksiyaning aniqlanish sohasi $D(f)$: $xy > 0$ tengsizlikni qanoatlantiruvchi nuqtalar to'plamidan iborat, qiymatlar to'plami esa $E(f)$: $-\infty < u < +\infty$ son o'qidan iborat.

3) $u = \sqrt{1 - x_1^2 - x_2^2 - \dots - x_m^2}$ funksiyaning aniqlanish sohasi $D(f)$: markazi $O(0, 0, \dots, 0)$ koordinatalar boshida joylashgan radiusi 1 ga teng bo'lgan donadan iborat bo'lib, uning qiymatlar to'plami $E(f)$: $[0; 1]$ kesmadan iborat.

4) $u = x_1^2 + x_2^2 + \dots + x_m^2$ funksiyaning aniqlanish sohasi $D(f)$: R^m fazodan iborat, uning qiymatlar to'plami $E(f)$: $[0; \infty]$ yarim segmentdan iborat.

10.7. Ko'p o'zgaruvchili funksiya limiti tushunchasi. $u = f(M)$ funksiya $\{M\} \subset R^m$ da aniqlangan. A nuqta $\{M\}$ to'plamning limit nuqtasi bo'lsin.

10.7.1. Ta'rif (Geyne). Agar $\{M\}$ to'plamning nuqtalaridan tuzilgan va A nuqtaga intiluvchi har qanday $\{M_n\}$ ($M_n \neq A, n=1,2,\dots$) ketma-ketlik olinganda ham unga mos kelgan funksiyaning qiymatlar ketma-ketlik $\{f(M_n)\}$ hamma vaqt yagona B songa (chekli yoki cheksiz) limitga intilsa, shu B $f(M)$ funksiyaning A nuqtadagi (yoki $M \rightarrow A$ dagi) *limiti* deb atala va u $\lim_{M \rightarrow A} f(M) = B$ yoki $M \rightarrow A$ da $f(M) \rightarrow B$ yoki $\lim_{\substack{x_1 \rightarrow a_1 \\ x_2 \rightarrow a_2 \\ \dots \\ x_m \rightarrow a_m}} f(x_1, x_2, \dots, x_m)$

kabi belgilanadi.

10.7.2. Ta'rif (Koshi). Agar $M \rightarrow A \Leftrightarrow \begin{cases} x_1 \rightarrow a_1 \\ x_2 \rightarrow a_2 \\ \dots \\ x_m \rightarrow a_m \end{cases} \quad \forall \varepsilon > 0 \text{ son olinganda}$

ham $\exists \delta = \delta(\varepsilon) > 0$ topilsaki, $0 < \rho(M, A) < \delta$ tengsizlikni qanoatlantiruvchi barcha $M \in \{M\}$ nuqtalarda

$$|f(M) - B| < \varepsilon$$

tengsizlik bajarilsa, shu B ga $f(M)$ funksiyaning A nuqtadagi ($M \rightarrow A$ dagi) *limiti* deyiladi.

10.7.1- va 10.7.2- ta'riflar o'zaro ekvivalent ta'riflardir. Buning isboti bir o'zgaruvchili funksiyalarning limitik qiymatining ikki ta'rif ekvivalentligining isboti kabi ko'rsatiladi.

10.7.3. Ta'rif. Agar $\forall \varepsilon > 0$ uchun $\exists \delta > 0$ bo'lib, $0 < \rho(M, A) < \delta$ tengsizlikni qanoatlantiruvchi barcha $M \in \{M\}$ nuqtalarda $|f(M)| > \varepsilon$ ($f(M) > \varepsilon$; $f(M) < -\varepsilon$) bo'lsa, $f(M)$ funksiya A nuqtadagi ($M \rightarrow A$ dagi) ∞ ($+\infty, -\infty$) deyiladi.

Funksiyaning limiti uning argumentlari $x_1, x_2, \dots, x_m$ larning bir vaqtda mos ravishda $a_1, a_2, \dots, a_m$ sonlarga intilgandagi limitdan iborat ekan. Biz bundan keyin, bu limitni *karrali limit* deb ataymiz.

Limitga ega bo'lgan m -o'zgaruvchili funksiyalar ustida arifmetik amallar, ya'ni quyiagi tasdiq o'rinli bo'ladi:

10.7.4. Teorema. $f(M)$ va $g(M)$ funksiyalar bir xil $\{M\}$ to'plamda aniqlangan va A nuqtada mos ravishda B va C limitlarga ega bo'lsa, u

Agar $f(M) \in g(M)$, $f(M)g(M)$ va $\frac{f(M)}{g(M)}$ funksiyalar ham A nuqtada

o'rinli bo'lishida $B \pm C$, BC va $\frac{B}{C}$ limitlarga ega bo'ladi.

Ushbu teoremlarning isboti ham bir o'zgaruvchili funksiya limiti haqidagi teoremlarning isboti kabi bo'ladi.

Albatta, bir o'zgaruvchili cheksiz kichik va cheksiz katta nuqtalar tushunchasi ([9], 3-bob, 3.21-bandga q.) berilgan edi. Xuddi shunday tushunchalarni ko'p o'zgaruvchili funksiyalar uchun ham kiritish mumkin va ularni o'rganish esa, bir o'zgaruvchili funksiyalardagiga o'xshash bo'lgani uchun cheksiz kichik va cheksiz katta ko'p o'zgaruvchili funksiyalar haqidagi ma'lumotlarni ham isbotsiz kiritib olish mumkin.

10.7.5. Eslatma. $f(M)$ va $g(M)$ funksiyalar yig'indisi, ayirmasi, ko'paytmasi va nisbatining limitga ega bo'lishidan, bu funksiyalarning har birining limitga ega bo'lishi har doim ham kelib chiqavermaydi.

$\alpha(M)$ funksiya $\{M\} \subset R^m$ to'plamda berilgan bo'lib, A ($A \in R^m$) nuqtada $\alpha(M)$ to'plamning limit nuqtasi bo'lsin.

10.7.6. Ta'rif. Agar $M \rightarrow A$ da $\alpha(M)$ funksiyaning limiti nol, ya'ni $\lim_{M \rightarrow A} \alpha(M) = 0$ bo'lsa, u holda $\alpha(M)$ funksiya $M \rightarrow A$ da *cheksiz kichik funksiya* deyiladi.

Masalan, $f(M) = (x_1 - a_1)^{n_1} + (x_2 - a_2)^{n_2} + \dots + (x_m - a_m)^{n_m}$ (bunda $n_1, n_2, \dots, n_m$ lar musbat sonlar) funksiya $A(a_1, a_2, \dots, a_m)$ nuqtada cheksiz kichik bo'ladi.

Agar $u = f(M)$ funksiya A nuqtada B limitga ega bo'lsa, u holda $\alpha(M) = f(M) - B$ funksiya A nuqtada cheksiz kichik bo'ladi.

Demak, limitga ega bo'lgan funksiyaning

$$f(M) = B + \alpha(M)$$

ko'rinishida tasvirlash mumkin ekan, bunda $\lim_{M \rightarrow A} \alpha(M) = 0$. Bundan $f(M)$ funksiyaning A nuqtaning atrofida B dan cheksiz kichik funksiyaga farq qilishi, ya'ni $f(M)$ funksiya A nuqtaning atrofida B ga asimptotik yaqinlashar ekan degan xulosa kelib chiqadi.

Cheksiz kichik funksiyalar quyidagi xossalarga ega:

1⁰. Agar $M \rightarrow A$ da $\alpha(M)$ va $\beta(M)$ funksiyalar cheksiz kichik funksiyalar bo'lsa, u holda ularning yig'indisi $\alpha(M) + \beta(M)$ ham nuqtada cheksiz kichik bo'ladi.

2⁰. Agar $M \rightarrow A$ da $\alpha(M)$ funksiya cheksiz kichik bo'lib, $\beta(M)$ funksiya chegaralangan bo'lsa, u holda $\alpha(M)$, $\beta(M)$ funksiya cheksiz kichik bo'ladi.

10.7.7. Ta'rif. Agar $h(M)$ funksiya $\{M\}$ to'plamda aniqlang bo'lib, $\lim_{M \rightarrow A} h(M) = \infty$ bo'lsa, $h(M)$ funksiya $M \rightarrow A$ da *cheksiz kichik* funksiya deyiladi.

3⁰. Agar $M \rightarrow A$ da $\alpha(M)$ ($\alpha(M) \neq 0$) cheksiz kichik funksiya bo'lsa, u holda $\frac{1}{\alpha(M)}$ funksiya $M \rightarrow A$ da cheksiz katta funksiya bo'ladi.

4⁰. Agar $M \rightarrow A$ da $h(M)$ cheksiz katta funksiya bo'lsa, $\frac{1}{h(M)}$ funksiya $M \rightarrow A$ da cheksiz kichik funksiya bo'ladi.

10.7.8. Teorema (Koshi kriteriyasi). $u = f(M)$ funksiyaning limit ega bo'lishi uchun $\forall \varepsilon > 0$ son olinganda ham $\exists \delta > 0$ mavjud bo'lib, ush

$$0 < \rho(M', A) < \delta, \quad 0 < \rho(M'', A) < \delta$$

tengsizliklarni qanoatlantiruvchi barcha $M', M'' \in \{M\}$ nuqtalarda

$$|f(M') - f(M'')| < \varepsilon$$

tengsizlikning bajarilishi zarur va yetarli.

Bu teoremaning isboti ham bir o'zgaruvchili funksiyaning limit ega bo'lishi haqidagi Koshi kriteriyasining isboti kabi bo'ladi.

10.8. Takroriy limitlar. Yuqorida biz $u = f(M) = f(x_1, x_2, \dots, x_m)$ funksiyaning $A = A(a_1, a_2, \dots, a_m)$ nuqtadagi limiti

$$\lim_{M \rightarrow A} f(M) = B \quad \left(\lim_{\substack{x_1 \rightarrow a_1 \\ x_2 \rightarrow a_2 \\ \dots \\ x_m \rightarrow a_m}} f(x_1, x_2, \dots, x_m) = B \right)$$

bu tushuncha edik. Bundan, funksiyani limit, uning argumentlari $x_1, x_2, \dots, x_m$ larning birdaniga mos ravishda $a_1, a_2, \dots, a_m$ sonlarga o'tishda limitidan iborat ekanligiga ishonch hosil qilamiz. Biz bundan foydalanib bu limitni *karrali limit* deb ataymiz. Faqat ko'p o'zgaruvchili o'tirayotgan xos bo'lgan boshqa ko'rinishdagi limit tushunchasini ham o'tirayotgan har bir argumenti bo'yicha (qolganlarini o'zgarmas deb qarab) bu limit tushunchasini kiritish mumkin. Bundan, takroriy limit deb atayotgan limit tushunchasi kelib chiqadi.

Uch $f(M)$ funksiya $\{M\} \subset R^m$ to'plamda berilgan bo'lib, $a_1, a_2, \dots, a_m$ nuqta $\{M\}$ to'plamning limit nuqtasi bo'lsin.

Uchdan funksiyani $x_1 \rightarrow a_1$ (qolgan barcha argumentlarini tayinlab) qolgan limit mavjud bo'lsin deb

$$\lim_{x_1 \rightarrow a_1} f(x_1, x_2, \dots, x_m)$$

qulaylik, bu limit $x_2, x_3, \dots, x_m$ o'zgaruvchilarga bog'liq bo'ladi:

$$\lim_{x_1 \rightarrow a_1} f(x_1, x_2, \dots, x_m) = \varphi_1(x_2, x_3, \dots, x_m).$$

Uch $\varphi_1(x_2, x_3, \dots, x_m)$ funksiyani x_2 argumentni o'zgaruvchi (qolgan barcha argumentlarini tayinlab) uning $x_2 \rightarrow a_2$ dagi limiti mavjud deb

$$\lim_{x_2 \rightarrow a_2} \varphi_1(x_2, x_3, \dots, x_m)$$

qulaymiz. Bu limit $x_3, x_4, \dots, x_m$ o'zgaruvchilarga bog'liq bo'ladi:

$$\lim_{x_2 \rightarrow a_2} \varphi_1(x_2, x_3, \dots, x_m) = \varphi_2(x_3, x_4, \dots, x_m).$$

Xuddi shunday birin-ketin $x_3 \rightarrow a_3, x_4 \rightarrow a_4, \dots, x_m \rightarrow a_m$ da limitga o'tib (bu limitlarni mavjud deb), natijada

$$\lim_{x_1 \rightarrow a_1} \lim_{x_2 \rightarrow a_2} \dots \lim_{x_m \rightarrow a_m} f(x_1, x_2, \dots, x_m)$$

hosil qilamiz. Bu limitga $f(x_1, x_2, \dots, x_m)$ funksiyani *takroriy limiti* deb ataladi.

Ravshanki, $f(x_1, x_2, \dots, x_m)$ funksiyani argumentlari $x_1, x_2, \dots, x_m$ lar mos ravishda $a_1, a_2, \dots, a_m$ sonlarga turli tartibda intilganda funksiyani takroriy limitlari hosil bo'ladi.

Masalan, $f(x_1, x_2, \dots, x_m)$ funksiyani $x_{i_1}, x_{i_2}, \dots, x_{i_k}$ argumentlari mos ravishda $a_{i_1}, a_{i_2}, \dots, a_{i_k}$ larga intilgandagi takroriy limitlari

$$\lim_{x_1 \rightarrow a_1} \lim_{x_2 \rightarrow a_2} \dots \lim_{x_m \rightarrow a_m} f(x_1, x_2, \dots, x_m)$$

ni qarash ham mumkin.

10.8.1. Misol. Ushbu

$$f(x_1, x_2) = \begin{cases} \frac{x_1 x_2}{\sqrt{x_1^2 + x_2^2}}, & x_1^2 + x_2^2 > 0 \text{ bo'lsa,} \\ 0, & x_1^2 + x_2^2 = 0 \text{ bo'lsa} \end{cases}$$

funksiyaning $M(x_1, x_2) \rightarrow A(0,0)$ ga karrali va takroriy limitlarini hisobla

Yechilishi. a) Berilgan funksiyaning $x_1 \rightarrow 0, x_2 \rightarrow 0$ karrali limit hisoblaymiz. Buni koshi ta'rif bo'yicha hisoblaymiz: $\forall \varepsilon > 0, \delta = 2\varepsilon$ dolsak, unda $0 < \rho(M, O) < \delta$ ni qanoatlantiruvchi $\forall M(x_1, x_2)$ va

$$|f(x_1, x_2) - O(0,0)| = \frac{|x_1 x_2|}{\sqrt{x_1^2 + x_2^2}} = \frac{|x_1| |x_2|}{\sqrt{x_1^2 + x_2^2}} \leq \frac{1}{2} \sqrt{x_1^2 + x_2^2} = \frac{1}{2} \rho(M, O) < \frac{1}{2} \delta$$

Tengsizlik o'rinli bo'ladi. Bundan esa,

$$\lim_{M \rightarrow O(0,0)} f(M) = \lim_{\substack{x_1 \rightarrow 0 \\ x_2 \rightarrow 0}} \frac{x_1 x_2}{\sqrt{x_1^2 + x_2^2}} = 0.$$

b) Endi berilgan funksiyaning takroriy limitlari mavjudligi va ul 0 ga teng ekanligini ko'rsatamiz:

$$\lim_{x_1 \rightarrow 0} f(x_1, x_2) = \lim_{x_1 \rightarrow 0} \frac{x_1 x_2}{\sqrt{x_1^2 + x_2^2}} = 0 \quad \lim_{x_2 \rightarrow 0} \lim_{x_1 \rightarrow 0} f(x_1, x_2) = 0,$$

$$\lim_{x_2 \rightarrow 0} f(x_1, x_2) = \lim_{x_2 \rightarrow 0} \frac{x_1 x_2}{\sqrt{x_1^2 + x_2^2}} = 0, \quad \lim_{x_1 \rightarrow 0} \lim_{x_2 \rightarrow 0} f(x_1, x_2) = 0.$$

Shunday qilib, berilgan funksiyaning takroriy limitlari mavjud va ul bir-biriga teng. Ular esa karrali limitlarga teng.

10.8.2. Misol. Ushbu $f(M) = \frac{x_1 x_2}{x_1^2 + x_2^2}$ funksiya

$M(x_1, x_2) \rightarrow M_0(0,0)$ dagi limitining mavjud emasligini ko'rsating.

Yechilishi. Ikkita $M_n(\frac{1}{n}, \frac{1}{n}) \rightarrow M_0(0,0)$, $M_n(\frac{2}{n}, \frac{1}{n}) \rightarrow M_0(0,0)$ ketma-ketliklarni olib, ularga mos kelgan funksiyaning qiymatlaridan tuzilg ketma-ketlikni qaraymiz;

$$f(M_n) = f(\frac{1}{n}, \frac{1}{n}) = \frac{1}{2}, \quad f(M_n) = f(\frac{2}{n}, \frac{1}{n}) = \frac{2}{5}.$$

funksiya $M(x_1, x_2) \rightarrow M_0(0,0)$ da berilgan funksiyaning limitga ega bo'lishi kelib chiqadi.

10.8.3. Misol. Ushbu

$$f(x_1, x_2) = \begin{cases} x_1 + x_2 \sin \frac{1}{x_1}, & x_1 \neq 0 \text{ bo'lganda,} \\ 0, & x_1 = 0 \text{ bo'lganda} \end{cases}$$

funksiyaning bitta takroriy limiti mavjud, ikkinchisi mavjud emas, karrali limiti esa mavjud ekanligini ko'rsating.

Yechilishi. Haqiqatan ham,

$$\lim_{x_2 \rightarrow 0} f(x_1, x_2) = x_1, \quad \lim_{x_1 \rightarrow 0} \lim_{x_2 \rightarrow 0} f(x_1, x_2) = 0$$

bo'lib,

$$\lim_{x_2 \rightarrow 0} \lim_{x_1 \rightarrow 0} f(x_1, x_2)$$

mavjud emas, lekin

$$|f(x_1, x_2) - 0| = \left| x_1 + x_2 \sin \frac{1}{x_1} \right| \leq |x_1| + |x_2| \quad (x_1 \neq 0)$$

amplitudadan ko'rinadiki $M(x_1, x_2) \rightarrow A(0,0)$ da $f(x_1, x_2)$ funksiyaning umumiy limiti mavjud va u

$$\lim_{\substack{x_1 \rightarrow 0 \\ x_2 \rightarrow 0}} f(x_1, x_2) = 0$$

ga teng.

Yuqoridagi keltirilgan misollardan ko'rinadiki, funksiyaning biror nuqtada karrali limitining mavjudligidan, uning shu nuqtada takroriy limitlarining mavjudligi va aksincha funksiyaning biror nuqtada takroriy limitlarining mavjudligidan, uning shu nuqtada karrali limitlarining mavjudligi har doim kelib chiqavermas ekan. Bundan tashqari takroriy limitlar har doim bir-biriga teng bo'lavermas ekan.

Biz quyida funksiyaning karrali limiti bilan takroriy limitlar masidagi bog'lanish hamda ularning ma'lum shartlarda o'zaro tengligi haqidagi teoremani keltiramiz va uni isbotlaymiz. $u = f(M)$ funksiya

$$\{M\} = \{(x_1, x_2) \in R^2, |x_1 - x_1^0| < d_1, |x_2 - x_2^0| < d_2\}$$

to'plamda berilgan bo'lsin.

10.8.4. Teorema. Agar 1) $M(x_1, x_2) \rightarrow M_0(x_1^0, x_2^0)$ da $f(x_1,$
funksiyaning karrali limiti mavjud:

$$\lim_{\substack{x_1 \rightarrow x_1^0 \\ x_2 \rightarrow x_2^0}} f(x_1, x_2) = B.$$

2) Har bir tayinlangan x_1 da quyidagi $\lim_{x_2 \rightarrow x_2^0} f(x_1, x_2) = \varphi(x_1)$ limit,
bir tayinlangan x_2 da $\lim_{x_1 \rightarrow x_1^0} f(x_1, x_2) = \varphi(x_2)$ limit mavjud bulsa, u holda
 $\lim_{x_1 \rightarrow x_1^0} \lim_{x_2 \rightarrow x_2^0} f(x_1, x_2)$ va $\lim_{x_2 \rightarrow x_2^0} \lim_{x_1 \rightarrow x_1^0} f(x_1, x_2)$ takroriy limitlar mavjud bo'ladi
ular B ga teng bo'ladi.

Isboti. Shartga ko'ra $u = f(M) = f(x_1, x_2)$ funksiya $M_0(x_1^0, x_2^0)$ nuqtada karrali limitga ega bo'lgani uchun $\forall \varepsilon > 0$ son olinganda ham shunday $\delta > 0$ mavjud bo'lib, $|x_1 - x_1^0| < \delta$, $|x_2 - x_2^0| < \delta$ tengsizliklarni qanoatlantiradigan barcha $M(x_1, x_2) \in \{M\}$ nuqtalar uchun

$$|f(x_1, x_2) - B| < \varepsilon \quad (10.8.5)$$

tengsizlik bajariladi. Teoremaning 2) shartiga asosan, x_1 o'zgaruvchini $|x_1 - x_1^0| < \delta$ tengsizlikni qanoatlantiradigan qiymatini tayinlab, (10.8.5) $x_2 \rightarrow x_2^0$ da limitga o'tib $|\varphi(x_1) - B| \leq \varepsilon$ ni topamiz. Bu tengsizlik $\lim_{x_1 \rightarrow x_1^0} \varphi(x_1) = B$ ekanligi kelib chiqadi. Bundan esa, $\lim_{x_1 \rightarrow x_1^0} \lim_{x_2 \rightarrow x_2^0} f(x_1, x_2) = B$ bo'lishi kelib chiqadi. Xuddi shunday $\lim_{x_2 \rightarrow x_2^0} \lim_{x_1 \rightarrow x_1^0} f(x_1, x_2)$ takroriy limitning mavjudligi va uning B ga tengligi isbot qilinadi. ■

10.8.6. Natija. 10.8.4-teoremaning shartlari bajarilganda

$$\lim_{\substack{x_1 \rightarrow x_1^0 \\ x_2 \rightarrow x_2^0}} f(x_1, x_2) = \lim_{x_1 \rightarrow x_1^0} \lim_{x_2 \rightarrow x_2^0} f(x_1, x_2) = \lim_{x_2 \rightarrow x_2^0} \lim_{x_1 \rightarrow x_1^0} f(x_1, x_2)$$

munosabat o'rinli bo'ladi. Xuddi yuqoridagidek $u = f(x_1, x_2, \dots, x_k)$ funksiyaning $x_{i_1}, x_{i_2}, \dots, x_{i_k}$ o'zgaruvchilari bo'yicha ham

$$\lim_{\substack{x_{i_1} \rightarrow x_{i_1}^0 \\ x_{i_2} \rightarrow x_{i_2}^0 \\ \dots \\ x_{i_k} \rightarrow x_{i_k}^0}} f(x_1, x_2, \dots, x_m)$$

karrali hamda

$$\lim_{x_1 \rightarrow x_1^0} \lim_{x_2 \rightarrow x_2^0} \dots \lim_{x_k \rightarrow x_k^0} f(x_1, x_2, \dots, x_m)$$

bu limitlarni va ular orasidagi bog'lanishlarni qarash mumkin.

44-§. Ko'p o'zgaruvchili funksiyalarning uzluksizligi

10.9 Ko'p o'zgaruvchili funksiyalarning uzluksizligi.

$f(M) = f(x_1, x_2, \dots, x_m)$ funksiya $\{M\} \subset R^m$ to'plamda aniqlangan bo'lib, $(a_1, a_2, \dots, a_m)$ nuqta $\{M\}$ to'plamning limit nuqtasi bo'lib, u $A \in \{M\}$ bo'lsa,

10.9.1. Ta'rif. Agar $\exists \lim_{M \rightarrow A} f(M) = f(A)$ bo'lsa, u holda $f(M)$ funksiya A nuqtada *uzluksiz* deyiladi.

U holda $\lim_{M \rightarrow A} M$ ekanligini e'tiborga olsak funksiyaning uzluksizlik belgisi ushbu

$$\lim_{M \rightarrow A} f(M) = f(\lim_{M \rightarrow A} M)$$

ko'rinishida ham yozish mumkin.

10.9.2. Ta'rif. Agar $\forall \varepsilon > 0$ son uchun $\exists \delta = \delta(\varepsilon)$ son topilsaki, ushbu ε uchun tengsizlikni qanoatlantiruvchi barcha $M \in \{M\}$ nuqtalarda

$$|f(M) - f(A)| < \varepsilon$$

qanoatlik bajarilsa, $f(M)$ funksiya A nuqtada *uzluksiz* deyiladi.

10.9.1-, 10.9.2-ta'riflar o'zaro ekvivalent ta'riflardir.

10.9.3. Ta'rif. Agar $f(M)$ funksiya $\{M\}$ to'plamning har bir nuqtasida uzluksiz bo'lsa, u holda u $\{M\}$ to'plamda *uzluksiz* deyiladi.

Ushbu

$$\Delta u = f(M) - f(A) \quad (10.9.4)$$

aytmaga $f(M)$ funksiya A nuqtadagi *ortirmasi* (to'liq) deyiladi, bunda $M \in \{M\}$.

Agar $x_1 - a_1 = \Delta x_1, x_2 - a_2 = \Delta x_2, \dots, x_m - a_m = \Delta x_m$ deb belgilasak, u holda (10.9.4) ni quyidagi

$$\Delta u = f(a_1 + \Delta x_1, a_2 + \Delta x_2, \dots, a_m + \Delta x_m) - f(a_1, a_2, \dots, a_m)$$

ko'rinishida ham yozish mumkin.

Ravshanki, $\Delta u = f(M) - f(A)$ funksiyaning A nuqtadagi uzluksizlik sharti $\lim_{M \rightarrow A} (f(M) - f(A)) = 0$ yoki $\lim_{\substack{\Delta x_1 \rightarrow 0 \\ \Delta x_2 \rightarrow 0 \\ \dots \\ \Delta x_n \rightarrow 0}} \Delta u = 0$ shartga ekvivalent.

$u = f(x_1, x_2, \dots, x_m)$ funksiyaning $x_k (k = \overline{1, m})$ argumentini o'zgaruvchi deb, qolgan argumentlarini o'zgarmas deb x_k argumentiga ixtiyoriy o'tirma beramiz. U holda $u = f(x_1, x_2, \dots, x_m)$ funksiya ham o'tirma oladi.

$$\Delta_{x_k} u = f(x_1, x_2, \dots, x_{k-1}, x_k + \Delta x_k, x_{k+1}, \dots, x_m) - f(x_1, x_2, \dots, x_m)$$

bu ayirmani funksiyaning A nuqtadagi xususiy o'tirmasi deyiladi.

10.9.5. Ta'rif. Agar

$$\lim_{\Delta x_k \rightarrow 0} \Delta_{x_k} u = 0$$

bo'lsa, u holda $u = f(x_1, x_2, \dots, x_m)$ funksiya A nuqtada x_k argumenti bo'yicha uzluksiz deyiladi.

$u = f(x_1, x_2, \dots, x_m)$ funksiyaning x_k argumentidan boshqa qolgan barcha argumentlari o'zgarmas deb olinganda funksiya bir argumentli funksiya aylanadi. Shu sababli, funksiyaning x_k argumenti bo'yicha uzluksizli, bir argumentli funksiyaning uzluksizligini anglatadi.

Ravshanki, $u = f(x_1, x_2, \dots, x_m)$ funksiyaning berilgan A nuqtadagi uzluksizligidan, funksiyaning shu nuqtadagi har bir argumentlari bo'yicha uzluksizligi kelib chiqadi, ya'ni quyidagi tasdiq o'rinli bo'ladi:

Agar $u = f(x_1, x_2, \dots, x_m)$ funksiya A nuqtaning biror atrofida aniqlang va A nuqtada uzluksiz bo'lsa, u holda u bu nuqtada har bir $x_1, x_2, \dots, x_m$ argumentlari bo'yicha ham uzluksiz bo'ladi.

Bu tasdiqning teskarisi o'rinli emas. Masalan, ushbu

$$f(x_1, x_2) = \begin{cases} \frac{2x_1x_2}{x_1^2 + x_2^2}, & x_1^2 + x_2^2 \neq 0, \\ 0, & x_1^2 + x_2^2 = 0 \end{cases}$$

funksiyani qaraylik. $x_2 \neq 0$ va x_0 - berilgan son bo'lsin. U holda

$$\lim_{x_1 \rightarrow x_0} f(x_1, x_2) = \lim_{x_1 \rightarrow x_0} \frac{2x_1x_2}{x_1^2 + x_2^2} = \frac{2x_0x_2}{x_0^2 + x_2^2} = f(x_0, x_2).$$

и наоборот, и holds $\forall x_0 \neq 0$ да $\lim_{x_1 \rightarrow x_0} f(x_1, 0) = f(x_0, 0) = 0$, agar $x_2 = 0$

$$\text{or by l.h.s. } \lim_{x_1 \rightarrow 0} f(x_1, 0) = f(0, 0) = 0.$$

Demak, har bir berilgan x_2 larda $f(x_1, x_2)$ funksiya x_1 bo'yicha
) tutalagan. Xuddi shunday har bir berilgan x_1 larda x_2 bo'yicha
) tutalagan ko'rsatiladi. Lekin bu funksiya $(0, 0)$ nuqtada hamma
) elementlari bo'yicha birdaniga uzluksiz emas.

Hopitaldan ham, ikkita $\left\{\frac{1}{n}, \frac{1}{n}\right\}$ va $\left\{\frac{2}{n}, \frac{1}{n}\right\}$, $n \rightarrow \infty$ da $(0, 0)$ nuqtaga

adətli ketma-ketlik olinganda, unga mos kelgan $\left\{f\left(\frac{1}{n}, \frac{1}{n}\right)\right\}, \left\{f\left(\frac{2}{n}, \frac{1}{n}\right)\right\}$

1) **har bir** $n \rightarrow \infty$ da har xil limitga ega:

$$f\left(\frac{1}{n}, \frac{1}{n}\right) = \frac{\frac{2}{n^2}}{\frac{1}{n^2} + \frac{1}{n^2}} \rightarrow 1, \quad f\left(\frac{2}{n}, \frac{1}{n}\right) = \frac{\frac{4}{n^2}}{\frac{4}{n^2} + \frac{1}{n^2}} \rightarrow \frac{4}{5}.$$

10.10. Uzlüksiz funksiyalar ustida arifmetik amallar. $u = f(M)$ va $v = g(M)$ funksiyalar $\{M\} \subset R^m$ aniqlangan. $A \in R^m$ nuqta esa, $\{M\}$ to'plamning limit nuqtasi bo'lib, $A \in \{M\}$.

10.10.1. Teorema. Agar $f(M)$ va $g(M)$ funksiyalar $A \in \{M\}$ nuqtada u lukats bo'lsa, $f(M) \pm g(M)$, $f(M)g(M)$, $\frac{f(M)}{g(M)}$ ($g(A) \neq 0$) funksiyalar ham shu nuqtada uzluksiz bo'ladi.

Bu teoremaning isboti 10.7.4-teoremadan bevosita kelib chiqadi.

10.11. Murakkab funksiyaning uzluksizligi.
 Agar $f(M) = f(x_1, x_2, \dots, x_m)$ funksiya $\{M\} \subset R^m$ to'plamda berilgan bo'lib,
 $x_1, x_2, \dots, x_m$ o'zgaruvchilarning har biri $\{N\} \subset R^k$ to'plamda berilgan
 funksiyalar bo'lsin:

$$\begin{cases} x_1 = \varphi_1(t_1, t_2, \dots, t_m), \\ x_2 = \varphi_2(t_1, t_2, \dots, t_m), \\ \dots\dots\dots \\ x_m = \varphi_m(t_1, t_2, \dots, t_m). \end{cases} \quad (10.11.1)$$

U holda har bir $N(t_1, t_2, \dots, t_k) \in \{N\}$ nuqtaga (10.11.1) formula yordamida $M(x_1, x_2, \dots, x_m) \in \{M\}$ nuqta mos qo'yiladi. Bu holda, $\{N\} \subset R^k$ to'plamda

$$u = f(\varphi_1(t_1, t_2, \dots, t_k), \dots, \varphi_m(t_1, t_2, \dots, t_k)) = \Phi(t_1, t_2, \dots, t_k)$$

murakkab funksiya aniqlangan bo'ladi.

10.11.2. Teorema. Agar $x_i = \varphi_i(t_1, t_2, \dots, t_k)$ ($i = \overline{1, m}$) funksiyaning biri $A(a_1, a_2, \dots, a_k)$ nuqtada uzluksiz, $u = f(x_1, x_2, \dots, x_m)$ funksiya $B(b_1, b_2, \dots, b_m)$ ($b_i = \psi(a_1, a_2, \dots, a_k), (i = \overline{1, m})$) nuqtada uzluksiz bo'lsa, u holda $\Phi(t_1, t_2, \dots, t_k)$ murakkab funksiya $A(a_1, a_2, \dots, a_k)$ nuqtada uzluksiz bo'ladi.

Isboti. $\forall \{N_n\} \subset \{N\} (N_n \neq A)$ A ga intiluvchi ketma-ketlikni qaraymiz. Unga mos kelgan $M_n(x_1^{(n)}, x_2^{(n)}, \dots, x_m^{(n)})$ ($x_i = \varphi_i(t_1^{(n)}, t_2^{(n)}, \dots, t_k^{(n)})$) ketma-ketlikni φ_i funksiyalarning A nuqtada uzluksiz bo'lgani uchun $\{M_n\}$ ketma-ketlikni $B(b_1, b_2, \dots, b_m)$ nuqtada yaqinlashadi (bunda $M_n = B$ bo'lishi ham mumkin).

$u = f(x_1, x_2, \dots, x_m)$ funksiyaning B nuqtada uzluksiz bo'lganligidan $\{f(M_n)\} \rightarrow f(B)$. Bu ketma-ketlik, A ga intiluvchi murakkab funksiyaning qiymatlaridan tuzilgan $\forall \{N_n\} \subset \{N\}$ ketma-ketlikni ifodalaydi. Bunday bo'lsa, murakkab funksiyaning uzluksizligi kelib chiqadi, ya'ni $\{f(M_n)\} \rightarrow f(B)$.

10.12. Ko'p o'zgaruvchili uzluksiz funksiyalarning xossalari. Ko'p o'zgaruvchili uzluksiz funksiyalar ham xuddi bir o'zgaruvchili uzluksiz funksiyalar kabi xossalarga ega. Bu xossalarning isboti to'g'ri o'zgaruvchili uzluksiz funksiyalarning isboti kabi bo'lganligi sabab ularni isbotsiz keltiramiz.

Nuqtada uzluksiz bo'lgan funksiyalarning (lokal) xossalari.

$u = f(M)$ funksiya $\{M\} \subset R^m$ to'plamda berilgan bo'lsin. E to'plamdan $M_0(x_1^0, x_2^0, \dots, x_m^0) \in \{M\}$ nuqtani olib, uning yetarli kichik atrofini qaraymiz.

1^o. Agar $f(M)$ funksiya M_0 nuqtada uzluksiz bo'lsa, u holda $u = f(M)$ nuqtaning yetarli kichik atrofida chegaralangan bo'ladi.

1^o Agar $f(M)$ funksiya M_0 nuqtada uzluksiz bo'lib, $f(M_0) > 0$ ($f(M_0) < 0$)

1^o M_0 nuqtaning yetarli kichik atrofidagi $\forall M$ nuqtalarda ham $f(M) > 0$ ($f(M) < 0$) bo'ladi.

2^o Agar $f(M)$ funksiya M_0 nuqtada uzluksiz bo'lsa, u holda M_0 nuqtaning yetarli kichik atrofidagi ixtiyoriy $M_1 \in \{M\}, M_2 \in \{M\}$ nuqtalar ham

$$|f(M_1) - f(M_2)| < \varepsilon$$

(ε ixtiyoriy istalgancha kichik son) tengsizlik o'rinli bo'ladi.

Ko'p o'zgaruvchili funksiyalarning global xossalari

10.1.1. Ta'rif. Agar shunday c va C sonlar mavjud bo'lib, $\forall M \in \{M\}$ ham $f(M) \leq C$ tengsizlik o'rinli bo'lsa, $u = f(M)$ funksiya shu plamda chegaralangan deyiladi.

1^o (**Bolsano – Koshining birinchi teoremasi**). $u = f(M)$ funksiya $u: K^n$ bog'lamli to'plamda berilgan va uzluksiz bo'lsin. Agar berilgan to'plamda $\{M\}$ to'plamning ikkita $A(a_1, a_2, \dots, a_n)$ va $B(b_1, b_2, \dots, b_n)$ nuqtalarida o'raldichorali qiymatlarga ega bo'lsa, u holda shunday $C(c_1, c_2, \dots, c_n) \in \{M\}$ nuqta topiladiki, shu nuqtada funksiya nolga aylanadi, ya'ni $u(C) = f(c_1, c_2, \dots, c_n) = 0$.

2^o (**Bolsano – Koshining ikkinchi teoremasi**). $u = f(M)$ funksiya $u: K^n$ bog'lamli to'plamda berilgan va uzluksiz bo'lib, $\{M\}$ to'plamning ikkita $A(a_1, a_2, \dots, a_n)$ va $B(b_1, b_2, \dots, b_n)$ nuqtalarida $f(A) \neq f(B)$ bo'lsa, u holda $f(A)$ va $f(B)$ qiymatlar orasida har qanday C son olinsa ham $\{M\}$ to'plamda shunday $C(c_1, c_2, \dots, c_n)$ nuqta topiladiki,

$$f(c) = f(c_1, c_2, \dots, c_n) = C$$

bo'ladi.

3^o (**Veyishtrassning birinchi teoremasi**). Agar $f(M)$ funksiya $M: \{M\} \subset R^n$ —chegaralangan yopiq to'plamda berilgan va uzluksiz bo'lsa, bu funksiya shu $\{M\}$ to'plamda chegaralangan bo'ladi.

4^o (**Veyishtrassning ikkinchi teoremasi**). Agar $f(M)$ funksiya chegaralangan yopiq $\{M\} \subset R^n$ to'plamda berilgan va uzluksiz bo'lsa, u bu to'plamda o'zining aniq yuqori hamda aniq kuyi chegarasiga erishadi.

10.13. Ko'p o'zgaruvchili funktsiyaning tekis uzluksizligi. 11
 funksiya $\{M\} \subset R^m$ to'plamda berilgan bo'lsin.

10.13.1. Ta'rif. Agar $\forall \varepsilon > 0$ son uchun, shunday $\delta > 0$ topilsaki, $\{M'\}$ to'plamning $\rho(M', M'') < \delta$ tengsizlikni qanoatlantiruvchi $\forall M', M'' \in \{M'\}$ nuqtalarida

$$|f(M') - f(M'')| < \varepsilon$$

tengsizlik bajarilsa, $f(M)$ funksiya $\{M\}$ to'plamda *tekis uzluksiz* deyil.

10.13.2. Eslatma. Funktsiyaning tekis uzluksizligi ta'rifidagi $\delta > 0$ son faqat $\varepsilon > 0$ songa bog'liq bo'ladi.

10.13.3. Teorema (Kantor teoremasi). Agar $f(M)$ funktsiya chegaralangan yopiq $\{M\}$ to'plamda uzluksiz bo'lsa, $f(M)$ funksiya to'plamda tekis uzluksiz bo'ladi.

10.13.4. Eslatma. Chegaralangan $\{M\}$ to'plamning $\forall M', M'' \in \{M\}$ nuqtalari orasidagi $\rho(M', M'')$ masofaning aniq yuqori chegarasiga *to'plamning diametri* deyiladi.

10.13.5. Natija. Agar $f(M)$ funksiya chegaralangan yopiq to'plamda berilgan va uzluksiz bo'lsa, u holda $\forall \varepsilon > 0$ son uchun, shunday δ topilib, $\{M\}$ to'plamni diametri δ dan kichik bo'lgan, $\{M_i\} \subset \{M\}$ $i = 1, 2, \dots$ bo'laklarga bolganimizda, bu bo'laklarning har birida, f funksiyaning tebranishi berilgan ε dan kichik bo'ladi.

10.13.6. Misol. Ushbu $u = f(x, y) = \arcsin \frac{x}{y}$ funksiyaning aniqlanish sohasida tekis uzluksizlikka tekshiring.

Yechilishi. Berilgan funktsiyaning aniqlanish sohasi:

$$\{M\} = \{(x, y) \in R^2 : |x| \leq |y|, y \neq 0\}.$$

Bu sohada $f(x, y) = \arcsin \frac{x}{y}$ funksiya uzluksiz funksiyalarning superpazitsiyasi sifatida uzluksiz. Endi bu funksiyaning tekis uzluksizligi tekshiramiz: $M_n \left(\frac{1}{n}, \frac{1}{n} \right)$.

$$M'_n \left(\frac{1}{n}, -\frac{1}{n} \right) (n = 1, 2, \dots) \text{ nuqtalar uchun}$$

$$\rho(M, M') = \sqrt{\left(\frac{1}{n} - \frac{1}{n}\right)^2 + \left(\frac{1}{n} + \frac{1}{n}\right)^2} = \frac{2}{n} \rightarrow 0.$$

Endilikdagi bu nuqtadagi qiymatlari

$$|u(M_n) - u(M'_n)| = |\arcsin 1 - \arcsin(-1)| = 2\arcsin 1 = \pi > 1 = \varepsilon$$

bu qanoatlaniradi, ya'ni tekis uzluksizlik ta'rifi bajarilmaydi.

Demak, berilgan funksiya $\{M\}$ to'plamda tekis uzluksiz emas.

10.13. Misol. Ushbu

$$u = \frac{x^3 + y^3}{x^2 + y^2}$$

funksiyaning $\{M\} = \{(x, y) \in R^2 : 0 < x^2 + y^2 \leq 1\}$ to'plamda tekis uzluksizlikka ega bo'lgan.

Yechilishi. Berilgan funksiyaning surati $x^3 + y^3$ va maxraji $x^2 + y^2$ u to'plamda uzluksiz.

Demak, funksiya $\{M\}$ to'plamda uzluksiz, lekin $\{M\}$ to'plam yopiq bo'lmaganligi uchun, bu holda Kantor teoremasini qo'llab bo'lmaydi. Endi funksiyaning $O(0,0)$ nuqtada qayta aniqlasak, ya'ni $u(0,0) = 0$ desak, u holda $u(x, y)$ funksiya $O(0,0)$ nuqtada uzluksiz bo'ladi. Haqiqatdan ham,

$x = \rho \cos \varphi$, $y = \rho \sin \varphi$ almashtirishni olsak, $x \rightarrow 0$, $y \rightarrow 0$ da $\rho \rightarrow 0$, ya'ni

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} u(x, y) = \lim_{\rho \rightarrow 0} u(\rho \cos \varphi, \rho \sin \varphi) = \lim_{\rho \rightarrow 0} \rho = 0.$$

Shunday qilib, $u(x, y)$ funksiyaning $O(0,0)$ nuqtada qayta aniqlasak, ya'ni $u(0,0) = 0$ desak, berilgan funksiya $\{\overline{M}\} = \{(x, y) \in R^2 : x^2 + y^2 \leq 1\}$ to'plamda uzluksiz bo'ladi.

Demak, Kantor teoremasiga asosan, berilgan funksiya markazi $O(0,0)$ olib tashlangan $\{\overline{M}\}$ - doirada, ya'ni $\{\overline{M}\}$ to'plamda tekis uzluksiz bo'ladi.

45-§. Ko'p o'zgaruvchili funksiyalarning hosila va differensial

10.14. Ko'p o'zgaruvchili funksiyalarning xususiy hosilalari.

Agar $f(M) = f(x_1, x_2, \dots, x_m)$ funksiya $\{M\} \subset R^m$ to'plamda berilgan bo'lib, $M(x_1, x_2, \dots, x_m) \in \{M\}$ to'plamning ichki nuqtasi bo'lsin. Bu $\{M\} \subset R^m$ to'plamning $M(x_1, x_2, \dots, x_m)$ nuqtasini olib, uning x_k koordinatasiga

$\Delta x_k \neq 0$ orttirma (qolganlarini o'zgarmas deb) belgilaylik
 $M_1(x_1, x_2, \dots, x_{k-1}, x_k + \Delta x_k, x_{k+1}, \dots, x_m) \in \{M\}$ bo'lsin. Natijada $u = f(x_1, x_2, \dots, x_m)$ funksiya ham $M(x_1, x_2, \dots, x_m)$ nuqtada x_k argumenti bo'yicha xususiy hosil $\Delta_{x_k} u$ orttirma oladi:

$$\Delta_{x_k} u = f(x_1, x_2, \dots, x_{k-1}, x_k + \Delta x_k, x_{k+1}, \dots, x_m) - f(x_1, x_2, \dots, x_m). \quad (10.14.1)$$

Uning xususiy $\Delta_{x_k} u$ orttirmasi (10.14.1) formula orqali ifoda qilinadi.

(10.14.1) ning ikkala tomonini Δx_k ga bo'lamiz, natijada

$$\frac{\Delta_{x_k} u}{\Delta x_k} = \frac{f(x_1, x_2, \dots, x_{k-1}, x_k + \Delta x_k, x_{k+1}, \dots, x_m) - f(x_1, x_2, \dots, x_m)}{\Delta x_k} \quad (10.14.2)$$

nisbatga ega bo'lamiz.

10.14.3. Ta'rif. Agar (10.14.2) nisbatning $\Delta x_k \rightarrow 0$ dagi limiti mavjud va chekli bo'lsa, bu limitga $u = f(x_1, x_2, \dots, x_m)$ funksiyani $M(x_1, x_2, \dots, x_m)$ nuqtadagi x_k argumenti bo'yicha xususiy hosil deyiladi va u

$$\frac{\partial f(x_1, \dots, x_m)}{\partial x_k}, \quad \frac{\partial f}{\partial x_k}, \quad u'_{x_k}, \quad f'_{x_k}(x_1, x_2, \dots, x_m), \quad f'_{x_k}$$

kabi belgilanadi.

Shunday qilib, ta'rifga ko'ra,

$$\frac{\partial u}{\partial x_k} = \lim_{\Delta x_k \rightarrow 0} \frac{\Delta_{x_k} u}{\Delta x_k} \quad (10.14.4)$$

bo'ladi.

10.14.5. Misol. 10.14.3-ta'rifdan foydalanib, ushbu

$$f(x, y) = 3^{2x+y}$$

funksiyaning $O(0,0)$ nuqtadagi f'_x, f'_y xususiy hosilalarini hisoblang.

Yechilishi. 10.14.3-ta'rifga ko'ra, f'_x, f'_y xususiy hosilalar topamiz:

$$\begin{aligned} \frac{\partial f(0,0)}{\partial x} &= \lim_{\Delta x \rightarrow 0} \frac{f(0 + \Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{3^{2\Delta x} - 1}{\Delta x} = 2 \ln 3 = \ln 9, \\ \frac{\partial f(0,0)}{\partial y} &= \lim_{\Delta y \rightarrow 0} \frac{f(0, 0 + \Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{3^{\Delta y} - 1}{\Delta y} = \ln 3. \end{aligned}$$

Demak, $f'_x(0,0) = \ln 9$, $f'_y(0,0) = \ln 3$.

10.14.6. Eslatma. $u = f(x_1, x_2, \dots, x_m)$ funksiyaning x_k argumenti bo'lgan xususiy hosilasi, bir argumentli funksiya hosilasi singari bo'ladi. Hozirgi uchun xususiy hosilasi hisoblanganda, oddiy hosilani hisoblash o'rinli bo'lmay qilinadi.

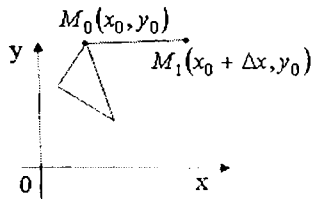
10.14.7. Eslatma. Berilgan nuqtada funksiyaning hamma xususiy hosilalarning mavjudligidan uning shu nuqtada uzluksizligi har doim olinib chiqavermaydi. Masalan, ushbu

$$u = \begin{cases} \frac{xy}{x^2 + y^2}, & x^2 + y^2 \neq 0 \text{ bo'lganda,} \\ 0, & \text{agar } x^2 + y^2 = 0 \text{ bo'lganda.} \end{cases}$$

chunki, bu funksiya $M_0(0,0)$ nuqtada uzluksiz emas, lekin bu funksiya berilgan nuqtada x va y argumentlari bo'yicha xususiy hosilalari mavjud

$$\left. \frac{\partial u}{\partial x} \right|_{(0,0)} = 0; \quad \left. \frac{\partial u}{\partial y} \right|_{(0,0)} = 0,$$

chunki $f(x,0) = 0$, $f(0,y) = 0$.



10.3-chizma.

10.14.8. Eslatma. Agar M nuqta funksiya aniqlanish sohasi chegarasida bo'lsa, u holda bu nuqtada funksiya xususiy hosilasi tarifi berilishi bo'lmaydi.

Masalan, $u = f(x,y)$ funksiyaning aniqlanish sohasi $\{M\}$ (10.3-chizma) uchburchakdan iborat bo'lib, $M_0(x_0, y_0)$ nuqta, uchburchakning chegaraviy nuqtasi bo'lsin. U holda, $M_0(x_0, y_0)$ nuqtada xususiy $\Delta_x u$ ortirma aniqlanmagan, chunki $\forall \Delta x \neq 0$ uchun $M_1(x_0 + \Delta x, y_0)$ nuqta funksiyaning aniqlanish sohasida yotmaydi. Shuning uchun $\frac{\partial u(M_0)}{\partial x}$ hosilasini 10.14.3-va 10.14.4-va bo'yicha aniqlab bo'lmaydi.

10.15. Ko'p o'zgaruvchili funksiyaning differensiallanuvchiligi. $u = f(M) = f(x_1, x_2, \dots, x_m)$ funksiya $\{M\} \subset R^m$ to'plamda aniqlangan bo'lib,

uning $\{M\}$ to'plamning ichki $M(x_1, x_2, \dots, x_m)$ nuqtasidagi Δu to'rttirmasini qaraylik:

$$\Delta u = f(x_1 + \Delta x_1, x_2 + \Delta x_2, \dots, x_m + \Delta x_m) - f(x_1, x_2, \dots, x_m).$$

10.15.1. Ta'rif. Agar $u = f(M)$ funksiyaning M nuqtadagi Δu to'liq to'liq orttirmasi ushbu

$$\Delta u = A_1 \Delta x_1 + A_2 \Delta x_2 + \dots + A_m \Delta x_m + \alpha_1 \Delta x_1 + \alpha_2 \Delta x_2 + \dots + \alpha_m \Delta x_m \quad (10.15.2)$$

ko'rinishda tasvirlansa (bunda $A_1, A_2, \dots, A_m$ lar mos ravishda $\Delta x_1, \Delta x_2, \dots, \Delta x_m$ larga bog'liq bo'lmagan sonlar, $\alpha_1, \alpha_2, \dots, \alpha_m$ lar $\Delta x_1 \rightarrow 0, \Delta x_2 \rightarrow 0, \dots, \Delta x_m \rightarrow 0$ da cheksiz kichik ifodalar bo'lib, $\Delta x_1 = \Delta x_2 = \dots = \Delta x_m = 0$ da nolga teng deb olinadi), u holda $u = f(M)$ nuqtada differensiallanuvchi deyiladi.

(10.15.2) ni boshqacha ham yozish mumkin. Funksiyaning $M(x_1, x_2, \dots, x_m)$ nuqtadagi differensiallanuvchilik sharti (10.15.2) quyidagicha

$$\Delta u = A_1 \Delta x_1 + A_2 \Delta x_2 + \dots + A_m \Delta x_m + o(\rho) \quad (10.15.3)$$

ko'rinishida ham ifodalash mumkinligini ko'rsatamiz, bunda $M(x_1, x_2, \dots, x_m)$ va $M_1(x_1 + \Delta x_1, x_2 + \Delta x_2, \dots, x_m + \Delta x_m)$ nuqtalar orasidagi masofa:

$$\rho = \sqrt{(\Delta x_1)^2 + (\Delta x_2)^2 + \dots + (\Delta x_m)^2},$$

bunda $\Delta x_1 \rightarrow 0, \Delta x_2 \rightarrow 0, \dots, \Delta x_m \rightarrow 0$ da $\rho \rightarrow 0$. Endi $\Delta x_1 \rightarrow 0, \Delta x_2 \rightarrow 0, \dots, \Delta x_m \rightarrow 0$

da (10.15.2) dagi $\alpha_1 \Delta x_1 + \alpha_2 \Delta x_2 + \dots + \alpha_m \Delta x_m$ miqdorning ρ ga nisbatan yuqori tartibli cheksiz kichik miqdor ekanligini ko'rsatamiz, ρ

bo'lganda $\frac{|\Delta x_i|}{\rho} \leq 1 (i = \overline{1, m})$ tengsizlik o'rinli, shuning uchun

$$\begin{aligned} |\alpha_1 \Delta x_1 + \alpha_2 \Delta x_2 + \dots + \alpha_m \Delta x_m| &\leq \rho \left\{ |\alpha_1| \frac{|\Delta x_1|}{\rho} + |\alpha_2| \frac{|\Delta x_2|}{\rho} + \dots + |\alpha_m| \frac{|\Delta x_m|}{\rho} \right\} \leq \\ &\leq (|\alpha_1| + |\alpha_2| + \dots + |\alpha_m|) \rho = o(\rho). \end{aligned}$$

Shunday qilib differensiallanuvchi funksiyaning (10.15.2) differensiallanuvchilik shartini (10.15.3) ko'rinishida ham ifodalash mumkin ekan. (10.15.2) shart bilan (10.15.3) o'zaro ekvivalent.

10.15.4. Teorema. Agar $u = f(M)$ funksiya M nuqtada differensiallanuvchi bo'lsa, u holda uning barcha argumentlari bo'yicha M nuqtadagi xususiy hosilalari mavjud va $A_i \in (i = \overline{1, m})$ bo'ladi, bunda $A_i (i = \overline{1, m})$ (10.15.2) yoki (10.15.3)

orinidan aniqlanadi.

Isboti. Shartga ko'ra $u = f(M)$ funksiya M nuqtada differensiallanuvchi bo'lgani uchun (10.15.2) shartdan

$$\Delta x_i \neq 0, \Delta x_1 = \Delta x_2 = \dots = \Delta x_{i-1} = \Delta x_{i+1} = \dots = \Delta x_m = 0$$

bo'lsak, u holda (10.15.2) dan $\Delta x_i u = A_i \Delta x_i + \alpha_i \Delta x_i$, bundan

" $1 + \alpha_i(\Delta x_i)$, $\Delta x_i \rightarrow 0$ da $\alpha_i(\Delta x_i) \rightarrow 0$ bo'lgani uchun

$$\lim_{\Delta x_i \rightarrow 0} \frac{\Delta x_i u}{\Delta x_i} = A_i, \frac{\partial u}{\partial x_i} = A_i (i = \overline{1, m}). \blacksquare$$

10.15.4. Natija. $u = f(M)$ funksiyaning M nuqtadagi (10.15.3) differensiallanuvchanlik shartini

$$\Delta u = \frac{\partial u}{\partial x_1} \Delta x_1 + \frac{\partial u}{\partial x_2} \Delta x_2 + \dots + \frac{\partial u}{\partial x_m} \Delta x_m + o(\rho)$$

orinida yozish mumkin.

10.5.5. Teorema. Agar $u = f(x_1, x_2, \dots, x_m)$ funksiya M nuqtada differensiallanuvchi bo'lsa, u shu nuqtada uzluksiz bo'ladi.

Isboti. $u = f(x_1, x_2, \dots, x_m)$ funksiya M nuqtada differensiallanuvchi bo'lsin. U holda, 10.15.1-ta'rifdagi (10.15.2) shart o'rinli bo'ladi. (10.15.3) dan

$$\lim_{\substack{\Delta x_1 \rightarrow 0 \\ \Delta x_2 \rightarrow 0 \\ \dots \\ \Delta x_m \rightarrow 0}} \Delta u = 0.$$

bundan $u = f(M)$ funksiyaning M nuqtada uzluksizligi kelib chiqadi. ■

10.5.6. Teorema. Agar $u = f(x_1, x_2, \dots, x_m)$ funksiya $\{M\} \subset R^m$ da aniqlangan bo'lib, funksiya $M_0(x_1^0, x_2^0, \dots, x_m^0) \in \{M\}$ nuqtaning biror atrofida barcha argumentlari bo'yicha xususiy hosilalarga ega bo'lib, bu xususiy

hosilalar M_0 nuqtada uzluksiz bo'lsa, u shu nuqtada differensiallanuvchi bo'ladi.

Isboti. Yozishni qisqartirish maqsadida, teoremaning isbotini o'zgaruvchili $u = f(x, y)$ funksiya uchun olib boramiz. Shartga ko'ra f va f' xususiy hosilalar $M_0(x_0, y_0)$ nuqtaning atrofida mavjud va ular M_0 nuqtada uzluksiz. x va y argumentlarga istalgancha kichik orttir beraylik, natijada $M(x_0 + \Delta x, y_0 + \Delta y)$ nuqta qarayotgan oraliq tashaqariga chiqib ketmasin. Bu holda, funksiyaning Δu orttirmasini quyidagicha ko'rinishda yozib olamiz:

$$\Delta u = [f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)] + [f(x_0, y_0 + \Delta y) - f(x_0, y_0)] \quad (10.15.7)$$

$[f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)]$ ifodani $[x_0, x_0 + \Delta x]$ oraliqda x ning o'zgaruvchili funksiyasi sifatida qarash mumkin. $u = f(x, y)$ funksiyasi xususiy hosilalarga ega bo'lganligi uchun, ko'rsatilgan $u = f(x, y_0 + \Delta y)$ funksiya differensiallanuvchi va uning x bo'yicha hosilasi f'_x dan iborat bo'ladi. $[f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)]$ ayirmaga chekli ayirma haqidagi Lagranj teoremasini qo'llaymiz:

$$\begin{aligned} & [f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)] = \\ & = f'_x(x_0 + \theta_1 \Delta x, y_0 + \Delta y) \Delta x, \quad (0 < \theta_1 < 1) \end{aligned} \quad (10.15.8)$$

Xuddi shunday mulohazadan so'ng:

$$[f(x_0, y_0 + \Delta y) - f(x_0, y_0)] = f'_y(x_0, y_0 + \theta_2 \Delta y) \Delta y, \quad (0 < \theta_2 < 1) \quad (10.15.9)$$

Shartga ko'ra f'_x va f'_y hosilalar $M_0(x_0, y_0)$ nuqtada uzluksiz bolgani uchun:

$$\begin{aligned} & f'_x(x_0 + \theta_1 \Delta x, y_0 + \Delta y) - f'_x(x_0, y_0) + \alpha, \\ & f'_y(x_0, y_0 + \theta_2 \Delta y) - f'_y(x_0, y_0) + \beta, \end{aligned} \quad (10.15.10)$$

bunda $\Delta x \rightarrow 0, \Delta y \rightarrow 0$ da α va β lar cheksiz kichik funksiyalar. (10.15.7), (10.15.8), (10.15.9) va (10.15.10) ifodalarni e'tiborga olgan hol funksiyaning orttirmasini ushbu

$$\Delta u = f'_x(x_0, y_0) \Delta x + f'_y(x_0, y_0) \Delta y + \alpha \Delta x + \beta \Delta y$$

ko'rinishga keltiramiz. Bundan $u = f(x, y)$ funksiyaning $M_0(x_0, y_0)$ nuqtada differensiallanuvchi ekanligi kelib chiqadi. ■

qaynay qilib, 10.15-banddagi teoremlardan quidagi xulosalarni
 olish mumkin:

1) Bu o'zgaruvchili funksiyalarda ham, ko'p o'zgaruvchili
 funksiyalarda ham funksiyaning biror nuqtada differensiallanuvchi
 bo'ladan uning chu nuqtada uzluksiz bo'lishi kelib chiqadi.

2) Ma'lumki, bir o'zgaruvchi funksiyalarda funksiyaning biror
 nuqtada differensiallanuvchi bo'lishidan uning shu nuqtada chekli
 hosilaga ega bo'lishi kelib chiqadi va aksincha funksiyaning biror nuqtada
 chekli hosilaga ega bo'lishidan uning shu nuqtada differensiallanuvchi
 bo'lishi kelib chiqadi.

3) Ko'p o'zgaruvchili funksiyalarda esa, funksiyaning biror nuqtada
 differensiallanuvchi bo'lishidan shu nuqtada barcha chekli xususiy
 hosilalarga ega bo'lishi kelib chiqadi, lekin funksiyaning biror nuqtada
 barcha chekli xususiy hosilalarga ega bo'lishidan uning shu nuqtada
 differensiallanuvchi bo'lishi har doim ham kelib chiqavermaydi

10.16. Ko'p o'zgaruvchili funksiyaning differensial
 tushunchasi. $u = f(M) = f(x_1, x_2, \dots, x_m)$ funksiya ochiq $\{M\} \subset R^m$ to'plamda
 aniqlangan bo'lib, u $M \in \{M\}$ nuqtada differensiallanuvchi bo'lsin.

10.16.1 ta'rifga ko'ra uning to'liq orttirmasi Δu uchun (10.15.3) formula

to'g'ri bo'ladi. Unda $A_i = \frac{\partial u(M)}{\partial x_i} (i = \overline{1, m})$. (10.15.3) tenglikning o'ng

qismidagi ikki qismdan iborat: 1) $\Delta x_1, \Delta x_2, \dots, \Delta x_m$ larga nisbatan chiziqli

$(A_1 \Delta x_1 + A_2 \Delta x_2 + \dots + A_m \Delta x_m)$ ifodadan; 2) $\Delta x_1 \rightarrow 0, \Delta x_2 \rightarrow 0, \dots, \Delta x_m \rightarrow 0$ da

$\alpha_1 \rightarrow 0, \alpha_2 \rightarrow 0, \dots, \alpha_m \rightarrow 0$, yoki $\rho \rightarrow 0$ da ρ ga nisbatan yuqori tartibli cheksiz

kichik ifodalar $o(\rho)$ dan iborat.

10.16.1. Ta'rif. $u = f(M)$ funksiya orttirmasi Δu ning $\Delta x_1, \Delta x_2, \dots, \Delta x_m$
 larga nisbatan chiziqli bosh qismi, $u = f(M)$ funksiyaning M nuqtadagi
 differensial (to'liq differensial) deb ataladi va du, df yoki $df(M)$ kabi
 belgilanadi.

Demak,

$$du = df = df(x_1, x_2, \dots, x_m) = A_1 \Delta x_1 + A_2 \Delta x_2 + \dots + A_m \Delta x_m.$$

$(t_1, t_2, \dots, t_m) \in \{N\}$ bo'lganda unga mos kelgan $(x_1, x_2, \dots, x_m) \in \{M\}$ bo'lib, natijada, ushbu

$$u = f(\varphi_1(t_1, t_2, \dots, t_k), \varphi_2(t_1, t_2, \dots, t_k), \dots, \varphi_m(t_1, t_2, \dots, t_k))$$

ni tabiiy funksiyaga ega bo'lamiz.

10.17.2. Teorema. Agar (10.17.1) fnksiyalarning har biri $(t_1^0, t_2^0, \dots, t_k^0) \in \{N\}$ nuqtada differensiallanuvchi bo'lib, $y = f(x_1, x_2, \dots, x_m)$ natijada unga mos $M_0(x_1^0, x_2^0, \dots, x_m^0) \in \{M\}$

$$(x_1^0 = \varphi_1(t_1^0, t_2^0, \dots, t_k^0), x_2^0 = \varphi_2(t_1^0, t_2^0, \dots, t_k^0), \dots, x_m^0 = \varphi_m(t_1^0, t_2^0, \dots, t_k^0))$$

nuqtada differensiallanuvchi bo'lsa, u holda murakkab funksiya $(t_1, t_2, \dots, t_k) \in \dots, \varphi_m(t_1, t_2, \dots, t_k)$, ham $N_0(t_1^0, t_2^0, \dots, t_k^0)$ nuqtada differensiallanuvchi bo'ladi va uning xususiy hosilalari ushbu

$$\begin{aligned} \frac{\partial u}{\partial t_1} &= \frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial t_1} + \frac{\partial u}{\partial x_2} \frac{\partial x_2}{\partial t_1} + \dots + \frac{\partial u}{\partial x_m} \frac{\partial x_m}{\partial t_1}, \\ \frac{\partial u}{\partial t_2} &= \frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial t_2} + \frac{\partial u}{\partial x_2} \frac{\partial x_2}{\partial t_2} + \dots + \frac{\partial u}{\partial x_m} \frac{\partial x_m}{\partial t_2}, \\ &\dots \dots \dots \\ \frac{\partial u}{\partial t_k} &= \frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial t_k} + \frac{\partial u}{\partial x_2} \frac{\partial x_2}{\partial t_k} + \dots + \frac{\partial u}{\partial x_m} \frac{\partial x_m}{\partial t_k}. \end{aligned} \quad (10.17.3)$$

Formula orqali topiladi.

Isboti. $N_0(t_1^0, t_2^0, \dots, t_k^0)$ nuqtaning argumentlariga shunday $(t_1^0 + \Delta t_1, t_2^0 + \Delta t_2, \dots, t_k^0 + \Delta t_k)$ orttirmalar beraylikki, natijada $(t_1^0 + \Delta t_1, t_2^0 + \Delta t_2, \dots, t_k^0 + \Delta t_k) \in \{N\}$ bo'lsin. U holda (10.17.1) dagi funksiyalarning har biri $\Delta x_1, \Delta x_2, \dots, \Delta x_m$ orttirmalarga ega bo'ladi. O'z navbatida $\Delta x_1, \Delta x_2, \dots, \Delta x_m$ orttirmalarga $u = f(x_1, x_2, \dots, x_m)$ funksiyaning $M_0(x_1^0, x_2^0, \dots, x_m^0)$ nuqtadagi Δu orttirmasi mos keladi. Shartga ko'ra $f(x_1, x_2, \dots, x_m)$ funksiya $M_0(x_1^0, x_2^0, \dots, x_m^0)$ nuqtada differensiallanuvchi bo'lgani uchun bu funksiyaning shu nuqtadagi Δu orttirmasini ushbu

$$\Delta u = \frac{\partial u}{\partial x_1} \Delta x_1 + \frac{\partial u}{\partial x_2} \Delta x_2 + \dots + \frac{\partial u}{\partial x_m} \Delta x_m + \alpha_1 \Delta x_1 + \alpha_2 \Delta x_2 + \dots + \alpha_m \Delta x_m \quad (10.17.4)$$

ko'rinishda yozish mumkin, bunda $\frac{\partial u}{\partial x_i} (i=1, 2, \dots, m)$ larning har biri M_0 nuqtada olingan $\Delta x_i \rightarrow 0 (i=1, 2, \dots, m)$ da α_i larning har biri cheksiz kichik

funksiyalar, $\Delta x_1 = \Delta x_2 = \dots = \Delta x_m = 0$ bo'lganda esa nolga teng. Ma'lum (10.17.4) ifodadagi $\Delta x_1, \Delta x_2, \dots, \Delta x_m$ lar $\Delta t_1, \Delta t_2, \dots, \Delta t_k$ larga mos kel (10.17.1) dagi funksiyalarning orttirmalari bo'lib hisoblanadi. (10.17.1) dagi funksiyalarning $N_0(t_1^0, t_2^0, \dots, t_k^0)$ nuqtada differensiallanuvchi bo'lishligida $\Delta x_i (i = 1, 2, \dots, m)$ orttirmani quyidagi

$$\Delta x_i = \frac{\partial x_i}{\partial t_1} \Delta t_1 + \frac{\partial x_i}{\partial t_2} \Delta t_2 + \dots + \frac{\partial x_i}{\partial t_k} \Delta t_k + o(\rho) \quad (10.17.4)$$

ko'rinishda yozish mumkin, bundagi $\frac{\partial x_i}{\partial t_1}, \frac{\partial x_i}{\partial t_2}, \dots, \frac{\partial x_i}{\partial t_k}$ xususiy hosil

$N_0(t_1^0, t_2^0, \dots, t_k^0)$ nuqtada olingan, $\rho = \sqrt{\Delta t_1^2 + \Delta t_2^2 + \dots + \Delta t_k^2}$ (10.17.5) ni

o'ng tomonini (10.17.4) ning o'ng tomoniga olib borib quyganimizda ning ko'rinishi quyidagi $\Delta u = A_1 \Delta t_1 + A_2 \Delta t_2 + \dots + A_k \Delta t_k + o(\rho)$ shaklga keladi bunda

$$A_i = \frac{\partial u}{\partial x_1} \cdot \frac{\partial x}{\partial t_i} + \frac{\partial u}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_i} + \dots + \frac{\partial u}{\partial x_m} \cdot \frac{\partial x_m}{\partial t_i} \quad (i = 1, 2, \dots, k). \quad (10.17.6)$$

Shunday qilib, (10.17.5) dan berilgan funksiyaning $N_0(t_1^0, t_2^0, \dots, t_k^0)$ nuqtada differensiallanuvchi ekanligi kelib chiqadi, (10.17.6) esa un xususiy hosilalarini topish formulasi. ■

10.17.7. Eslatma. 1) (10.17.4) dagi hamma $\frac{\partial u}{\partial x_i} (i = 1, 2, \dots, m)$ xususiy hosilalar $M_0(x_1^0, \dots, x_m^0)$ nuqtada olingani uchun ularning hammasi bitta o'zgarmas sonlarni ifoda qiladi. Shuning uchun ularning $o(\rho)$ ko'paytmasi yana $o(\rho)$ ni ifoda qiladi. 2) $\Delta x_i (i = 1, 2, \dots, m)$ larning hammasi $|\Delta x_i| \leq \text{Const} \cdot \rho$ tengsizlikni qanoatlantiradi. 3) (10.17.4) dagi α_i lar hammasi $\rho \rightarrow 0$ da cheksiz kichik funksiyalar. 4) Har bir $\alpha_i \Delta x_i$ ko'paytmasi $o(\rho)$ ni ifoda qiladi. Bu 2) va 3) punktlardan bevosita kelib chiqadi.

10.17.8. Eslatma. Xususiy holda, (10.17.1) dagi funksiyalarning biri faqat bitta t ga bog'liq bo'lsa, u holda biz faqat t ga bog'liq bo'lgan $u = f(x_1, x_2, \dots, x_m)$, $x_i = \varphi_i(t)$ murakkab funksiyaga ega bo'lamiz. Murakkab funksiyaning hosilasi

$$\frac{du}{dt} = \frac{du}{dx_1} \cdot \frac{dx_1}{dt} + \frac{du}{dx_2} \cdot \frac{dx_2}{dt} + \dots + \frac{du}{dx_m} \cdot \frac{dx_m}{dt} \quad (10.17.7)$$

qada to'lan topiladi.

10.18. Bir jinsli funksiya tushunchasi. $u = f(x_1, x_2, \dots, x_m)$

$x_1, x_2, \dots, x_m \in \{M\} \subset R^m$ to'plamda aniqlangan bo'lib, $M(x_1, x_2, \dots, x_m)$ nuqta to'ldan $M_1(t, x_1, t, x_2, \dots, t, x_m) (-\infty < t < \infty)$ nuqta ham $\{M\}$ to'plamga kirishi bo'lsin.

10.18.1. Ta'rif. Agar $\{M\}$ to'plamning har bir nuqtasida va har bir t uchun

$$f(tx_1, tx_2, \dots, tx_m) = t^p f(x_1, x_2, \dots, x_m) \quad (10.18.2)$$

qabul bo'lsa, $\{M\}$ to'plamda $u = f(x_1, x_2, \dots, x_m)$ funksiya p -darajali bir jinsli funksiya deyiladi.

Misolalar. 1) $f(x, y) = x \frac{\sqrt{x^4 + y^4}}{x - y} \ln \frac{x}{y}$ funksiya ikkinchi darajali bir jinsli funksiya bo'ladi. Haqiqatan ham, ya'ni

$$f(tx, ty) = tx \frac{\sqrt{(tx)^4 + (ty)^4}}{tx - ty} \ln \frac{tx}{ty} = t^2 f(x, y).$$

2) $f(x, y) = e^{\sin xy}$ funksiya uchun (10.18.2) shart bajarilmaydi.

Demak bu funksiya bir jinsli funksiya emas.

Agar $u = f(x_1, x_2, \dots, x_m)$ funksiya nolinchi tartibli birjinsli funksiya

bo'lsa, ya'ni $f(tx_1, tx_2, \dots, tx_m) = f(x_1, x_2, \dots, x_m)$, u holda, $t = \frac{1}{x_1}$ deb olsak,

$$f(x_1, x_2, \dots, x_m) = f\left(1, \frac{x_2}{x_1}, \dots, \frac{x_m}{x_1}\right)$$

qabul bo'lamiz. Agar $n-1$ argumentli ushbu

$$\varphi(x_1, x_2, \dots, x_{n-1}) = f(1, u_1, u_2, \dots, u_{n-1})$$

funktsiyani kiritsak, u holda $f(x_1, x_2, \dots, x_n) = \varphi\left(\frac{x_2}{x_1}, \frac{x_3}{x_1}, \dots, \frac{x_n}{x_1}\right)$ bo'ladi.

Demak, har qanday nolinchi tartibli bir jinsli funksiya hamma argumentlarning ularning birortasiga nisbati shaklida tasvirlanar ekan.

Agar $u = f(x_1, x_2, \dots, x_m)$ funksiya m -tartibli bir jinsli funksiya bo'lsa,

u holda, $\frac{f(x_1, x_2, \dots, x_m)}{x_1^m}$ funksiya nolinchi tartibli bir jinsli funksiya bo'ladi,

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$$\frac{f(x_1, x_2, \dots, x_m)}{x_1^m} = \varphi\left(\frac{x_2}{x_1}, \frac{x_3}{x_1}, \dots, \frac{x_m}{x_1}\right).$$

Shunday qilib, m -tartibli bir jinsli funksiyani umumiy ko'rin

$$f(x_1, x_2, \dots, x_m) = x_1^m \varphi\left(\frac{x_2}{x_1}, \frac{x_3}{x_1}, \dots, \frac{x_m}{x_1}\right)$$

shaklida bo'lar ekan.

10.18.3. Teorema (Bir jinsli funksiyalar haqida Eyler teoremi)

Agar $u = f(x_1, x_2, \dots, x_m)$ funksiya ochiq $\{M\}$ da differensiallanuvchi darajali bir jinsli funksiya bo'lsa, u holda $\{M\}$ to'plamning har $M(x_1, x_2, \dots, x_m)$ nuqtasida

$$\frac{\partial u}{\partial x_1} x_1 + \frac{\partial u}{\partial x_2} x_2 + \dots + \frac{\partial u}{\partial x_m} x_m = pu \quad (10.18)$$

tenglik o'rinli bo'ladi.

Isboti. Ixtiyoriy $M_0(x_1^0, x_2^0, \dots, x_m^0) \in \{M\}$ nuqta bo'lsin. Ushbu $u = f(x_1, x_2, \dots, x_m)$ murakkab funksiyani qaraymiz, bu $x_i = tx_i^0$ ($i = 1, 2, \dots, m$), ya'ni $u = f(tx_1^0, tx_2^0, \dots, tx_m^0)$ funksiyani qaraymiz. $u = f(x_1, x_2, \dots, x_m)$ funksiya M_0 nuqtada differensiallanuvchi bo'lgan uchun, u holda 10.17.2-teorema va unga keltirilgan 10.17.8-eslatma e'tiborga olsak, u holda $t = 1$ bo'lganda berilgan murakkab funksiyani $\frac{du}{dt}$ hosilasini (10.17.9) formula bilan hisoblaymiz: $\frac{dx_i}{dt} = x_i^0$ bo'lgani uchun

$$\left. \frac{du}{dt} \right|_{t=1} = \frac{\partial u}{\partial x_1} x_1^0 + \frac{\partial u}{\partial x_2} x_2^0 + \dots + \frac{\partial u}{\partial x_m} x_m^0, \quad (10.18.5)$$

bunda $\frac{\partial u}{\partial x_i}$ hosila M_0 nuqtada xisoblangan. $u = f(tx_1, tx_2, \dots, tx_m)$ murakkab funksiya (10.18.2) ga asosan ushbu

$$u = f(tx_1^0, tx_2^0, \dots, tx_m^0) = t^p f(x_1^0, x_2^0, \dots, x_m^0) \quad (10.18.6)$$

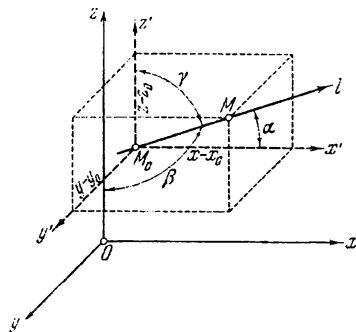
ko'rinishda tasvirlanadi. (10.18.6) dan

$$\frac{du}{dt} = pt^{p-1} f(x_1^0, x_2^0, \dots, x_m^0), \text{ ya'ni } \left. \frac{du}{dt} \right|_{t=1} = pf(x_1^0, x_2^0, \dots, x_m^0) = pu. \quad (10.18.7)$$

(10.18.5) bilan (10.18.7) ni solishtirish natijasida $t = 1$ bo'lgan (10.18.4) ni hosil qilamiz. $M_0 \in \{M\}$ to'plamda ixtiyoriy nuqta bo'lgan uchun teorema isbot bo'ladi. ■

10.19. Yo'nalish bo'yicha hosila. Gradiyen. Ma'lumki, bir o'zgaruvchili $f(x)$ funksiyaning $(x \in R, y \in R) \frac{df}{dx}$ hosilasi berilgan bo'lsa, u o'zgarish tezligini bildiradi. Ko'p o'zgaruvchili $f(x_1, x_2, \dots, x_m)$ funksiyaning $((x_1, x_2, \dots, x_m) \in R^m, u \in R)$ hususiy nuqtada ham bir o'zgaruvchili funksiyaning hosilasi kabi ekanligini ko'rib olib, bu $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_m}$ funksiyalar ham $u = f(x_1, x_2, \dots, x_m)$ bo'lgan holda mos ravishda $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_m}$ o'qlar bo'yicha R^m da o'zgarish tezligini ifodalaydi deb qabul qilish mumkin. Endi ko'p o'zgaruvchili funksiyaning ixtiyoriy nuqtada bo'yicha o'zgarish tezligini ifodalovchi tushuncha bilan shu hamiz. Soddalik uchun uch o'zgaruvchili funksiyani qaraymiz.

$u(x, y, z) = f(M)$ funksiya ochiq



10.4-chizma.

$u: R^3$ to'plamda berilgan bo'lsin. Birlik vektori $\vec{r}(\cos\alpha, \cos\beta, \cos\gamma)$ bilan aniqlanadigan biror ℓ yo'nalishni qaraylik. $\{M\}$ to'plamda ixtiyoriy $M_0(x_0, y_0, z_0)$ nuqtani olib, bu nuqta orqali o'tuvchi, yo'nalishi $(\cos\alpha, \cos\beta, \cos\gamma)$ vektor yo'nalishga mos kelgan $\vec{\ell}$ yo'nalishni qaraylik (10.4-chizma). $M_0(x_0, y_0, z_0)$ nuqtaga yaqin $\vec{\ell}$ yo'nalishda yotgan ixtiyoriy o'zgaruvchi $M(x, y, z)$ nuqtani olamiz ($M \in \{M\}$). M_0M kesma $u: R^3$ to'plamga tegishli.

Ma'lumki,

$$\frac{x - x_0}{\rho(M_0, M)} = \cos\alpha, \quad \frac{y - y_0}{\rho(M_0, M)} = \cos\beta, \quad \frac{z - z_0}{\rho(M_0, M)} = \cos\gamma \quad (10.19.1)$$

10.19.2. Ta'rif. Agar $M(x, y, z)$ nuqta $\vec{\ell}$ yo'nalgan to'g'ri chiziq bo'ylab $M_0(x_0, y_0, z_0)$ nuqtaga intilganda ($M \rightarrow M_0$) ushbu nisbatning

$$\frac{f(M) - f(M_0)}{\rho(M_0, M)} = \frac{f(x, y, z) - f(x_0, y_0, z_0)}{\rho((x_0, y_0, z_0), (x, y, z))}$$

limiti mavjud bo'lsa, bu limitga $f(x, y, z) = f(M)$ funksiya $M_0(x_0, y_0, z_0)$ nuqtadagi $\bar{\ell}$ yo'nalish bo'yicha hosilasi deb ataladi va

$$\frac{\partial f(M_0)}{\partial \ell} \text{ yoki } \frac{\partial f(x_0, y_0, z_0)}{\partial \ell}$$

kabi belgilanadi.

Demak, ta'rifga ko'ra,

$$\frac{\partial f}{\partial \ell} = \lim_{M \rightarrow M_0} \frac{f(M) - f(M_0)}{\rho(M_0, M)}$$

bo'ladi.

10.19.3. Teorema. Agar $u = f(x, y, z) = f(M)$ funksiya o' $\{M\} \subset R^3$ to'plamda berilgan bo'lib, u $M_0(x_0, y_0, z_0) \in \{M\}$ nuq differensiallanuvchi bo'lsa, u holda funksiya shu nuqtada har qanda yo'nalish bo'yicha hosilaga ega bo'ladi va bu hosila ushbu

$$\begin{aligned} \frac{\partial f(M_0)}{\partial \ell} &= \frac{\partial f(x_0, y_0, z_0)}{\partial \ell} = \\ &= \frac{\partial f(x_0, y_0, z_0)}{\partial x} \cos \alpha + \frac{\partial f(x_0, y_0, z_0)}{\partial y} \cos \beta + \frac{\partial f(x_0, y_0, z_0)}{\partial z} \cos \gamma \end{aligned} \quad (10.19.4)$$

formula orqali topiladi.

Isboti. $u = f(x, y, z) = f(M)$ funksiya $M_0(x_0, y_0, z_0)$ nuqt differensiallanuvchi bo'lgani uchun uning to'liq orttirmasi uchun

$$\begin{aligned} \Delta u &= f(M) - f(M_0) = \\ &= \frac{\partial f(M_0)}{\partial x} (x - x_0) + \frac{\partial f(M_0)}{\partial y} (y - y_0) + \frac{\partial f(M_0)}{\partial z} (z - z_0) + o(\rho) \end{aligned}$$

formula o'rinli bo'ladi, bunda

$$\rho = \rho(M_0, M) = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$$

ning ikkala tomonini ρ ga bo'lamiz:

$$\begin{aligned} \frac{f(M) - f(M_0)}{\rho} &= \\ &= \frac{\partial f(M_0)}{\partial x} \cdot \frac{x - x_0}{\rho} + \frac{\partial f(M_0)}{\partial y} \cdot \frac{y - y_0}{\rho} + \frac{\partial f(M_0)}{\partial z} \cdot \frac{z - z_0}{\rho} + \frac{o(\rho)}{\rho} \end{aligned} \quad (10.19.5)$$

(10.19.1) ni e'tiborga olsak, (10.19.5) tenglik ushbu

$$\frac{f(M) - f(M_0)}{\rho(M_0, M)} = \frac{\partial f(M_0)}{\partial x} \cos \alpha + \frac{\partial f(M_0)}{\partial y} \cos \beta + \frac{\partial f(M_0)}{\partial z} \cos \gamma + \frac{o(\rho)}{\rho}$$

o'ziga keladi. Bu tenglikdan $M \rightarrow M_0$ da $\rho \rightarrow 0$ limitga o'tsak, unda

$$\lim_{M \rightarrow M_0} \frac{f(M) - f(M_0)}{\rho(M_0, M)} = \lim_{\rho \rightarrow 0} \frac{f(M) - f(M_0)}{\rho} =$$

$$= \frac{\partial f(M_0)}{\partial x} \cos \alpha + \frac{\partial f(M_0)}{\partial y} \cos \beta + \frac{\partial f(M_0)}{\partial z} \cos \gamma$$

Endi shunday qilib,

$$\frac{\partial f(M_0)}{\partial \ell} = \frac{\partial f(x_0, y_0, z_0) - f(M_0)}{\partial \ell} =$$

$$= \frac{\partial f(x_0, y_0, z_0)}{\partial x} \cos \alpha + \frac{\partial f(x_0, y_0, z_0)}{\partial y} \cos \beta + \frac{\partial f(x_0, y_0, z_0)}{\partial z} \cos \gamma$$

Endi ■

Misol: 1) Ushbu $u = f(x, y) = \arctg \frac{x}{y}$ funksiyani qaraylik. $\bar{\ell}$

nuqtakvadratning $M_0(1,1)$ nuqtasidan o'tuvchi va $(0,0)$ nuqtadan $M_0(1,1)$ nuqtasigacha qarab yo'nalgan bissektressadan iborat bo'lsin. Berilgan funksiyaning $M_0(1,1)$ nuqtadagi $\bar{\ell}$ yo'nalish bo'yicha hosilasi topilsin. Ushbu funksiya $M_0(1,1)$ nuqtada differensiallanuvchi bo'lgani uchun uning yo'nalish bo'yicha hosilasini (10.19.4) formula bo'yicha topamiz:

$$\frac{\partial f(1,1)}{\partial \ell} = \frac{\partial f(1,1)}{\partial x} \cos \frac{\pi}{4} + \frac{\partial f(1,1)}{\partial y} \cos \frac{\pi}{4} = \left(\frac{y}{x^2 + y^2} - \frac{x}{x^2 + y^2} \right)_{x=1, y=1} = \frac{\sqrt{2}}{2} = 0$$

ya'ni $\frac{\partial f(1,1)}{\partial \ell} = 0$

2) Ushbu $f(x, y) = \sqrt{x^2 + y^2}$ funksiyaning $M(0,0)$ nuqtada ixtiyoriy yo'nalish bo'yicha hosilasi $\frac{\partial f(0,0)}{\partial \ell} = 1$ ekanligini ko'rsating. Haqiqatan ham

bu

$$\frac{f(M) - f(M_0)}{\rho(M_0, M)} = \frac{\sqrt{x^2 + y^2}}{\rho} = \frac{\rho}{\rho} = 1,$$

ya'ni $\lim_{\rho \rightarrow 0} \frac{f(M) - f(M_0)}{\rho(M_0, M)} = 1.$

3) Ushbu $f(x, y) = x + |y|$ funksiyaning $M_0(0,0)$ nuqtada Ox o'qida o'qi bo'yicha hosilasi 1 ga teng bo'lib, Oy koordinata o'qi bo'yicha hosilasi mavjud emas.

10.19.6. Eslatma. Funksiya biror nuqtada differensiallanuvchi bo'lmasa ham, u shu nuqtada biror yo'nalish bo'yicha va hatto har qanday yo'nalish bo'yicha hosilaga ega bo'lishi mumkin. Masalan, $u = \sqrt{x^2 + y^2}$ funksiya $M_0(0,0)$ nuqtada differensiallanuvchi bo'lmaydi, lekin biz yuqorida ko'rdikki, bu funksiya $M_0(0,0)$ nuqtada ixtiyoriy yo'nalish bo'yicha hosilaga ega.

10.19.7. Ta'rif. Komponentlari (koordinatalari) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ bo'lgan vektorga, $u = f(x, y, z) = f(M)$ funksiyaning $M_0(x_0, y_0, z_0)$ nuqtada *gradiyenti* deb ataladi va u $gradu = \frac{\partial u}{\partial x} \vec{i} + \frac{\partial u}{\partial y} \vec{j} + \frac{\partial u}{\partial z} \vec{k}$ kabi belgilanadi, bunda $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ xususiy hosilalar $M_0(x_0, y_0, z_0)$ nuqtada hisoblanadi.

$\vec{r}(\cos\alpha, \cos\beta, \cos\gamma)$ ni e'tiborga olsak, u holda (10.19.4) ni ushbu

$$\frac{\partial u}{\partial \ell} = \left(\vec{r}, gradu \right) \quad (10.19.8)$$

ko'rinishda yozish mumkin. (10.19.8) formulani

$$\frac{\partial u}{\partial \ell} = |\vec{r}| \cdot |gradu| \cdot \cos\varphi$$

ko'rinishda ham yozish mumkin, bunda

$$|gradu| = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2},$$

$\varphi, \vec{r}$ bilan $gradu$ ning orasidagi burchak. $|\vec{r}|=1$ bo'lgani uchun $\frac{\partial u}{\partial \ell} = |gradu| \cdot \cos\varphi$. Bu formuladan ko'rinadiki, yo'nalish bo'yicha hosilaning

o'zining maksimum $\left(\frac{\partial u}{\partial \ell}\right)_{\max}$ qiymatiga $\cos\varphi=1$ bo'lganda erishadi, ya'ni

$\vec{r}$ vektorning yo'nalishi $gradu$ ning yo'nalishiga mos tushganda, $\left(\frac{\partial u}{\partial \ell}\right)_{\max} = |gradu|$. Agar $\vec{r}$ ning yo'nalishi $gradu$ ga perpendikulyar bo'lsa,

$\frac{\partial u}{\partial \ell} = 0$ bo'ladi, chunki $\varphi = \frac{\pi}{2}$, $\cos\frac{\pi}{2} = 0$.

10.10 Birinchi tartibli differensial formasining invariantligi.

10.10.16 bandda $u = f(x_1, x_2, \dots, x_m)$ funksiyaning argumentlari

bir tekli o'zgaruvchi bo'lganda uning differensial (10.16.2)

qanday holatda qilinishini ko'rgan edik. Endi biz bu bandda argument

erksiz bo'lib, ularning o'zlari ham $t_1, t_2, \dots, t_m$ larning

differensiallanuvchi funksiyalari bo'lgan holda ham (10.16.2) formula

qanday holatda qilinishini ko'rsatamiz. Odatda birinchi tartibli differensialning bu

sharti *birinchi tartibli differensial formasi invariantlig xossasi* deb

ataadi.

10.10.17 $u = f(x_1, x_2, \dots, x_m)$ funksiya $\{M\} \subset R^m$ to'plamda berilgan

10.10.18 Biz yuqorida ko'rgan edikki, agar $x_1, \dots, x_m$ argumentlar erkli

o'zgaruvchilar bo'lganda uning differensial (to'liq differensial)

$$du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 + \dots + \frac{\partial u}{\partial x_m} dx_m \quad (10.20.1)$$

shunda tasvirlanadi. Endi $x_1, x_2, \dots, x_m$ argumentlarni erksiz, ya'ni biror

N_0 nuqtada, $u = f(x_1, x_2, \dots, x_m)$ funksiya esa, $M_0(x_1^0, x_2^0, \dots, x_m^0)$

nuqtada differensiallanuvchi bo'lsin deb faraz qilaylik, bunda

$t_1, t_2, \dots, t_k$). Bu holda $u = f(x_1, x_2, \dots, x_m)$ funksiyaning $t_1, t_2, \dots, t_k$ larning

qanday holatda funksiyasi deb qaraymiz. 10.17.2-teoreмага asosan, bu

nuqtada funksiya N_0 nuqtada differensiallanuvchi bo'ladi va uning

differensial ushbu

$$du = \frac{\partial u}{\partial t_1} dt_1 + \frac{\partial u}{\partial t_2} dt_2 + \dots + \frac{\partial u}{\partial t_m} dt_m \quad (10.20.2)$$

shunda tasvirlanadi, bunda $\frac{\partial u}{\partial t_i}$ (10.17.3) formula orqali topiladi. $\frac{\partial u}{\partial t_i}$

(10.17.3) dan (10.20.2) ga olib kelib qo'ysak va $\frac{\partial u}{\partial x_i}$ larning

qanday holatda jamlash natijasida

$$\begin{aligned} du = & \frac{\partial u}{\partial x_1} \left(\frac{\partial x_1}{\partial t_1} dt_1 + \frac{\partial x_1}{\partial t_2} dt_2 + \dots + \frac{\partial x_1}{\partial t_k} dt_k \right) + \dots + \\ & + \frac{\partial u}{\partial x_m} \left(\frac{\partial x_m}{\partial t_1} dt_1 + \frac{\partial x_m}{\partial t_2} dt_2 + \dots + \frac{\partial x_m}{\partial t_k} dt_k \right) \end{aligned}$$

ni hosil qilamiz. Ma'lumki $\frac{\partial u}{\partial x_i}$ ($i = 1, 2, \dots, m$) ning ko'rsatishidir.
 $x_i = \varphi_i(t_1, t_2, \dots, t_k)$ funksiyaning dx_i differensialini ifodalaydi.

Demak, $x_1, x_2, \dots, x_m$ lar erksiz uzgaruvchilar bo'lganda $u = f(x_1, x_2, \dots, x_m)$ funksiyaning differensial (10.20.1) ko'rinishda bo'lgan, ya'ni ko'p o'zgaruvchili murakkab funksiya birinchi tartibli differensial formasi invariantligi (ko'rinishi) saqlanar ekan. Bir tartibli differensial formasining invariantligi quyidagi qoidalarining o'rinli ekanini ko'rsatishga imkon beradi u va v ko'p o'zgaruvchi funksiya differensiallanuvchi bo'lsa, u holda

$$d(Cu) = Cdu \quad (C = \text{const}),$$

$$d(u \pm v) = du \pm dv,$$

$$d(uv) = u dv + v du,$$

$$d\left(\frac{u}{v}\right) = \frac{v du - u dv}{v^2}$$

formulalar o'rinli bo'ladi. Bu formulalarning uchinchisining o'rinli ekanini (qolganlari xam shu usulda ko'rsatiladi) ko'rsatamiz. $w = f(u, v)$ funksiyaning qaraylik, bu funksiyaning u va v o'zgaruvchilarning funksiyasi deb qaralsa, u holda bu funksiyaning differensial

$$dw = \frac{\partial w}{\partial u} du + \frac{\partial w}{\partial v} dv$$

bo'ladi.

$$\frac{\partial w}{\partial u} = v, \quad \frac{\partial w}{\partial v} = u$$

bo'lgani uchun $dw = v du + u dv$. Birinchi tartibli differensial formasi invariantligining saklanish xossasiga asosan, $u dv + v du$ ifoda, u va v differensiallanuvchi funksiya bulganda $w = uv$ funksiya differensialini ifodalaydi.

16.3. Ko'p o'zgaruvchili funksiyaning yuqori tartibli xususiy hosilalari va differentsiallari

10.21 Yuqori tartibli xususiy hosilalar. $u = f(x_1, x_2, \dots, x_m) = f(M)$

bu erda M to'plam $\{M\} \subset R^m$ to'plamda berilgan bo'lib, uning har bir nuqtasida $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_m}$ xususiy hosilalarga ega bo'lsin. Bu xususiy hosilalari o'z navbatida $x_1, x_2, \dots, x_m$ o'zgaruvchilarning funksiyasi bo'lsa, $\{M\}$ to'plamda aniqlangan bo'lsin.

Endi $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_m}$ funksiyalar ham biror $M \in \{M\}$ nuqtada x_k argument bo'yicha xususiy hosilaga ega bo'lishi mumkin. Bu x_k argument bo'yicha olingan hosila berilgan $u = f(x_1, x_2, \dots, x_m)$ funksiyaning *k -tartibli xususiy hosilasi* deb ataladi va u $u^{(k)} = u^{(k)}_{x_1, x_2, \dots, x_m}$ ($i = 1, 2, \dots, m; k = 1, 2, \dots, m$) kabi belgilanadi, bunda $i \neq k$.

U holda $\frac{\partial^2 u}{\partial x_k \partial x_i}$ xususiy hosilaga *aralash xususiy hosila* deyiladi,

bu holda $\frac{\partial^2 u}{\partial x_k \partial x_k}$ deb yozish o'rniga $\frac{\partial^2 u}{\partial x_k^2} = f''_{x_k^2}$ deb yoziladi.

Xuddi shunday, $f(x_1, x_2, \dots, x_m)$ funksiyaning uchinchi, to'rtinchi, va boshqa tartibli xususiy hosilalari ta'riflanadi. $f(x_1, x_2, \dots, x_m)$ funksiya $x_{i_1}, x_{i_2}, \dots, x_{i_{n-1}}$ argumentlari bo'yicha $(n-1)$ tartibli xususiy hosilalarga ega bo'ladi. Bu $(n-1)$ tartibli xususiy hosilalar ham $M(x_1, x_2, \dots, x_m) \in \{M\}$ nuqtada x_{i_n} argumenti bo'yicha hosilaga ega bo'lsin. Bu hosila $f(x_1, x_2, \dots, x_m)$ funksiyaning $x_{i_1}, x_{i_2}, \dots, x_{i_{n-1}}$ argumentlar bo'yicha $M \in \{M\}$ nuqtadagi *n -tartibli xususiy hosilasi* deyiladi. Shunday qilib $x_{i_1}, x_{i_2}, \dots, x_{i_n}$ argumentlar bo'yicha n -tartibli xususiy hosilani

$$\frac{\partial^n u}{\partial x_{i_1} \partial x_{i_2} \dots \partial x_{i_n}} = \frac{\partial}{\partial x_{i_n}} \left(\frac{\partial^{n-1} u}{\partial x_{i_1} \dots \partial x_{i_{n-1}}} \right)$$

deb yozish mumkin.

Agar $i_1, i_2, \dots, i_n$ indekslarning hammasi birdaniga bir-biriga bo'lmasa, u holda $\frac{\partial^n u}{\partial x_{i_1} \dots \partial x_{i_2} \partial x_{i_1}}$ xususiy hosila n -tartibli *aralish* xususiy hosila deyiladi.

10.21.1. Misol. Ushbu

$$u = \operatorname{arctg} \frac{x}{y}$$

funksiyaning 2-tartibli xususiy hosilalarini toping.

Yechilishi. Berilgan funksiyaning 1- va 2-tartibli xususiy hosilalarini topamiz:

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{y}{x^2 + y^2}, \quad \frac{\partial u}{\partial y} = -\frac{x}{x^2 + y^2}, \\ \frac{\partial^2 u}{\partial x \partial y} &= \frac{x^2 - y^2}{(x^2 + y^2)^2}, \quad \frac{\partial^2 u}{\partial x^2} = -\frac{2xy}{(x^2 + y^2)^2}, \\ \frac{\partial^2 u}{\partial y^2} &= \frac{x^2 - y^2}{(x^2 + y^2)^2}, \quad \frac{\partial^2 u}{\partial x^2} = -\frac{2x}{(x^2 + y^2)^2}. \end{aligned}$$

Bu 10.21.1-misolda $\frac{\partial^2 u}{\partial y \partial x}$ va $\frac{\partial^2 u}{\partial x \partial y}$ aralash xususiy hosilalar bir-biriga teng bo'ladi. Umumiy holda bu aralash xususiy hosilalar bir-biriga teng bo'lmashligi ham mumkin.

10.21.2. Misol. Ushbu

$$f(x, y) = \begin{cases} xy \frac{x^2 - y^2}{x^2 + y^2}, & x^2 + y^2 \neq 0, \\ 0, & x^2 + y^2 = 0 \end{cases}$$

funksiyaning $0(0;0)$ nuqtada $f''_{xx}(x, y)$, $f''_{xy}(x, y)$ hosilalarga ega va ular bir-biriga teng emasligini isbot qiling.

Yechilishi. Haqiqatan ham,

$$f'_x(x, y) = y \frac{x^2 - y^2}{x^2 + y^2} + \frac{4x^2 y^3}{(x^2 + y^2)^2}; \quad f'_y(x, y) = x \frac{x^2 y^2}{x^2 + y^2} - \frac{4x^3 y^2}{(x^2 + y^2)^2}.$$

Agar $x = y = 0$ bo'lsa, u holda $f'_x(0,0)$ va $f'_y(0,0)$ hosilalarni bevosita hisoblab, ularning ta'rifiga ko'ra topamiz:

$$f'_x(0,0) = \lim_{\Delta x \rightarrow 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{0}{\Delta x} = 0,$$

$$f'_y(0,0) = \lim_{\Delta y \rightarrow 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{0}{\Delta y} = 0$$

6) Quyidagi formulalardan foydalanib aralash hosilalarni topamiz:

$$f''_{xy}(0,0) = \lim_{\Delta y \rightarrow 0} \frac{f'_x(0, \Delta y) - f'_x(0,0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{-\Delta y^3}{\Delta y^3} = -1;$$

$$f''_{yx}(0,0) = \lim_{\Delta x \rightarrow 0} \frac{f'_y(\Delta x, 0) - f'_y(0,0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta x^3}{\Delta x^3} = 1.$$

7) Endan $f''_{xy}(0,0) \neq f''_{yx}(0,0)$ ekanligini topamiz.

Quyidagi keltirilgan misollardan ko'rinadiki, $\frac{\partial^2 f}{\partial x \partial y}$ va $\frac{\partial^2 f}{\partial y \partial x}$

darajali hosilalar teng bo'lishi ham teng bo'lmاسligi ham mumkin ekan.

Endi shunday savol tug'iladi, qanday shartlarda aralash hosilalar o'zaro ekanligini qanday tartibda olib borishga bog'liq bo'lmaydi. Bu savol quyidagi teorema javob beradi.

10.21.3. Teorema. $u = f(x, y)$ funksiya $\{M\} \subset R^2$ ochiq to'plamda ekanligi bo'lib, shu to'plamda $f'_x, f'_y, f''_{xy}, f''_{yx}$ hosilalarga ega bo'lsin. U holda aralash xususiy hosilalar $M_0(x_0, y_0) \in \{M\}$ nuqtada uzluksiz bo'lsa, u nuqtada bu nuqtada

$$f''_{xy}(x_0, y_0) = f''_{yx}(x_0, y_0) \quad (10.21.4)$$

o'rinli.

Isboti. Shartga ko'ra $u = f(x, y)$ funksiya $M_0(x_0, y_0)$ nuqtada ikki marta differentsiallanuvchi bo'lgani uchun f'_x va f'_y lar M_0 nuqtaning atrofida aniqlangan bo'lib, u bu nuqtada differentsiallanuvchi bo'ladi.

U holda

$$\Delta \varphi = f(x_0 + h, y_0 + h) - f(x_0 + h, y_0) - f(x_0, y_0 + h) + f(x_0, y_0) \quad (10.21.5)$$

ifodani qaraymiz, bunda h - ixtiyoriy kichik son, $M(x_0 + h, y_0 + h) \in \{M\}$.

(10.21.5) ifodani $[x_0, x_0 + h]$ oraliqda differentsiallanuvchi

funksiya $\varphi(t) = f(x_0 + t, y_0 + h) - f(x_0 + t, y_0)$ funksiyani $\Delta \varphi = \varphi(x_0 + h) - \varphi(x_0)$ ortirmasi ko'rinishida qarash mumkin. U holda Lagranj formulasiga asosan,

$$D = \Delta\varphi = \varphi'(x_0 + \theta h)h = [f'_x(x_0 + \theta h, y_0 + h) - f'_x(x_0 + \theta h, y_0)]h = \\ = \{f'_x(x_0 + \theta h, y_0 + h) - f'_x(x_0, y_0)\} - [f'_x(x_0 + \theta h, y_0) - f'_x(x_0, y_0)]h, \quad (10.21.6)$$

bunda $0 < \theta < 1$ dagi son. f'_x hosila $M_0(x_0, y_0)$ nuqtada differensiallan bo'lgani uchun (10.21.6) ning o'ng tomondagi o'rta qavus quyidagicha yozish mumkin:

$$[f'_x(x_0 + \theta h, y_0 + h) - f'_x(x_0, y_0)] = f''_{xx}(x_0, y_0)\theta h + f''_{xy}(x_0, y_0)h + \alpha_1\theta h + \beta_1h, \\ [f'_x(x_0 + \theta h, y_0) - f'_x(x_0, y_0)] = f''_{xx}(x_0, y_0)\theta h + \alpha_2\theta h, \quad (10.21.7)$$

bunda $h \rightarrow 0$ da $\alpha_1, \beta_1, \alpha_2$ - cheksiz kichik ifodalar. (10.21.7) (10.21.6) ga qo'shib

$$D = [f''_{xy}(x_0, y_0) + \alpha]h^2 \quad (10.21.8)$$

ni topamiz, bunda, $h \rightarrow 0$ da $\alpha = \alpha_1\theta + \beta_1 - \alpha_2\theta$ cheksiz kichik ifoda (10.21.5) bilan aniqlanadigan D ifodani $[y_0, y_0 + h]$ oralig'ida differensiallanuvchi $\psi(y) = f(x_0 + h, y) - f(x_0, y_0)$ funksiya $\Delta\psi = \psi(y_0 + h) - \psi(y_0)$ ortirmasi shaklida qarash mumkin. Bundan funksiya M_0 nuqtada differensiallanuvchi ekanligi e'tiborga olinib, holda Lagranj formulasini qo'llab, xuddi yuqoridagi singari D ifoda uchun

$$D = [f''_{yx}(x_0, y_0) + \beta]h^2, \quad (10.21.9)$$

$h \rightarrow 0$ da β - cheksiz kichik ifoda (10.21.8) bilan (10.21.9) ning o'ng tomonlarini tenglashtiramiz:

$$f''_{xy}(x_0, y_0) + \alpha = f''_{yx}(x_0, y_0) + \beta$$

$h \rightarrow 0$ da α va β - cheksiz ifodalar bo'lganligi uchun keyingi tenglik (10.21.4) tenglikni hosil qilamiz. ■

Aralash hosilalar uchun quyidagi umumiy teorema o'rinni tutadi.

10.21.10. Teorema. $u = f(x_1, x_2, \dots, x_m)$ funksiya $\{M\} \subset R^m$ ochiq to'plamda aniqlangan bo'lib, bu to'plamda mumkin bo'lgan ham $(n-1)$ tartibgacha xususiy hosilalarga va n -tartibli aralash xususiy hosilaga ega bo'lib, bu hosilalar $\{M\}$ to'plamda uzluksiz bo'lsa, u xususiy n -tartibli aralash hosila, hosilani topish tartibiga bogliq bo'lmaydi.

10.22. Yuqori tartibli differensiyallar. $u = f(x_1, x_2, \dots, x_m) = f(M)$ ochiq $\{M\} (\{M\} \subset R^m)$ to'plamda $f'_{x_1}, f'_{x_2}, \dots, f'_{x_m}$ xususiy hosilalarga bo'lsin.

Agar bu xususiy hosilalar $M_0 \in \{M\}$ nuqtada differensiallanuvchi bo'lsa, $f(M)$ funksiya shu nuqtada *ikki marta differensiallanuvchi* bo'lsa deyiladi.

Agar $u = f(x_1, x_2, \dots, x_m)$ funksiya $\{M\}$ to'plamda barcha $n-1$ tartibli xususiy hosilalarga ega bo'lib, bu xususiy hosilalar $M_0 \in \{M\}$ nuqtada differensiallanuvchi bo'lsa, $f(M)$ funksiya M_0 nuqtada *n -marta differensiallanuvchi funksiya* deyiladi.

10.22.1. Teorema. Agar ochiq $\{M\}$ to'plamda $f(M)$ funksiyaning barcha n -tartibli xususiy hosilalari mavjud va $M_0 \in \{M\}$ nuqtada uzluksiz bo'lsa, $f(M)$ funksiya M_0 nuqtada n -marta differensiallanuvchi bo'ladi.

Agar $u = f(x_1, x_2, \dots, x_m)$ funksiya ochiq $\{M\} (\{M\} \subset R^m)$ to'plamda differensiallanuvchi bo'lib, $u = f(x_1, x_2, \dots, x_m) \in \{M\}$ nuqtada differensiallanuvchi bo'lsa, holda bu funksiyaning M nuqtadagi differensial

$$du = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \dots + \frac{\partial f}{\partial x_m} dx_m \quad (10.22.2)$$

shunday bo'lishini yuqoridagi 10.16-bandda ko'rgan edik, bunda $dx_1, dx_2, \dots, dx_m$ lar $x_1, x_2, \dots, x_m$ o'zgaruvchilarning ixtiyoriy o'zgarishlaridir. Faraz qilaylik $f(M)$ funksiya $M \in \{M\}$ nuqtada ikki marta differensiallanuvchi bo'lsin.

10.22.3. Ta'rif. $u = f(M)$ funksiyaning M nuqtadagi differensial $df(M)$ ning differensialiga berilgan $f(M)$ funksiyaning ikkinchi tartibli differensial deyiladi va $u = d^2u$ yoki d^2f kabi belgilanadi. (10.22.2) ni qaytadan differensiallash qoidalaridan foydalanib quyidagini topamiz:

$$\begin{aligned} d^2f &= d(df) = d\left(\frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \dots + \frac{\partial f}{\partial x_m} dx_m\right) = \\ &= dx_1 d\left(\frac{\partial f}{\partial x_1}\right) + dx_2 d\left(\frac{\partial f}{\partial x_2}\right) + \dots + dx_m d\left(\frac{\partial f}{\partial x_m}\right) = \end{aligned}$$

$$\begin{aligned}
&= \left(\frac{\partial^2 f}{\partial x_1^2} dx_1 + \frac{\partial^2 f}{\partial x_1 \partial x_2} dx_2 + \dots + \frac{\partial^2 f}{\partial x_1 \partial x_m} dx_m \right) dx_1 + \left(\frac{\partial^2 f}{\partial x_2 \partial x_1} dx_1 + \frac{\partial^2 f}{\partial x_2^2} dx_2 + \dots + \frac{\partial^2 f}{\partial x_2 \partial x_m} dx_m \right) \\
&+ \dots + \left(\frac{\partial^2 f}{\partial x_m \partial x_1} dx_1 + \frac{\partial^2 f}{\partial x_m \partial x_2} dx_2 + \dots + \frac{\partial^2 f}{\partial x_m^2} dx_m \right) dx_m = \\
&= \frac{\partial^2 f}{\partial x_1^2} dx_1^2 + \frac{\partial^2 f}{\partial x_2^2} dx_2^2 + \dots + \frac{\partial^2 f}{\partial x_m^2} dx_m^2 + 2 \frac{\partial^2 f}{\partial x_1 \partial x_2} dx_1 dx_2 + 2 \frac{\partial^2 f}{\partial x_1 \partial x_3} dx_1 dx_3 + \dots \\
&+ 2 \frac{\partial^2 f}{\partial x_1 \partial x_m} dx_1 dx_m + \frac{2 \partial^2 f}{\partial x_2 \partial x_3} dx_2 dx_3 + 2 \frac{\partial^2 f}{\partial x_2 \partial x_4} dx_2 dx_4 + \dots + 2 \frac{\partial^2 f}{\partial x_2 \partial x_m} dx_2 dx_m \\
&+ \dots + 2 \frac{\partial^2 f}{\partial x_{m-1} \partial x_m} dx_{m-1} dx_m. \tag{10}
\end{aligned}$$

$f(x_1, x_2, \dots, x_m)$ funksiyaning $M(x_1, x_2, \dots, x_m)$ nuqtadagi uch to'rtinchi va hokazo differensiallari ham xuddi yuqoridagidek ta'rif Umumiy holda $f(M)$ funksiyaning M nuqtadagi $(n-1)$ ta differensial $d^{n-1}f(M)$ ning differensialligiga berilgan $f(M)$ funksiy shu nuqtadagi n -tartibli differensial deyiladi va u $d^n f$ kabi belgila

Shunday qilib, $d^n f = d(d^{n-1}f)$. Biz yuqorida $f(M)$ funksiy ikkinchi tartibli differensialining (10.22.4) ko'mishda tasvirla ko'rdik. Funksiyaning keyingi tartibli differensiallarining uning x1 hosilalari orqali ifodasi borgan sari murakkablashib boradi. Sh uchun yuqori tartibli differensiallarini, simvolik ravishda, sod qulayroq formada ifoda qilamiz:

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \dots + \frac{\partial f}{\partial x_m} dx_m$$

ni simvolik ravishda (f ni formal ravishda qavs tauqarisiga chiqar quyidagicha

$$df = \left(\frac{\partial}{\partial x_1} dx_1 + \frac{\partial}{\partial x_2} dx_2 + \dots + \frac{\partial}{\partial x_m} dx_m \right) f$$

yozamiz. U holda funksiyaning ikkinchi tartibli differensialini

$$d^2 f = \left(\frac{\partial}{\partial x_1} dx_1 + \frac{\partial}{\partial x_2} dx_2 + \dots + \frac{\partial}{\partial x_m} dx_m \right)^2 f$$

qavslar bo'yicha yozish mumkin. Bunda qavs ichidagi yig'indilar kvadratga ko'paytiriladi va f ga ko'paytiriladi. Bunda daraja ko'rsatgichlar xususiy qavslar tartibini bildiradi.

total d'ya usuli bo'yicha funksiyaning n -darajali differensialini quyidagi formula bilan ifodalash mumkin.

$$d^n f = \left(\frac{\partial}{\partial x_1} dx_1 + \frac{\partial}{\partial x_2} dx_2 + \dots + \frac{\partial}{\partial x_m} dx_m \right)^n f$$

qavslar bo'yicha yozish mumkin.

Quyidagi hollarda, ikki o'zgaruvchili funksiyalar uchun ushbu formula umumlashadi:

$$d^2 f = d(dz) = \frac{\partial^2 f(x,y)}{\partial x^2} dx^2 + 2 \frac{\partial^2 f(x,y)}{\partial x \partial y} dx dy + \frac{\partial^2 f(x,y)}{\partial y^2} dy^2,$$

$$d^3 f = d(d^2 z) = \frac{\partial^3 f(x,y)}{\partial x^3} dx^3 + 3 \frac{\partial^3 f(x,y)}{\partial x^2 \partial y} dx^2 dy +$$

$$+ 3 \frac{\partial^3 f(x,y)}{\partial x \partial y^2} dx dy^2 + \frac{\partial^3 f(x,y)}{\partial y^3} dy^3 = \left(dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y} \right)^3 f.$$

$$\dots \dots \dots$$

$$d^n z = \left(dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y} \right)^n f.$$

10.23. Ko'p o'zgaruvchili funksiyaning Teylor formulasi.

$f(x_1, x_2, \dots, x_m) = f(M)$ funksiya ochiq bog'lamli $\{M\} \subset R^n$ to'plamda aniqlangan bo'lsin. Bu funksiyaning k -tartibli differensialining $u \in \{M\}$ nuqtadagi qiymatini $d^k u|_{M_0}$ deb belgilaymiz.

10.23.1. Teorema. $u = f(M)$ funksiya $M_0(x_1^0, x_2^0, \dots, x_m^0)$ nuqtaning atrofidagi $U_\delta(M_0) \subset \{M\}$ $\delta > 0$ atrofida aniqlangan va $n+1$ marta differentsiyalanuvchi bo'lsin. U holda uning M_0 nuqtadagi to'liq differensial $\Delta u = f(M) - f(M_0)$ uchun ushbu

$$\Delta u = du|_{M_0} + \frac{1}{2!} d^2 u|_{M_0} + \dots + \frac{1}{n!} d^n u|_{M_0} + \frac{1}{(n+1)!} d^{n+1} u|_N \quad (10.23.2)$$

tenglik o'rinli bo'ladi, bunda $N - U_\delta(M_0)$ atrofida $M(x_1, x_2, \dots)$ nuqtaga bog'liq bo'lgan biror nuqta, $d^k u|_{M_0}$ va $d^{n+1} u|_N$ ifodalar qatnashuvchi dx_i lar $\Delta x_i = x_i - x_i^0$ ga teng.

(10.23.2) formulaga $u = f(x_1, x_2, \dots, x_m)$ funksiyaning M_0 nuqtasidagi Teylor formulasi deyiladi.

Isboti. Yozishni qiaqartirish maqsadida, isbotni $u = f(x, y)$ funksiyasi uchun olib boramiz. Ma'lumki, (5.29-bandga q.) biror t_0 nuqtasidagi atrofida $n+1$ marta differensiyalanuvchi $F(t)$ funksiyaning t_0 nuqtasidagi Teylor formulasi ushbu

$$F(t) = F(t_0) + F'(t_0)(t - t_0) + \frac{1}{2!} F''(t_0)(t - t_0)^2 + \dots + \frac{1}{n!} F^{(n)}(t_0)(t - t_0)^n + \frac{1}{(n+1)!} F^{(n+1)}(t_0 + \theta(t - t_0))(t - t_0)^{n+1} \quad (0 < \theta < 1) \quad (10.23.3)$$

ko'rinishida bo'ladi. Bunda, t -erkli o'zgaruvchi bo'lgani uchun u o'rtirtirasi $\Delta t = t - t_0$ ni dt ga almashtiramiz.

Natijada,

$$F^{(k)}(t_0)(t - t_0)^k = F^{(k)}(t_0)dt^k = d^k F(t_0) = d^k u|_{t_0},$$

$$F^{(n+1)}(t_0 + \theta(t - t_0))(t - t_0)^{n+1} = d^{n+1} u|_{t_0 + \theta(t - t_0)}. \quad (10.23.4)$$

Agar $F(t) - F(t_0)$ ayirmaning Δu deb belgilasak, u holda (10.23.4) formulani e'tiborga olgan holda (10.23.3) Teylor formulasini quyidagi

$$\Delta u = du|_{t_0} + \frac{1}{2!} d^2 u|_{t_0} + \dots + \frac{1}{n!} d^n u|_{t_0} + \frac{1}{(n+1)!} d^{n+1} u|_{t_0 + \theta(t - t_0)} \quad (10.23.5)$$

ko'rinishda yozamiz. $M_0(x_0, y_0)$ nuqtaning $U_\delta(M_0)$ atrofida $M(x_0 + \Delta x, y_0 + \Delta y)$ nuqtani olib, M_0 bilan M nuqtani to'g'ri chiziqli yordamida birlashtiramiz. Ravshanki, bu to'g'ri chiziqli kesma U_δ atrofida qarashli bo'ladi. $M_0 M$ kesmadagi nuqtalarning x va y koordinatlarini yangi o'zgaruvchi t ning chiziqli funksiyasi bo'lib, ushbu

$$x = x_0 + t\Delta x, \quad y = y_0 + t\Delta y \quad (0 \leq t \leq 1) \quad (10.23.6)$$

Uchun M_0 M kesmaning nuqtalari t o'zgaruvchining $[0,1]$ segmentdagi nuqtalarga mos keladi, $t_0=0$ qiymatga M_0 nuqta, $t=1$ ga M nuqta mos keladi.

Shunday qilib ko'ra, $u=f(x,y)$ funksiya $U_\delta(M_0)$ atrofda $n+1$ marta differentsiallanuvchi bo'lgani uchun (10.23.6) dan M_0 M kesmada bu funksiya $t(t \in [0,1])$ ning $n+1$ marta differentsiallanuvchi murakkab funksiyaga aylanadi. Hosil bo'lgan murakkab funksiyani $F(t)$ bilan qoplaymiz. Bu funksiya $t_0=0$ nuqtadagi Teylor formulasi (10.23.5) ko'rinishida bo'ladi, bunda

$$F(t) = F(1) - F(0) = f(M) - f(M_0).$$

Uchun t dagi qatnashgan har xil tartibdagi differentsiallar $F(t) = f(x(t), y(t)) = F(t)$ murakkab funksiyadan olingan differentsiallar bo'lib hisoblanadi. **10.22-** bandga asosan, $u = f(x, y)$ funksiyasining istalgan tartibdagi differentsialini (10.23.5) ko'rinishda ifodalash mumkin bo'ladi. Bunda,

$$d^k u|_{t=t_0} = \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy \right)^k u|_{M_0(x_0, y_0)} = d^k u|_{M_0}. \quad (10.23.7)$$

$$d^{n+1} u|_{t_0} = \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy \right)^{n+1} u|_{N(x_0 + \theta \Delta x, y_0 + \theta \Delta y)} = d^{n+1} u|_N$$

bu yerda dx va dy lar $dt = \Delta t = 1 - 0 = 1$ da (10.23.6) dan topiladi. (10.23.6)

shartlari bilan

$$dx = dt \Delta x = \Delta x, \quad dy = dt \Delta y = \Delta y. \quad (10.23.8)$$

Uchun (10.23.7) ni e'tiborga olib, (10.23.7) dagi $d^k u|_{t_0}$ va $d^{n+1} u|_{t_0 + \theta(t-t_0)}$

ifodalarini (10.23.5) ga qo'ysak, natijada Teylor formulasi

hosil qilamiz. ■

10.24. Funktsiya ekstremumining zaruriy sharti. Agar $u = f(x_1, x_2, \dots, x_m) = f(M)$ funktsiya ochiq $\{M\} \subset R^m$ to'plamda aniqlanib, $M_0(x_1^0, x_2^0, \dots, x_m^0) \in \{M\}$ bo'lsin.

10.24.1. Ta'rif. Agar M_0 nuqtaning shunday $U_\delta(M_0) \subset \{M\}$ atrofi mavjud bo'lib, $\forall M \in U_\delta(M_0)$ uchun $f(M) \leq f(M_0)$ ($f(M) \geq f(M_0)$) bo'lsa, $f(M)$ funktsiya M_0 nuqtada lokal maksimumga (minimumga) deyiladi.

10.24.2. Ta'rif. Agar M_0 nuqtaning shunday $U_\delta(M_0) \subset \{M\}$ atrofi mavjud bo'lsaki, $\forall M \in U_\delta(M_0) \setminus \{M_0\}$ uchun $f(M) < f(M_0)$ ($f(M) > f(M_0)$) bo'lsa, $f(M)$ funktsiya M_0 nuqtada qat'iy maksimumga (qat'iy minimumga) ega deyiladi.

Funktsiyaning lokal maksimumi, lokal minimumi umumiy nom funktsiyaning lokal ekstremumi deyiladi. Bunda M_0 nuqta funktsiyaning lokal ekstremum nuqtasi deyiladi, $f(M_0)$ ga funktsiyaning lokal ekstremum qiymati deyiladi.

Funktsiyaning maksimum(minimum) qiymatlari

$$f(M_0) = \max_{M \in U_\delta(M_0)} \{f(M)\} \quad f(M_0) = \min_{M \in U_\delta(M_0)} \{f(M)\}$$

kabi belgilanadi.

Ma'lumki, $\Delta u = f(M) - f(M_0)$ ayirma funktsiyaning M_0 nuqta to'liq orttirmasi deyilar edi.

$f(M)$ funktsiyaning M_0 nuqtada lokal maksimum(minimum) erishsa, u holda $\forall M \in U_\delta(M)$ da $\Delta f(M_0) \leq 0$ ($\Delta f(M_0) \geq 0$) bo'ladi.

10.24.3. Teorema (funktsiya ekstremumining zaruriy sharti). Agar $f(M)$ funktsiya M_0 nuqtada lokal ekstremumga erishib, bu nuqtada ba

$f'_{x_1}, f'_{x_2}, \dots, f'_{x_m}$ xususiy hosilalarga ega bo'lsa, u holda

$$f'_{x_1}(M_0) = 0, f'_{x_2}(M_0) = 0, \dots, f'_{x_m}(M_0) = 0$$

bo'ladi.

Isbot. $u = f(x_1, x_2, \dots, x_m) = f(M)$ funksiya $M_0(x_1^0, x_2^0, \dots, x_m^0) \in \{M\}$ da lokal maksimum(minimum)ga ega bo'lsin. **10.24.1-ta'rifga**

$$f(x_1, x_2, \dots, x_m) \leq f(x_1^0, x_2^0, \dots, x_m^0) \quad f(x_1, x_2, \dots, x_m) \geq f(x_1^0, x_2^0, \dots, x_m^0)$$

shart o'rinli bo'ladi. $x_2 = x_2^0, \dots, x_m = x_m^0$ deb belgilasak, u holda x_1 argumentga bog'liq bo'lgan bir argumentli $u = f(x, x_2^0, \dots, x_m^0)$ funksiyaga ega bo'lamiz. Xususiyl holda, bu funksiya x_1^0 nuqtada lokal maksimumga ega, ya'ni $u(x_1) \leq u(x_1^0) (u(x_1) \geq u(x_1^0))$ bo'ladi. Agarda $f(x_1, x_2^0, \dots, x_m^0)$ funksiya x_1^0 nuqtada xususiyl hosilaga ega bo'lsa, u holda (10.24.1 bandga q.) teoremgaga asosan,

$$\frac{\partial f(M_0)}{\partial x_1} = 0.$$

Shu e'lon shunday

$$\frac{\partial f(M_0)}{\partial x_2} = 0, \frac{\partial f(M_0)}{\partial x_3} = 0, \dots, \frac{\partial f(M_0)}{\partial x_m} = 0 \quad (10.24.4)$$

to'liq e'lonlanadi.

10.24.5. Natija. Agar $u = f(M)$ funksiya M_0 nuqtada ekstrimumga ega bo'lsin, u shu nuqtada differensiyallanuvchi bo'lsa, u holda $du(M_0) = 0$ bo'ladi.

10.24.6. Eslatma. (10.24.4) shart, funksiyaning M_0 nuqtada ekstrimumga ega bo'lishi uchun yetarli shart bo'la olmaydi. Masalan:

$u(x, y) = x^2 y$ funksiya xususiyl hosilalarga ega: $\frac{\partial u}{\partial x} = 2xy$, $\frac{\partial u}{\partial y} = x^2$. Bu

hosilalar $M_0(0,0)$ nuqtada nolga aylanadi:

$$\left. \frac{\partial u}{\partial x} \right|_{M_0} = 0, \left. \frac{\partial u}{\partial y} \right|_{M_0} = 0.$$

Shu e'lon $M_0(0,0)$ nuqta berilgan funksiyaning ekstrimum nuqtasi bo'la olmaydi. Chunki, $M_0(0,0)$ nuqtaning ixtiyoriyl $U_\delta(M_0)$ atrofida

$u(x, y) = 0 = x^2 y$ ifoda faqat manfiyl yoki faqat musbat qiymatlarni qabul

etadi.

$u = f(x_1, x_2, \dots, x_m) = f(M)$ funksiya xususiyl hosilalarni nolga ega bo'ladigan nuqtalar uning *statsionar nuqtalari* (ekstrimumga shubhali

nuqtalari) deyiladi. Ekstremumga shubhali nuqtalarni , ya'ni stat
nuqtalarni topish uchun quyidagi

$$\begin{cases} \frac{\partial f(M)}{\partial x_1} = 0, \\ \frac{\partial f(M)}{\partial x_2} = 0, \\ \dots\dots\dots, \\ \frac{\partial f(M)}{\partial x_m} = 0 \end{cases}$$

tenglamalar sistemasini yechish zarur.

10.25. Funksiyaning ekstremumga ega bolishning etarli sh

$u = f(x_1, x_2, \dots, x_m) = f(M)$ funksiyaning ekstremumga ega bo'lish
yetarli shartini keltirishdan , avvalo, algebra kursida ba
o'rganiladigan kvadratik forma haqida qisqacha tushuncha beramiz:

$$Q(x_1, x_2, \dots, x_m) = a_{11}x_1^2 + a_{12}x_1x_2 + \dots + a_{1m}x_1x_m + a_{21}x_2x_1 + a_{22}x_2^2 + \dots + a_{mm}x_m^2$$

yoki

$$Q(x_1, x_2, \dots, x_m) = \sum_{i=1}^m \sum_{j=1}^m a_{ij}x_ix_j \quad (10.24)$$

funksiyaga $x_1, x_2, \dots, x_n$ o'zgaruvchilarning *kvadratik formasi* dey
bunda $a_{ij} = a_{ji}$, $(i, j = \overline{1, m})$ sonlar kvadratik formaning *koeffitsiye*
deyiladi. Agar $x_1 = x_2 = \dots = x_n = 0$ bo'lsa, har qanday kvadratik f
uchun $Q(0, 0, \dots, 0) = 0$ bo'ladi.

Kvadratik formaning koeffitsiyentlari a_{ij} lardan tuzulgan ushb

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{pmatrix}$$

simmetrik matritsaga kvadratik formaning *matritsasi* deyiladi.

Ushbu

$$\delta_1 = a_{11}, \quad \delta_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \delta_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \quad \dots, \quad \delta_m = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{vmatrix}$$

minimallarga A matritsaning *bosh minorlari* deyiladi.

Agarda birdaniga hammasi nolga aylanmaydigan $x_1, x_2, \dots, x_m$ nuqtachilarning harqanday qiymatida $Q(x_1, x_2, \dots, x_m)$ kvadratik forma faqat musbat (qat'iy manfiy) qiymat, ya'ni $Q(x_1, x_2, \dots, x_m) > 0$ ($Q(x_1, x_2, \dots, x_m) < 0$) qabul qilsa, u holda *kvadratik forma musbat aniqlangan (manfiy aniqlangan)* deyiladi. Masalan, $Q(x_1, x_2) = x_1^2 + 2x_2^2$ kvadratik forma musbat aniqlangan, chunki $(0,0)$ nuqtadan tashqari hamma (x_1, x_2) nuqtalarda $Q(x_1, x_2) > 0$ bo'ladi.

Agar kvadratik forma faqat musbat yoki faqat manfiy aniqlangan bo'lsa, u holda u *ishora saqlaydi* deyiladi.

Agar kvadratik forma faqat barcha nuqtalarda $Q(x_1, x_2, \dots, x_m) \geq 0$ ($Q(x_1, x_2, \dots, x_m) \leq 0$) qiymatni qabul qilsa, ular orasida shunday $(x_1, x_2, \dots, x_m)$ nuqtalar ham borki $x_1 = x_2 = \dots = x_m = 0$ dan tashqari $Q(x_1, x_2, \dots, x_m) = 0$. Bu holda kvadratik forma *yarim musbat (yarim manfiy) aniqlangan* deyiladi. Masalan, $Q(x_1, x_2) = x_1^2 + 2x_1x_2 + x_2^2$ kvadratik forma yarim musbat aniqlangan, chunki hamma (x_1, x_2) nuqtalarda $Q(x_1, x_2) = (x_1 + x_2)^2 \geq 0$, kvadratik forma nafaqat $(0,0)$ nuqtada $Q(x_1, x_2) = 0$ bo'ladi, $(1, -1)$ nuqtada ham $Q(1, -1) = 0$ bo'ladi.

Agar ba'zi $(x_1, x_2, \dots, x_m)$ nuqtalar uchun $Q(x_1, x_2, \dots, x_m) > 0$ bo'lsa, barcha $(x_1, x_2, \dots, x_m)$ nuqtalar uchun $Q(x_1, x_2, \dots, x_m) < 0$ bo'lsa, kvadratik forma *noaniq* deyiladi. Masalan, $Q(x_1, x_2) = x_1^2 - x_1x_2 - x_2^2$ kvadratik forma noaniq, chunki u $Q(1,0) = 1 > 0$, $Q(0,1) = -1 < 0$.

Ekstremumning yetarli shartini aniqlashda kvadratik formaning musbat yoki manfiy aniqlanganligini aniqlash muhim ro'l o'ynaydi. Kvadratik formaning musbat va manfiy aniqlanganligini algebra kursidan o'rganilgan Sil'vestr alomatidan foydalanib topish mumkin. Bu alomatni ham qisqacha isbotsiz keltiramiz.

10.25.2. Sil'vestr alomati. Ushbu $Q(x_1, x_2, \dots, x_m) = \sum_{i=1}^m \sum_{j=1}^m a_{ij}x_ix_j$ kvadratik formaning musbat aniqlangan bo'lishi uchun, ushbu $\delta_1, \delta_2, \dots, \delta_m > 0$ tengsizliklarning bajarilishi zarur va yetarli;

$Q(x_1, x_2, \dots, x_m)$ kvadratik formaning manfiy aniqlangan bo'lishi uchun, ushbu $\delta_1 < 0, \delta_2 > 0, \delta_3 < 0, \dots, (-1)^m \delta_m > 0$ tengsizliklarning bajarilishi zarur va yetarli.

Biz yuqorida 10.24-bandda $f(M)$ funksiyaning M_0 nuqtasidagi ekstremumga ega bo'lishining zaruriy shartini ko'rsatdik. Endi 10.25-bandda funksiyaning ekstremumga erishishning yetarli shartini ko'rsatamiz.

$f(M)$ funksiya $M_0 \in \{M\} \subset R^m$ nuqtaning biror $U_\delta(M_0)$ ($\delta > 0$) atrofini aniqlangan bo'lib, M_0 nuqta $f(M)$ funksiyaning statsionar nuqtasi bo'lsin.

$u = f(x_1, x_2, \dots, x_m) = f(M)$ funksiyaning $x_1, x_2, \dots, x_m$ lar o'zgaruvchilar bo'lganda, $M_0(x_1^0, x_2^0, \dots, x_m^0)$ nuqtadagi ikkinchi tartibli differensiyalini ushbu

$$d^2u|_{M_0} = \sum_{i=1}^m \sum_{j=1}^m \frac{\partial^2 f(M_0)}{\partial x_i \partial x_j} dx_i dx_j \quad (10.25.3)$$

ko'rinishda yozish mumkin. Bu (10.25.3) ifoda, $u = f(x_1, x_2, \dots, x_m)$ funksiyaning M_0 nuqtadagi ikkinchi tartibli differensiyali bo'lgan $dx_1, dx_2, \dots, dx_m$ o'zgaruvchilarga nisbatan esa, kvadratik forma ekanligini ifodalaydi. Kvadratik formaning koeffitsiyentlarini

$$a_{ij} = \frac{\partial^2 f(M_0)}{\partial x_i \partial x_j} (i, j = \overline{1, m}) \quad (10.25.4)$$

deb belgilaymiz.

10.25.5. Teorema (ekstremumning yetarli sharti)

$u = f(x_1, x_2, \dots, x_m) = f(M)$ funksiya $M_0(x_1^0, x_2^0, \dots, x_m^0)$ nuqtaning biror $U_\delta(M_0)$ ($\delta > 0$) atrofida differensiallanuvchi va M_0 nuqtada ikkinchi tartibli differensiallanuvchi hamda M_0 nuqta statsionar nuqta, ya'ni $du|_{M_0} = 0$ bo'lsin. U holda, agar $dx_1, dx_2, \dots, dx_m$ o'zgaruvchilarga nisbatan kvadratik forma ikkinchi tartibli $d^2u|_{M_0}$ musbat (manfiy) aniqlangan bo'lsa, $u = f(M)$ funksiya M_0 nuqtada minimum (maksimum) ega bo'ladi.

Agar $d^2u|_{M_0}$ ifoda noaniq aniqlangan kvadratik forma bo'lsa, $u = f(M)$ funksiya M_0 nuqtada ekstremumga ega bo'lmaydi.

10.25.6. Eslatma. Agar M_0 statsionar nuqtda ikki marta differentsiallanuvchi $u = f(M)$ funksiyaning ikkinchi tartibli differentsiali ham musbat (yarim manfiy) aniqlangan bo'lsa, u holda funksiyaning M_0 nuqtada ekstremumga ega bo'lishi yoki ekstremumga ega bo'lmashligi haqida hech narsa aytib bo'lmaydi. Masalan, $u_1 = x^3 + y^3$ va $u_2 = x^3 - y^3$ ikkala funksiylarning ham ikkinchi tartibli differentsialari $M_0(0,0)$ -statsionar nuqtalarda aynan nulgga teng (ya'ni ular yarim musbat ham manfiy) aniqlangan), lekin ularning ikkinchisi ekstremumga ega.

Bu teoremani xususiyl holda, funksiya ikki o'zgaruvchiga bo'g'liq bo'lgan holda qaraylik.

$u = f(M)$ funksiya $M_0(x_0, y_0)$ nuqtaning biror $U_\delta(M_0)$ ($\delta > 0$) atrofida differentsiallanuvchi va M_0 nuqtada ikki marta differentsiallanuvchi (M_0 -statsionar nuqta, ya'ni $du|_{M_0} = 0$) bo'lsin. Quyidagicha belgilashlarni

qilamiz: $a_{11} = \frac{\partial^2 u}{\partial x_1^2}(M_0)$, $a_{12} = \frac{\partial^2 u}{\partial x_1 \partial x_2}(M_0)$, $a_{22} = \frac{\partial^2 u}{\partial x_2^2}(M_0)$. U holda,

10.25.5. -teorema va 10.25.2-Silvestr alomatiga asosan,

1) agar $D = a_{11}a_{22} - a_{12}^2 > 0$ va $a_{11} > 0$ bo'lsa, $f(M)$ funksiya M_0 nuqtada minimumga erishadi;

2) agar $D > 0$ va $a_{11} < 0$ bo'lsa, $f(M)$ funksiya M_0 nuqtada maksimumga erishadi.

3) $D < 0$ bo'lsa, $f(M)$ funksiya M_0 nuqtada ekstremumga erishmaydi.

4) $D = 0$ bo'lsa, $f(M)$ funksiya M_0 nuqtada ekstremumga erishishi ham mumkin, erishmasligi ham mumkin. Bu hol qo'shimcha tekshirish yordamida aniqlanadi.

Isboti. Shartga ko'ra, $M_0(x_0, y_0)$ - statsionar nuqta bo'lgani uchun $f'_x(M_0) = 0$, $f'_y(M_0) = 0$ yoki, $du(M_0) = 0$. Berilgan $u = f(x, y)$ funksiyaning $M_0(x_0, y_0)$ nuqtada lokal ekstremumga ega bo'lishi yoki ega bo'lmashligi

$$\Delta u = \Delta f(x_0, y_0) = f(x, y) - f(x_0, y_0)$$

aynaning ishorasiga bog'liq: agar $\forall (x, y) \in U_\delta(M_0)$ da $\Delta f(x_0, y_0) > 0$ bo'lsa, funksiya $M_0(x_0, y_0)$ nuqtada lokal minimum, agar $\Delta f(x_0, y_0) < 0$

bo'lsa, funksiya shu nuqtada lokal maksimumga erishadi. Ayirmishorasini aniqlash uchun uni Teylor formulasiga yoyib, uning ikkil bilan chegaralanamiz, ya'ni (10.23.2) Teylor formulasidan xususiy $n=1$ bo'lganda

$$\Delta u = \Delta f(x_0, y_0) = du|_{M_0} + \frac{1}{2} d^2 u|_N \quad (10.25.7)$$

ni hosil qilamiz. $M_0(x_0, y_0)$ nuqta statsionar nuqta bo'lgani uchun $du|_{M_0} = 0$ bo'ladi. Unda (10.25.7) dan

$$\Delta u = \frac{1}{2} d^2 u|_N, \quad (10.25.8)$$

bunda N -qaralayotgan atrofda $M_0(x_0, y_0)$ nuqtaga bog'liq bo'lgan nuqta. $\frac{1}{2} d^2 u|_N$ ning tarkibidagi dx va dy lar esa, x va y o'zgaruvchilarning differensiyallari bo'lib, ular mos ravishda Δx va Δy larga teng. $d^2 u|_N$ ni yozib olsak, u holda (10.25.8) dan

$$\Delta u = \Delta f(x_0, y_0) = \frac{1}{2} \{ f''_{xx}(x_0 + \theta \Delta x, y_0 + \theta \Delta y) \Delta x^2 + 2 f''_{xy}(x_0 + \theta \Delta x, y_0 + \theta \Delta y) \Delta x \Delta y + f''_{yy}(x_0 + \theta \Delta x, y_0 + \theta \Delta y) \Delta y^2 \} \quad (10.25.9)$$

ga ega bo'lamiz. Ikkinchi tartibli xususiy hosilalar uzluksiz bo'lgan uchun hamda (10.25.4) ni e'tiborga olib,

$$\begin{aligned} f''_{xx}(x_0 + \theta \Delta x, y_0 + \theta \Delta y) &= f''_{xx}(x_0, y_0) + \alpha_{11} = a_{11} + \alpha_{11}, \\ f''_{xy}(x_0 + \theta \Delta x, y_0 + \theta \Delta y) &= f''_{xy}(x_0, y_0) + \alpha_{12} = a_{12} + \alpha_{12}, \\ f''_{yy}(x_0 + \theta \Delta x, y_0 + \theta \Delta y) &= f''_{yy}(x_0, y_0) + \alpha_{22} = a_{22} + \alpha_{22}, \end{aligned} \quad (10.25.10)$$

bunda $\Delta x \rightarrow 0$ va $\Delta y \rightarrow 0$ da hamma $\alpha_{11} \rightarrow 0, \alpha_{12} \rightarrow 0, \alpha_{22} \rightarrow 0$. (10.25.10) va (10.25.9) lardan

$$\Delta u = \frac{1}{2} \{ a_{11} \Delta x^2 + 2a_{12} \Delta x \Delta y + a_{22} \Delta y^2 + \alpha_{11} \Delta x^2 + 2\alpha_{12} \Delta x \Delta y + \alpha_{22} \Delta y^2 \}.$$

Keyingi mulohazalarda soddalashtirish maqsadida $\Delta x = \rho \cos \varphi$ va $\Delta y = \rho \sin \varphi$ deb olamiz (10.5-chizma), bunda, $\rho = \sqrt{\Delta x^2 + \Delta y^2}$

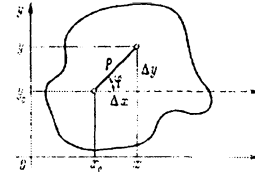
$$\Delta u = \frac{1}{2} \{ a_{11} \cos^2 \varphi + 2a_{12} \cos \varphi \sin \varphi + a_{22} \sin^2 \varphi + \alpha_{11} \cos^2 \varphi + 2\alpha_{12} \cos \varphi \sin \varphi + \alpha_{22} \sin^2 \varphi \}. \quad (10.25.11)$$

o'zingiz ishorasi $a_{11}a_{22} - a_{12}^2 > 0$ ifodaning

to'lariga bog'liq quyidagi hollarni

qaraymiz:

1^o. $a_{11} > 0, a_{11}a_{22} - a_{12}^2 > 0$ bo'lsin. Bu holda $a_{11}a_{22} > 0, a_{11} \neq 0$.



10.5-chizma.

(10.25.11) dagi $a_{11} \cos^2 \varphi + 2a_{12} \cos \varphi \sin \varphi + a_{22} \sin^2 \varphi$ ifodani quyidagicha ko'rinishda tasvirlaymiz:

$$\frac{1}{a_{11}} [(a_{11} \cos \varphi + a_{12} \sin \varphi)^2 + (a_{11}a_{22} - a_{12}^2) \sin^2 \varphi] \quad (10.25.12)$$

Hundan, [...] kabi ichidagi ifoda har doim musbat. [...] kabi ichidagi uch had φ ning barcha qiymatida nolga aylanmaydi. Uning ichidagi koeffitsiyent a_{11} ning ishorasiga bog'liq. [...] kabi ichidagi holda φ ning funksiyasi sifatida $[0, 2\pi]$ da uzluksiz bo'lib, Veyersstrass teoremasiga asosan, y o'zining eng kichik qiymati m ga ega bo'ladi.

$$|a_{11} \cos^2 \varphi + 2a_{12} \cos \varphi \sin \varphi + a_{22} \sin^2 \varphi| \geq m > 0.$$

Ikkinchi tomondan, $\Delta x \rightarrow 0$ va $\Delta y \rightarrow 0, \rho \rightarrow 0$ da $\alpha_{12} \rightarrow 0, \alpha_{22} \rightarrow 0$ bo'lganligi uchun ρ ning yetarli kichik qiymatida

$$|\alpha_{11} \cos^2 \varphi + 2\alpha_{12} \cos \varphi \sin \varphi + \alpha_{22} \sin^2 \varphi| \leq |\alpha_{11}| + 2|\alpha_{12}| + |\alpha_{22}| < m$$

bo'ladi.

Shunday qilib, $a_{11} > 0, a_{11}a_{22} - a_{12}^2 > 0$ bo'lganda, (10.25.11) ning o'ng tomonidagi ifoda musbat bo'ladi, ya'ni, $\Delta u = \Delta f(x_0, y_0) > 0, f(x, y) > f(x_0, y_0)$.

Demak, bu holda, $f(x, y)$ funksiya $M_0(x_0, y_0)$ nuqtada lokal minimumga erishadi.

$a_{11} < 0, a_{11}a_{22} - a_{12}^2 > 0$ bo'lganda, (10.25.11) ning o'ng tomoni manfiy ya'ni, $\Delta f(x_0, y_0) < 0$ bo'lib, $f(x, y)$ funksiya $M_0(x_0, y_0)$ nuqtada lokal maksimumga erishadi.

2⁰. $a_{11}a_{22} - a_{12}^2 < 0$ bo'lsa, $a_{11} \neq 0$ hol bilan chegaralanamiz. U (10.25.12) dan foydalanamiz.

$\varphi = \varphi_1 = 0$ bo'lganda [...] kabi ichidagi ifoda musbat bo'lsa. Aksincha agar $\varphi = \varphi_2$ ni $a_{11}\cos\varphi_2 + a_{12}\sin\varphi_2$ ($\sin\varphi_2 \neq 0$) shartini aniqlasak, u holda [...] kabi ichidagi ifoda $(a_{11}a_{22} - a_{12}^2)\sin^2\varphi_2$ keltiriladi va u manfiy bo'ladi. Istalgancha kichik ρ da {...} ichidagi ikkinchi uchhad $\varphi = \varphi_1$ va $\varphi = \varphi_2$ larda istalgancha kabi bo'ladi va ΔU ning ifodasi birinchi uch hadning ifodasi bilan aniqlanadi.

Shunday qilib, $M_0(x_0, y_0)$ nuqtaning atrofidagi $\varphi = \varphi_1$ va $\varphi = \varphi_2$ nurlarda ΔU ning ifodasi qarama-qarshi ifodalarga ega bo'ladi. Shunday uchun bu holda funksiya qaralayotgan nuqtada ekstremumga ega bo'lmaydi.

Agar $a_{11} = 0$ bo'lsa, {...} kabining birinchi uchhadni tekshirish

$$2a_{12}\cos\varphi\sin\varphi + a_{22}\sin^2\varphi = \sin\varphi(2a_{12}\cos\varphi + a_{22}\sin\varphi)$$

ga keltiriladi, u holda $a_{12} \neq 0$ ekanligini e'tiborga olgan holda $\varphi_1 \neq 0$ ni

$$|a_{22}|\sin\varphi_1 < 2|a_{12}|\cos\varphi_1$$

ni qanoatlantiradigan qilib aniqlash mumkin. U holda $\varphi = \varphi_1$ va $\varphi = \varphi_2 = -\varphi_1$ da qaralayotgan uchhad qarama-qarshi ifodalarga ega bo'ladi. Ya'ni ΔU qarama-qarshi ifodalarga ega bo'ladi. Bundan $f(x, y)$ funksiyaning $M_0(x_0, y_0)$ nuqtada ekstremumga erishmasligi kelib chiqadi.

Shunday qilib: agar $a_{11}a_{22} - a_{12}^2 > 0$ bo'lsa, $f(x, y)$ funksiyasi qaralayotgan statsionar $M_0(x_0, y_0)$ nuqtada ekstremumga ega bo'lib, $a_{11} > 0$ da maksimumga, $a_{11} < 0$ bo'lganda esa minimumga ega bo'ladi.

Agar $a_{11}a_{22} - a_{12}^2 < 0$ bo'lsa, ekstremumga ega bo'lmaydi.

3⁰. $a_{11}a_{22} - a_{12}^2 = 0$ bo'lsa funksiya ekstremumga ega bo'lishi ham bo'lmaydi ham mumkin.

10.25.13. Misol. Ushbu $f(x, y, z) = x^2 + 2xy + 4xz + 8yz + 5y^2 + 3z^2$ funksiyani ekstremumga tekshiring:

Yechilishi. Berilgan funksiyaning aniqlanish sohasi C koordinatalar tekislikdan iborat $D(f) = R^3$. $u = f(x, y, z)$ funksiya C

shuntlari tekislikning har bir $M(x,y,z) \in D(f)$ nuqtasida bajarilganuvchi.

1. Stanssioanar nuqta topamiz:

$$\begin{cases} \frac{\partial f}{\partial x} = 2x + 2y + 4z = 0, \\ \frac{\partial f}{\partial y} = 2x + 10y + 8z = 0, \\ \frac{\partial f}{\partial z} = 4x + 8y + 18z = 0. \end{cases} \Rightarrow \begin{cases} 2x + 2y + 4z = 0, \\ 2x + 10y + 8z = 0, \\ 4x + 8y + 18z = 0. \end{cases}$$

bu sistemani determinanti hisoblaymiz:

$$\begin{vmatrix} 2 & 2 & 4 \\ 2 & 10 & 8 \\ 4 & 8 & 18 \end{vmatrix} = 128 > 0.$$

bu ushbu sistemani determinanti noldan farqli bo'lganligi uchun, bu sistemani yagona $x=0, y=0, z=0$ yechimga ega. Berilgan $u=f(x,y,z)$ funktsiyaning bitta stanssioanar nuqtasi bor ekan: $O(0,0,0)$.

Endi $d^2f(0,0,0)$ ni topamiz:

$$d^2f(0,0,0) = 2dx^2 + 4dxdy + 8dxdz + 16dydz + 10dy^2 + 18dz^2.$$

Kvadratik formaning koeffitsiyentlaridan tuzilgan bosh minorlarini hisoblaymiz:

$$\Delta_1 = 2 > 0, \Delta_2 = \begin{vmatrix} 2 & 2 \\ 2 & 10 \end{vmatrix} > 0, \Delta_3 = \begin{vmatrix} 2 & 2 & 4 \\ 2 & 10 & 8 \\ 4 & 8 & 18 \end{vmatrix} = 128 > 0.$$

Sylvestr kriteriysiga asosan, $d^2f(0,0,0)$ kvadratik forma musbat aniqlangan.

Demak, 10.25.5-teorema (ekstremumning yetarli sharti) ko'ra, berilgan funktsiya $O(0,0,0)$ nuqtada qat'iy minimumga ega bo'ladi.

48-§. Oshkormas funksiyalar

10.26. Oshkormas funktsiya haqida tushuncha. x va y

o'zgaruvchilar o'zaro ushbu

$$F(x,y) = 0 \quad (10.26.1)$$

tenglama orqali bog'langan bo'lsin. Bunda $F(x, y)$ funksiya, ikki o'zgaruvchining funksiyasi sifatida $M = \{(x, y) \in R^2 : a < x < b, c < y < d\}$ sohada aniqlangan. Biror $x_0 \in (a, b)$ sonni olib, uni (10.26.1) tenglamadagi x ning o'rniga qo'ysak, natijada y ni topish uchun, quyidagi

$$F(x_0, y) = 0 \quad (10.26.2)$$

tenglamaga ega bo'lamiz. Bu tenglamani yechishda quyidagi hollar bo'lishi mumkin:

- 1^o. (10.26.2) tenglama yagona y_0 yechimga ega bo'lishi mumkin;
- 2^o. (10.26.2) tenglama bitta ham haqiqiy yechimga ega bo'lmaydi;
- 3^o. (10.26.2) tenglama bir nechta haqiqiy, hatto cheksiz ko'p yechimga ega bo'lishi mumkin.

Masalan. 1) $F(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ tenglama $[-a, a]$ kesmada y ni x ning ikki qiymatli funksiyasi sifatida aniqlaydi: $y = \pm \frac{b}{a} \sqrt{a^2 - x^2}$. Bu $F(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ tenglamadagi y ning o'rniga qo'ysak, natijada ayniyat hosil bo'ladi.

2) $F(x, y) = y\sqrt{x^2 - 3} - 2 = 0$ tenglama x ning ixtiyoriy $R \setminus \{x \in R : -\sqrt{3} \leq x \leq \sqrt{3}\}$ dan olingan har bir qiymatida yagona $y = \frac{2}{\sqrt{x^2 - 3}}$ yechimga ega, bundan $F(x, \frac{2}{\sqrt{x^2 - 3}}) \equiv 0$ hosil bo'ladi.

Yuqorida keltirilgan 1) va 2) misollarda, y ni x orqali ifodalash mumkin bo'ldi. Lekin har doim ham y ni x orqali ifodalash oson bo'lavermaydi.

Masalan, ushbu

$$F(x, y) = y - x - \varepsilon \sin y = 0 \quad (0 < \varepsilon < 1) \quad (10.26.3)$$

tenglamani qaraylik. Bunda y ni x orqali elementar funksiya yordamida aniqlab bo'lmaydi. Lekin, (10.26.3) tenglama, umumiy holda y ni x ning bir qiymatli funksiyasi sifatida aniqlaydi. Haqiqatdan ham (10.26.3) tenglamani $x = y - \varepsilon \sin y = \varphi(y)$ ($y \in (-\infty, \infty)$) ko'rinishida yoz

olib, $\varphi(y)$ funksiyaning $(-\infty, \infty)$ da uzluksizligi hamda $\varphi'(y) = 1 - \varepsilon \sin y > 0$ bo'lganiga ega ekanligiga ishonch hosil qilamiz. Bunda teskari funksiya mavjudligi haqidagi teorema asosan, $y = \varphi^{-1}(x)$ funksiya mavjud, (10.26.3) tenglama Kepler tenglamasini ifodalaydi.

Demak (10.26.3) tenglama yagona $y = \varphi^{-1}(x)$ yechimga ega, ya'ni, $F(\varphi^{-1}(x)) = 0$.

3) $F(x, y) = x^2 + y^2 - \ln y = 0$ ($y > 0$) tenglama x ning $(-\infty, \infty)$ oraliqdan olingan hech qanday qiymatida yechimga ega emas. Chunki doimo $x^2 + y^2 - \ln y > 0$ munosabat o'rinli. Bu holda berilgan tenglama bitta ham yechimga ega emas.

$F(x, y) = 0$ tenglama uchun 1^o- holning o'rinli bo'lishi muhim ahamiyatga ega. Uning yordamida oshkormas funksiyani aniqlash mumkin. x o'zgaruvchining qiymatlaridan iborat shunday X to'plamni qayavlikki, u to'plamdan olingan x ning har bir qiymatida $F(x, y) = 0$ tenglama yagona yechimga ega bo'lsin.

X to'plamdan olingan ixtiyoriy x ga $F(x, y) = 0$ tenglamaning yagona y yechimini mos qo'yamiz. Natijada X to'plamda aniqlangan funksiya hosil bo'ladi. Odatda bunday aniqlangan funksiyani oshkormas funksiya deyiladi va u $x \rightarrow y: F(x, y) = 0$ kabi belgilanadi.

Masalan, 2) va 3) misollardagi oshkormas funksiyalar:

$$x \rightarrow y = \frac{2}{\sqrt{x^2 - 3}}: F\left(x, \frac{2}{\sqrt{x^2 - 3}}\right) = 0, \quad x \rightarrow y = \varphi^{-1}(x): F(x, \varphi^{-1}(x)) = 0$$

kabi belgilanadi.

10.26.4. Eslatma. Agar $F(x, y) = 0$ tenglama oshkormas ko'rinishdagi funksiyani aniqlamasa, ba'zi hollarda, y ga ma'lum shart qo'yish natijasida berilgan tenglama oshkormas funksiyani aniqlashi mumkin.

Masalan, $F(x, y) = x^2 + y^2 - 1 = 0$ tenglama x ning $(-1; 1)$ oraliqdan olingan har bir qiymatida, ikkita, $y = -\sqrt{1 - x^2}$, $y = \sqrt{1 - x^2}$ yechimlarga ega.

Agar y ga uning qiymatlari $[-1;0]$ kesmada bo'lsin, degan shart qo'yilsa, u holda, berilgan tenglama yordamida aniqlangan,

$$x \rightarrow y = -\sqrt{1-x^2} : F(x, -\sqrt{1-x^2}) = 0$$

ko'rinishdagi oshkormas funksiya hosil bo'ladi.

10.27. Oshkormas funksiyaning mavjudligi va uni differensiyallash.

10.27.1. Teorema. $F(x, y)$ funksiya $M_0(x_0, y_0) \in R^2$ nuqtaning biror atrofida berilgan bo'lib, quyidagi shartlarni qanoatlantirsin:

- 1) Uzlüksiz va uzluksiz $F'_x(x, y), F'_y(x, y)$ xususiy hosilalarga ega;
- 2) $F(x, y)$ funksiya $M_0(x_0, y_0)$ nuqtada $F(x_0, y_0) = 0$;
- 3) $F'_y(x, y)$ hosila $M_0(x_0, y_0)$ nuqtada $F'_y(x_0, y_0) \neq 0$ bo'lsa, u holda, $M_0(x_0, y_0)$ nuqtaning shunday

$$K = \{M(x, y) \in R^2 : x_0 - a \leq x \leq x_0 + a, y_0 - b \leq y \leq y_0 + b\}$$

atrofi topiladiki, bunda $F(x, y) = 0$ tenglama y ni x ning $y = f(x)$ oshkormas funksiyasi sifatida aniqlaydi va bu $y = f(x)$ funksiya $(x_0 - a, x_0 + a)$ oraliqda uzluksiz differensiyalanuvchi va $y' = -\frac{F'_x(x, y)}{F'_y(x, y)}$

bo'ladi.

Isboti. Teoremaning isbotini ikki qismga ajratamiz:

a) **Oshkormas funksiya mavjudligining isboti.** Teoremaning 1) shartiga ko'ra uzluksiz $F'_y(x, y)$ mavjud va u 3) shartga asosan, $F'_y(x_0, y_0) \neq 0$. Faraz qilaylik, $y = F'_y(x_0, y_0) > 0$ bo'lsin. $F'_y(x, y)$ funksiya $M_0(x_0, y_0)$ nuqtada uzluksiz va $F'_y(x_0, y_0) > 0$ bo'lgani uchun uzluksiz funksiyaning lokal xossasiga asosan, $M_0(x_0, y_0)$ nuqtaning shunday

$$K_1 = \{M(x, y) \in R^2 : x_0 - a_1 \leq x \leq x_0 + a_1, y_0 - b_1 \leq y \leq y_0 + b_1\} (0 < a < a_1)$$

atrofi topiladiki, unda $F'_y(x_0, y_0) > 0$ bo'ladi. Bundan esa, belgilangan $x \in (x_0 - a_1, x_0 + a_1)$ larda $F(x, y)$ funksiya y o'zgaruvchining funksiyasi sifatida o'suvchiligi kelib chiqadi.

Ushbu $\psi(y) = F(x_0, y), y \in (y_0 - b, y_0 + b)$ funksiyani qaraylik. (10.27.1) $F'_y(x_0, y) > 0$ bo'lgani uchun $\psi(y)$ funksiya $[y_0 - b, y_0 + b]$ kesmada qat'iy o'suvchi. Bundan tashqari 2) shartga asosan $\psi(y_0) = F(x_0, y_0) = 0$.

Demak, $\psi(y_0 - b) = F(x_0, y_0 - b) < 0$, $\psi(y_0 + b) = F(x_0, y_0 + b) > 0$ bo'ladi. $F(x, y)$ funksiya uzluksiz bo'lgani uchun $(x_0, y_0 - b)$ va $(x_0, y_0 + b)$ nuqtalarning atrofida ishora saqlaydi. Shuning uchun $(0, a_1)$ $x \in (x_0 - a, x_0 + a)$ lar uchun

$$F(x_0, y_0 - b) < 0, F(x_0, y_0 + b) > 0 \quad (10.27.2)$$

bo'ladi. Endi

$K = \{(x, y) \in R^2 : x_0 - a \leq x \leq x_0 + a, y_0 - b \leq y \leq y_0 + b\}$ o'rtburchakda $F(x, y) = 0$ tenglama y ni x ning oshkormas funksiyasi sifatida aniqlashni ko'rsatamiz. Buning uchun $\forall \bar{x} \in (x_0 - a, x_0 + a)$ nuqtani olib, $[y_0 - b, y_0 + b]$ kesmada uzluksiz bo'lgan bir argumentli $\phi(y) = F(\bar{x}, y)$ funksiyani qaraymiz. (10.27.2) ga asosan, bu funksiya $[y_0 - b, y_0 + b]$ kesmaning uchlarida har xil ishorali qiymatlarni qabul qiladi: $\phi(y_0 - b) = F(\bar{x}, y_0 - b) < 0$, $\phi(y_0 + b) = F(\bar{x}, y_0 + b) > 0$. U holda Koshi teoremasiga asosan, $\exists \bar{y} \in [y_0 - b, y_0 + b]$ nuqta topiladiki, $\phi(\bar{y}) = F(\bar{x}, \bar{y}) = 0$ bo'ladi. $\phi'(y) = F'_y(\bar{x}, y) > 0$ bo'lgani uchun $\phi(y)$ funksiya $[y_0 - b, y_0 + b]$ kesmada qat'iy o'suvchi bo'ladi va u kesmada bitta nuqtadan tashqari boshqa nuqtaga nolga aylanmaydi. Shuning uchun $\forall x \in [x_0 - a, x_0 + a]$ uchun yagona $\exists \bar{y} \in [y_0 - b, y_0 + b]$ topiladiki, $F(x, y) = 0$ bo'ladi. Bundan, K o'rtburchakda $F(x, y) = 0$ tenglamaning y ni x ning oshkormas funksiyasi sifatida aniqlashi kelib chiqadi.

b). Oshkormas funksiyaning uzluksiz differensiyalanuvchiligining isboti. $F'_y(x, y)$ funksiya K yopiq to'g'ri burchakda uzluksiz bo'lgani uchun Veyershtass teoremasiga asosan, y bu to'g'ri burchakda o'zining eng kichik qiymati α ga erishadi. K da $F'_y(x, y) > 0$ bo'lgani uchun

$$F'_y(x, y) \geq \alpha > 0, \forall (x, y) \in K_1. \quad (10.27.3)$$

$F'_x(x, y)$ funksiya K_1 va uzluksiz bo'lgani uchun Veyershtass teoremasi asosan, y chegaralangan bo'ladi, ya'ni

$$|F'_x(x, y)| < \beta, \forall (x, y) \in K_1. \quad (10.27.4)$$

$y = f(x)$ funksiya D to'rtburchakda $F(x, y) = 0$ tenglama bilan aniqlanadigan oshkormas funksiya bo'lsin. $y = f(x)$ funksiya grafiga yotgan ikkita $M(x, y)$ va $M_1(x + \Delta x, y + \Delta y)$ nuqtani olamiz. U holda $F(x, y) = 0$, $F(x + \Delta x, y + \Delta y) = 0$ bo'ladi

$$\Delta F(x, y) = F(x + \Delta x, y + \Delta y) - F(x, y) = 0.$$

Bunga chekli orttirmalar haqidagi Lagranj teoremasini qo'llash natijasida

$$F'_x(x + \theta \Delta x, y + \theta \Delta y) \Delta x + F'_y(x + \theta \Delta x, y + \theta \Delta y) \Delta y = 0 \quad (10.27.5)$$

$$\Delta y = -\frac{F'_x(x + \theta \Delta x, y + \theta \Delta y) \Delta x}{F'_y(x + \theta \Delta x, y + \theta \Delta y)}, \quad 0 < \theta < 1 \quad (10.27.6)$$

(10.27.3) va (10.27.4) tengsizliklarni e'tiborga olgan holda (10.27.5) dan

$$|\Delta y| \leq \frac{\beta}{\alpha} |\Delta x|.$$

Bundan, $\Delta x \rightarrow 0$ da $\Delta y \rightarrow 0$.

Demak, $y = f(x)$ -oshkormas funksiyaning $\forall x \in [x_0 - a, x_0 + a]$ da uzluksiz ekanligi kelib chiqadi. Xususiyl hosilalarning uzluksizligini e'tiborga olgan holda (10.27.6) dan

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \left(-\frac{F'_x(x + \theta \Delta x, y + \theta \Delta y) \Delta x}{F'_y(x + \theta \Delta x, y + \theta \Delta y)} \right) = -\frac{F'_x(x, y)}{F'_y(x, y)}$$

ekanligini topamiz, ya'ni $f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = -\frac{F'_x(x, y)}{F'_y(x, y)}$.

10.27.7. Misol. Ushbu $ye^{-x} - x \ln y - 1 = 0$ tenglama, $x = 0$ nuqtasida, uzluksiz $y = f(x)$ funksiyaning aniqlaydimi? Agar aniqlasa, funksiya hosilaga egami? Agar hosilaga ega bo'lsa, uning hosilasi hisoblang.

Yechilishi. $ye^{-x} - x \ln y - 1 = 0$ tenglamada $x = 0$ qiymatni qo'yib $y = 1$ bo'lishini topamiz. Endi $M_0(0; 1)$ nuqta atrofi

$f(x, y) = ye^x - x \ln y - 1$ funksiyaning teoremaning hamma shartlarini qanoatlantirishini tekshirib ko'ramiz:

1) $F(x, y) = ye^x - x \ln y - 1$ funksiya, $y > 0$ yarim tekislikda aniqlangan va uzluksiz. Uning xususiy hosilalari mavjud va uzluksiz:

$$F'_x(x, y) = ye^x - \ln y, \quad F'_y(x, y) = e^x - \frac{x}{y};$$

$$2) F'_{xy}(0, 1) = 1 \neq 0.$$

Shunday qilib-teoremaning hamma shartlari bajariladi, shuning uchun $ye^x - x \ln y - 1 = 0$ tenglama $M_0(0; 1)$ nuqtaning atrofida $x \rightarrow y(x)$: $ye^x - x \ln y - 1 = 0$ ko'rinishdagi oshkormas uzluksiz funksiyani aniqlaydi. Bu oshkormas funksiya, quyidagi uzluksiz hosilaga ega va bu hosila

$f'(x) = -\frac{F'_x(x, y)}{F'_y(x, y)}$ formula orqali topiladi:

$$f'(x) = -\frac{ye^x - \ln y}{e^x - \frac{x}{y}} = \frac{y(\ln y - ye^x)}{ye^x - x}.$$

10.27.8. Eslatma. Ushbu $F(x, y) = 0$ tenglama berilgan bo'lib, $f(x, y)$ funksiya- teoremaning hamma shartlarini qanoatlantirsin. U holda, bu tenglama yordamida aniqlangan $y = f(x)$ oshkormas funksiyaning hosilasini quyidagicha (y o'zgaruvchini x ning funksiyasi ekanini e'tiborga olib) ham hisoblash mumkin: $F(x, y) = 0$ (10.26.1) tenglamani differensiyalab topamiz:

$$F'_x(x, y) + F'_y(x, y)y' = 0, \quad \text{bundan } y' = f'(x) = -\frac{F'_x(x, y)}{F'_y(x, y)}. \quad (10.27.9)$$

Faraz qilaylik, $F(x, y)$ funksiya $M_0(x_0, y_0)$ nuqtaning biror atrofida uzluksiz ikkinchi tartibli $F''_{xx}(x, y)$, $F''_{xy}(x, y)$, $F''_{yy}(x, y)$ xususiy hosilalarga ega bo'lsin. (10.27.9) ni differensiyalash natijasida

$$y'' = -\frac{(F'_x(x, y))'_x \cdot F'_y(x, y) - (F'_y(x, y))'_x \cdot F'_x(x, y)}{(F'_y(x, y))^2},$$

bunda,

$$(F'_x(x, y))'_x = F''_{xx}(x, y) + F''_{xy}(x, y) \cdot y', \quad (F'_y(x, y))'_x = F''_{yx}(x, y) + F''_{yy}(x, y) \cdot y'$$

ekanligini hamda, (10.27.9) ni e'tiborga olgan holda,

$$y'' = - \frac{2F'_x(x,y) \cdot F'_y(x,y)F''_{xy}(x,y) - (F'_y(x,y))^2 F''_{xx}(x,y) - (F'_x(x,y))^2 F''_{yy}(x,y)}{(F'_y(x,y))^2}$$

10.28. Ko'p o'zgaruvchili oshkormas funksiyalar. $F(x,y)$ tenglamaga o'xshash x,y,z o'zgaruvchilarning qiymatlari o'zaro ush

$$F(x,y,z) = 0 \quad (10.28.1)$$

ko'rinishda bog'langan bo'lsin. $F(x,y,z)$ funksiya uch o'zgaruvchili funksiya sifatida $\{M\} = \{(x,y,z) \in R^3 : a < x < b, c < y < d, e < z < h\}$ soh aniqlangan bo'lsin.

(x,y) nuqtalardan iborat shunday E to'plamni qaraylikki, to'plamdan olingan har bir nuqtada (10.28.1) tenglama yagona haq yechimga ega bo'lsin. E to'plamdan $\forall (x,y)$ nuqtani olib, unga (10.28.1) tenglamaning yagona yechimi z ni mos qo'yamiz. Natijada, to'plamdan olingan har bir nuqtaga yuqoridagi qoidaga ko'ra, bitta z qo'yilib, funksiya hosil bo'ladi va unga ikki o'zgaruvchili *oshkor funksiya* deyiladi: $(x,y) \rightarrow z = g(x,y) : F(x,y,z) = 0$ kabi belgilanadi. Xi shunday m -o'zgaruvchili oshkormas funksiyaning ta'rifi beriladi:

$$(x_1, x_2, \dots, x_m) \rightarrow x_m = g(x_1, x_2, \dots, x_{m-1}) : F(x_1, x_2, \dots, x_m) = 0.$$

10.28.2. Misol. Ushbu $F(x,y,z) = x^2y - y^2z + xz$ funksiya qaraylik. $F(x,y,z) = x^2y - y^2z + xz = 0$ tenglama $M = R^2 \setminus \{(x,y) \in R^2 : x = 0\}$ to'plamdan olingan har bir (x,y) nuqtada $z = g(x,y) = \frac{x^2y}{y^2 - x}$ yechi

ega:

$$(x,y) \rightarrow z = \frac{x^2y}{y^2 - x} : F\left(x, y, \frac{x^2y}{y^2 - x}\right) = 0.$$

Ko'p o'zgaruvchili oshkormas ko'rinishdagi funksiyaning mavjud uzluksizligi va hosilalarga ega bo'lishi haqidagi teoremani o'zgaruvchili oshkormas funksiya uchun keltiramiz.

10.28.3. Teorema. $F(x,y,z)$ funksiya :

1) $M_0(x_0, y_0, z_0)$ nuqtaning biror atrofida uzluksiz va uzluksiz F'_x, F'_y, F'_z xususiy hosilalarga ega;

2) $F(x, y, z)$ funksiya $M_0(x_0, y_0, z_0)$ nuqtada $F(x_0, y_0, z_0) = 0$;

3) $F'_z(x, y, z)$ funksiya $M_0(x_0, y_0, z_0)$ nuqtada $F'_z(x_0, y_0, z_0) \neq 0$ bo'lsa, u holda;

a) $M_0(x_0, y_0, z_0)$ nuqtaning shunday atrofi mavjud bo'ladiki:

$$U = \{ (x, y, z) \in R^3 : x_0 - \delta_1 < x < x_0 + \delta_1, y_0 - \delta_2 < y < y_0 + \delta_2, z_0 - \delta_3 < z < z_0 + \delta_3 \}$$

Unda (10.28.1) tenglama z ni x va y o'zgaruvchilarning oshkormas

$f(x, y)$ funksiyasi sifatida aniqlaydi;

b) $x = x_0, y = y_0$ bo'lganda, bu funksiya $f(x_0, y_0) = z_0$ bo'ladi;

c) oshkormas ko'rinishda aniqlangan funksiya K to'plamda uzluksiz bo'ladi;

2) $z = f(x, y)$ funksiya uzluksiz $f'_x(x, y), f'_y(x, y)$ hosilalarga ega bo'ladi.

Bu teoremaning isbotini o'quvchiga havola etamiz.

11 bob. FUNKSIONAL KYETMA-KYETLIKLAR VA QATORLAR

49-§. Funksional ketma-ketliklar, funksional qatorlar va ularning yaqinlashuvchiligi

**11.1. Funksional ketma-ketliklar va ularning yaqinlash
chiligi.** Elementlari biror $X \subset R$ to'plamda aniqlangan

$$f_1(x), f_2(x), \dots, f_n(x), \dots \quad (11.1.1)$$

funksiyalar ketma-ketligi berilgan bo'lsin. Bu ketma-ketlik *funksional ketma-ketlik* deb ataladi va qisqacha $\{f_n(x)\}$ kabi belgilanadi. Umumiy holda $\{f_n(x)\}$ ketma-ketlik turli hadlarining aniqlanish sohasi, umumiy aytganda, turlicha bo'lishi ham mumkin. Biz buyerda X sifatida soxalarning umumiy qismini olamiz. (11.1.1) ketma ketlikdagi f_n funksiya shu ketma-ketlikning umumiy hadi deyiladi. X to'plam x_0 ($x_0 \in X$) nuqtani olib, (11.1.1) ketma-ketlik har bir hadining nuqtadagi qiymatini hisoblab, natijada

$$f_1(x_0), f_2(x_0), \dots, f_n(x_0), \dots$$

sonlar ketma-ketligini hosil qilamiz.

11.1.2. Ta'rif. Agar $\{f_n(x_0)\}$ sonlar ketma-ketligi yaqinlashuvchi (uzoqlashuvchi) bo'lsa, $\{f_n(x)\}$ funksional ketma-ketlik x_0 nuqtasida *yaqinlashuvchi (uzoqlashuvchi)* deyiladi.

11.1.3. Ta'rif. Agar $\{f_n(x)\}$ funksional ketma-ketlik X to'plamida har bir nuqtasida yaqinlashuvchi (uzoqlashuvchi) bo'lsa, u X to'plamida *yaqinlashuvchi (uzoqlashuvchi)* deyiladi.

11.1.4. Eslatma. $\{f_n(x)\}$ funksional ketma-ketlikning yaqinlashuvchi sohasi $\{f_n(x)\}$ funksional ketma-ketlikning aniqlanish sohasiga teng yoki uning bir qismi, yoki bo'sh to'plam ham bo'lishi mumkin.

Faraz qilaylik, $\{f_n(x)\}$ funksional ketma-ketlik $X \subset R$ to'plamida yaqinlashuvchi bo'lsin. U holda $\forall x_0 \in X$ uchun unga mos kelgan,

$$f_1(x_0), f_2(x_0), \dots, f_n(x_0), \dots$$

ketma-ketlik chekli limitga ega bo'ladi, ya'ni $\lim_{n \rightarrow \infty} f_n(x_0) = f(x_0)$.

Agar X to'plamdan olingan har bir x ga, unga mos kelgan $f_1(x), f_2(x), \dots, f_n(x), \dots$ ketma-ketlikning limitini mos qo'ysak, ya'ni $f(x) = \lim_{n \rightarrow \infty} f_n(x)$, unda X to'plamda aniqlangan biror $f(x)$ funksiya hosil bo'ladi. $f(x)$ funksiya $\{f_n(x)\}$ funksional ketma-ketlikning limit funksiyasi deb ataladi va uni

$$\lim_{n \rightarrow \infty} f_n(x) = f(x) \quad (x \in X) \quad (11.1.5)$$

kabi yozamiz yoki qisqacha $f_n(x) \xrightarrow{X} f(x)$ deb belgilaymiz. (11.1.5) munosabatida quyidagicha ham yozish mumkin:

$$\varepsilon > 0 \quad \exists n_0 = n_0(\varepsilon, x) \in \mathbb{N}: \quad \forall n \geq n_0, \quad \forall x \in X \Rightarrow |f_n(x) - f(x)| < \varepsilon.$$

Uu yaqinlashishni “nuqtaviy” yaqinlashish deb ham yuritiladi.

11.1.6. Misol. Ushbu $\{f_n(x)\} = \left\{ \frac{n^2 + 1}{n^2 + x^2} \right\}$ funksional ketma-

ketlikning limit funksiyasini toping.

Yechilishi. Berilgan ketma-ketlikning hamma hadlari $X = \mathbb{R}$ da aniqlangan. Shuning uchun bu ketma-ketlikning aniqlanish sohasi $X = \mathbb{R}$ dan iborat. Umumiy hadi $f_n(x) = \frac{n^2 + 1}{n^2 + x^2}$. Endi limit funksiyani topamiz

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \frac{1 + \frac{1}{n^2}}{1 + \frac{x^2}{n^2}} = 1.$$

Demak, berilgan funksional $\left\{ \frac{n^2 + 1}{n^2 + x^2} \right\}$ ketma-ketlikning limit

funksiyasi $f(x) = 1$ bo'ladi, ya'ni $\frac{n^2 + 1}{n^2 + x^2} \xrightarrow{\mathbb{R}} 1$.

11.1.7. Misol. Ushbu $\{f_n(x)\} = \left\{ n^2 \sin \frac{1}{n^2 x} \right\}$ funksional ketma-

ketlikning yaqinlashish sohasi va limit funksiyasi topilsin.

Yechilishi. Berilgan ketma-ketlikning hamma hadlari $(-\infty; 0) \cup (0; \infty)$ to'plamda aniqlangan. Bu ketma-ketlikning umumiy hadi $f_n(x) = n^2 \sin \frac{1}{n^2 x}$. Limit funksiyani topamiz: $\forall x \in (-\infty; 0) \cup (0; \infty)$ lar uchun

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \frac{\sin \frac{1}{n^2 x}}{\frac{1}{x}} = \frac{1}{x}.$$

Demak, berilgan funksional ketma-ketlik $(-\infty; 0) \cup (0; \infty)$ da yaqinlashuvchi, uning limit funksiyasi $\frac{1}{x}$ ga teng:

$$n^2 \sin \frac{1}{n^2 x} \xrightarrow{(-\infty; 0) \cup (0; \infty)} \frac{1}{x}.$$

11.2. Funksional qatorlar va ularning yaqinlashuvchiligi. Biror $(X \subset \mathbb{R})$ to'plamda $u_1(x), u_2(x), \dots, u_n(x), \dots$ funksiyalar ketma-ketligi berilgan bo'lsin.

11.2.1. Ta'rif. Ushbu

$$u_1(x) + u_2(x) + \dots + u_n(x) + \dots$$

ifodaga *funksional qator* deyiladi va u $\sum_{n=1}^{\infty} u_n(x)$ kabi belgilanadi:

$$u_1(x) + u_2(x) + \dots + u_n(x) + \dots = \sum_{n=1}^{\infty} u_n(x). \quad (11.2.2)$$

Bunda $u_1(x), u_2(x), \dots, u_n(x), \dots$ lar qatorning hadlari, $u_n(x)$ esa, funksional qatorning umumiy hadi deb ataladi. (11.2.2) funksional qatorning hadlaridan tuzilgan ushbu

$$\begin{aligned} S_1(x) &= u_1(x) \\ S_2(x) &= u_1(x) + u_2(x) \\ &\dots \dots \dots \\ S_n(x) &= u_1(x) + u_2(x) + \dots + u_n(x) \\ &\dots \dots \dots \end{aligned} \quad (11.2.3)$$

yig'indilar ketma-ketligi (11.2.2) funksional qatorning qisman yig'indi ketma-ketligi deyiladi va u $\{S_n(x)\}$ kabi belgilanadi.

11.2.4. Eslatma. Ushbu $\sum_{n=1}^{\infty} u_n(x)$ funksional qator turli hadlarining aniqlanish sohalari (to'plamlari), umuman aytganda, turlicha bo'ladi. Biz bu yerda X to'plam sifatida shu sohalarning umumiy qismini tutishnamiz.

X to'plamdan $x_0 (x_0 \in X)$ nuqtani olib, (11.2.2) funksional qator har bir $u_n(x) (n=1,2,\dots)$ hadlarining shu nuqtadagi qiymatini hisoblab, ushbu

$$\sum_{n=1}^{\infty} u_n(x_0) = u_1(x_0) + u_2(x_0) + \dots + u_n(x_0) + \dots \quad (11.2.5)$$

sonli qatorni hosil qilamiz.

11.2.6. Ta'rif. Agar (11.2.5) sonli qator yaqinlashuvchi (uzoqlashuvchi) bo'lsa, (11.2.2) funksional qator x_0 nuqtada yaqinlashuvchi (uzoqlashuvchi) deyiladi.

11.2.7. Ta'rif. Agar (11.2.2) funksional qator X to'plamning har bir nuqtasida yaqinlashuvchi (uzoqlashuvchi) bo'lsa, (11.2.2) funksional qator X to'plamda yaqinlashuvchi (uzoqlashuvchi) deyiladi. Bu holda (11.2.2) funksional qator "nuqtaviy" yaqinlashish deyiladi.

Faraz qilaylik, (11.2.2) funksional qator X to'plamda yaqinlashuvchi bo'lsin. U holda $\forall x_0 \in X$ uchun unga mos kelgan (11.2.5) sonli qator yaqinlashuvchi bo'ladi va uning yig'indisi biror S_0 soniga teng bo'ladi. Agar X to'plamdan olingan har bir x ga, unga mos kelgan

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + \dots + u_n(x) + \dots$$

qatorning yig'indisini mos qo'ysak, u holda X to'plamda aniqlangan biror $S(x)$ funksiya hosil bo'ladi. Bu $S(x)$ funksiya $\sum_{n=1}^{\infty} u_n(x)$ funksional qatorning yig'indisi deyiladi va u

$$S(x) = \sum_{n=1}^{\infty} u_n(x)$$

kabi yoziladi.

Sonli qatorlarning yaqinlashish (uzoqlashish) ta'rifiga asosan funksional qatorning x_0 nuqtadagi yaqinlashish (uzoqlashish) ta'riflari quyidagicha ham berish mumkin.

11.2.8. Ta'rif. Agar $n \rightarrow \infty$ da (11.2.3) funksional ketma-ketlik $\{S_n(x)\}$ nuqtada yaqinlashuvchi (uzoqlashuvchi) bo'lsa, (11.2.2) funksional qator x_0 nuqtada *yaqinlashuvchi (uzoqlashuvchi)* deyiladi.

Agar $n \rightarrow \infty$ da $\{S_n(x)\}$ funksional ketma-ketlik X to'plamida $S(x)$ limit funksiyaga ega bo'lsa, ya'ni

$$\lim_{n \rightarrow \infty} S_n(x) = S(x) \text{ yoki } n \rightarrow \infty \text{ da } S_n(x) \xrightarrow{x} S(x)$$

bo'lsa, $S(x)$ funksiya (11.2.2) qatorning *yig'indisi* deyiladi.

11.2.9. Ta'rif. Agar

$$\sum_{n=1}^{\infty} |u_n(x)| = |u_1(x)| + |u_2(x)| + \dots + |u_n(x)| + \dots \quad (11.2.10)$$

funksional qator $x = x_0$ nuqtada yaqinlashuvchi bo'lsa, (11.2.10) funksional qator x_0 nuqtada *absolyut yaqinlashuvchi* deyiladi.

11.2.10. Ta'rif. Agar X to'plamning har bir nuqtasida (11.2.10) qator yaqinlashuvchi bo'lsa, (11.2.2) funksional qator X to'plamida *absolyut yaqinlashuvchi* deb ataladi.

Agar $x = x_0$ nuqtada (11.2.10) qator uzoqlashuvchi bo'lib, (11.2.2) qator yaqinlashuvchi bo'lsa, (11.2.2) qator $x = x_0$ nuqtada *shartli yaqinlashuvchi* deyiladi.

(11.2.2) va (11.2.10) qatorlar yaqinlashadigan nuqtalar to'plamini mos ravishda (11.2.2) qatorning *yaqinlashish va absolyut yaqinlashish sohasi* deyiladi.

11.2.12. Eslatma. Berilgan (11.2.2) funksional qatorni yaqinlashish va absolyut yaqinlashish sohasini topishda sonli qator mavzusida ko'rib o'tilgan Dalamber va Koshi alomatlaridan foydalani mumkin.

11.2.13. Misol. Ushbu

$$\sum_{n=1}^{\infty} \frac{nx}{(1+x)(1+2x) \cdots (1+nx)}$$

funksional qatorning $X = (0; \infty)$ da yig'indisini toping.

Yechilishi. 8.1.4-teoremaga ko'ra, funksional qatorning umumiy hadini

quyidagi ko'rinishda ifodalaymiz:

$$u_n(x) = \frac{nx}{(1+x)(1+2x)\cdots(1+nx)} = \frac{1}{(1+x)(1+2x)\cdots(1+(n-1)x)} - \frac{1}{(1+x)(1+2x)\cdots(1+n)x}.$$

U holda funksional qatorning qisman yig'indisi

$$S_n(x) = 1 - \frac{nx}{(1+x)(1+2x)\cdots(1+nx)}$$

bo'ladi. Endi $n \rightarrow \infty$ da limitga o'tamiz:

$$\lim_{n \rightarrow \infty} S_n(x) = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{(1+x)(1+2x)\cdots(1+nx)} \right) = 1.$$

Demak, berilgan funksional qatorning yig'indisi $S(x) = 1$ bo'ladi.

50-§. Funksional ketma-ketlik va qatorlarning tekis yaqinlashuvchiligi

11.3. Funksional ketma-ketliklarning tekis yaqinlashuvchiligi.

Ushbu

$$f_1(x), f_2(x), \dots, f_n(x), \dots \quad (11.3.1)$$

funksional ketma-ketlik X ($X \subseteq R$) to'plamda yaqinlashuvchi va uning limit funksiyasi $f(x)$ bo'lsin.

11.3.2. Ta'rif. Agar $\forall \varepsilon > 0$ son olganda ham $\exists n_0(\varepsilon) \in N$ nomer topilib, $\forall n > n_0(\varepsilon)$ va $\forall x \in X$ lar uchun bir vaqtda

$$|f_n(x) - f(x)| < \varepsilon$$

tengsizlik bajarilsa, $\{f_n(x)\}$ funksional ketma-ketlik x to'plamda $f(x)$ ga tekis yaqinlashadi deyiladi va u qisqacha $n \rightarrow \infty$ da

$$f_n(x) \xrightarrow{X} f(x)$$

kabi belgilanadi.

11.3.3. Eslatma. 11.3.2-ta'rifdagi n_0 natural son faqat ε ga bog'liq bo'lib, x larga bog'liq bo'lmaydi. «Nuqtaviy» yaqinlashish (11.1.5) da

esa, n_0 natural son ε hamda x ga bog'liq bo'ladi. Shunday qilib, tekis yaqinlashishdan «nuqtaviy» yaqinlashish kelib chiqadi. Buning o'rinli emas.

11.3.4. Ta'rif. $\forall m \in N$ olinganda ham, $\exists \varepsilon_0 > 0$, $\exists n \geq m$ va x_0 mavjud bo'lib,

$$|f_n(x_0) - f(x_0)| \geq \varepsilon_0$$

tengsizlik bajarilsa, $\{f_n(x)\}$ funksional ketma-ketlik X to'plamda $f(x)$ funksiyaga *tekis yaqinlashmaydi* deyiladi va u qisqacha

$$f_n(x) \not\stackrel{X}{\rightarrow} f(x)$$

kabi belgilanadi.

11.3.5. Ta'rif. Agar $n \rightarrow \infty$ da $f_n(x) \stackrel{X}{\rightarrow} f(x)$ bo'lib, lekin $f_n(x) \not\stackrel{X}{\rightarrow} f(x)$ bo'lsa, $\{f_n(x)\}$ ketma-ketlik X da $f(x)$ funksiyaga *tekis yaqinlashmaydi (notekis yaqinlashadi)* deyiladi.

Xususiylashtirish holda, agar $f_n(x) \stackrel{X}{\rightarrow} f(x)$ va $\exists \varepsilon_0 > 0$, $\forall m \in N$ $\exists n \geq m$ va $\exists x_n \in X$

$$|f_n(x_n) - f(x_n)| \geq \varepsilon_0 \quad (11.3.6)$$

shart bajarilsa, $\{f_n(x)\}$ ketma-ketlik X da $f(x)$ ga tekis yaqinlashmaydi deyiladi.

Misol. $f_n(x) = x^n$, $0 \leq x < 1$. Bu $\{f_n(x)\}$ ketma-ketlik $\forall x \in [0, 1)$ uchun $n \rightarrow \infty$ da $f_n(x) \rightarrow 0$. Lekin, u $\forall x \in [0, 1)$ da $f(x) = 0$ limit funksiyaga tekis yaqinlashmaydi. Haqiqatan ham,

$$\sup_{x \in [0, 1)} |x^n - 0| = 1, \quad f_n \not\stackrel{[0, 1)}{\rightarrow} 0.$$

Berilgan ketma-ketlik $[0, q]$ ($0 < q < 1$) kesmada $f(x) = 0$ funksiyaga tekis yaqinlashadi. Haqiqatan ham,

$$\sup_{x \in [0, q]} |x^n - 0| = q^n \xrightarrow{n \rightarrow \infty} 0.$$

Tekis yaqinlashuvchi funksional ketma-ketliklar quyidagi xossalarga ega.

1⁰. Agar $\{f_n(x)\}$ va $\{g_n(x)\}$ funksional ketma-ketliklar $n \rightarrow \infty$ da

$$f_n \xrightarrow{X} f(x), \quad g_n \xrightarrow{X} g(x)$$

bo'lsa, u holda $\{\lambda f_n(x) + \mu g_n(x)\}$ (bunda λ, μ ixtiyoriy haqiqiy sonlar) funksional ketma-ketlik ham $n \rightarrow \infty$ da

$$\lambda f_n(x) + \mu g_n(x) \xrightarrow{X} \lambda f(x) + \mu g(x)$$

bo'ladi.

2⁰. Agar $\{f_n(x)\}$ funksional ketma-ketlik $f_n(x) \xrightarrow{X} f(x)$ bo'lib, $g(x)$

funksiya esa X to'plamda chegaralangan bo'lsa, ya'ni

$$\exists M > 0: \forall x \in X \text{ uchun } |g(x)| \leq M$$

bo'lsa, u holda $\{g(x)f_n(x)\}$ ketma-ketlik ham $n \rightarrow \infty$ da

$$g(x)f_n(x) \xrightarrow{X} g(x)f(x) \text{ bo'ladi.}$$

11.3.7. Teorema (tekis yaqinlashish sharti). (11.3.1) funksional ketma-ketlik X to'plamda $f(x)$ ga tekis yaqinlashishi uchun

$$\lim_{n \rightarrow \infty} \sup_{x \in X} |f_n(x) - f(x)| = 0 \quad (11.3.8)$$

shartning bajarilishi zarur va yetarli, ya'ni $n \rightarrow \infty$

$$\text{da } f_n(x) \xrightarrow{X} f(x) \Leftrightarrow \lim_{n \rightarrow \infty} \sup_{x \in X} |f_n(x) - f(x)| = 0.$$

Isboti. Zarurligi. Teorema shartiga ko'ra, $n \rightarrow \infty$

da $f_n(x) \xrightarrow{X} f(x): \forall \varepsilon > 0$ son olganda ham $\exists n_0(\varepsilon) \in N$ nomer

topilib, $\forall n > n_0(\varepsilon)$ va $\forall x \in X$ lar uchun bir vaqtda $|f_n(x) - f(x)| < \varepsilon$

tengsizlik bajarilsin. Bundan $\sup_{x \in X} |f_n(x) - f(x)| \leq \varepsilon$ ($\lim_{n \rightarrow \infty} \sup_{x \in X} |f_n(x) - f(x)| = 0$)

bo'ladi.

Yetarliligi. Faraz qilaylik (11.3.8) shart o'rinli bo'lsin. U holda limitning ta'rifiga asosan, $\forall \varepsilon > 0$ son olganda ham $\exists n_0(\varepsilon) \in N$ nomer topilib, $\forall n > n_0(\varepsilon)$ boshlab $\sup_{x \in X} |f_n(x) - f(x)| < \varepsilon$ tengsizlik bajariladi.

Bundan esa, $\forall x \in X$ lar uchun bir vaqtda $|f_n(x) - f(x)| < \varepsilon$ tengsizlik o'rinli bo'ladi.

Demak, $\forall \varepsilon > 0$ son olganda ham $\exists n_0(\varepsilon) \in \mathbb{N}$ nomer topilib, $\forall n > n_0$ va $\forall x \in X$ lar uchun bir vaqtda $|f_n(x) - f(x)| < \varepsilon$ tengsizlik o'rinli. Bu $n \rightarrow \infty$ da $f_n(x) \xrightarrow{X} f(x)$ ekanligini anglatadi.

Shunday qilib, $n \rightarrow \infty$ da $f_n(x) \xrightarrow{X} f(x) \Leftrightarrow \lim_{n \rightarrow \infty} \sup_{x \in X} |f_n(x) - f(x)| = 0$ bo'ladi.

11.3.9. Teorema. (11.3.1) funksional ketma-ketlik X to'planidagi $f(x)$ ga tekis yaqinlashishi uchun shunday $\{a_n\}$ sonli ketma-ketlik (bunda $\lim_{n \rightarrow \infty} a_n = 0$) va shunday m nomer mavjud bo'lib, barcha $n > m$ va barcha $x \in X$ lar uchun

$$|f_n(x) - f(x)| \leq a_n \quad (10.3.10)$$

tengsizlikning bajarilishi zarur va yetarli.

Isboti. Yetarliligi. (10.3.10) shart o'rinli bo'lsin. U holda $\forall n \in \mathbb{N}$ va $\forall x \in X$ lar uchun $0 \leq |f_n(x) - f(x)| \leq a_n$ tengsizlik o'rinli. 2.8.4-xossaga asosan, $\lim_{n \rightarrow \infty} \sup_{x \in X} |f_n(x) - f(x)| = 0$ bo'ladi. Tekis yaqinlashish shartiga

asosan, $n \rightarrow \infty$ da $f_n(x) \xrightarrow{X} f(x)$ kelib chiqadi.

Zarurligi. Faraz qilaylik $n \rightarrow \infty$ da $f_n(x) \xrightarrow{X} f(x)$ bo'lsin. U holda $a_n = \sup_{x \in X} |f_n(x) - f(x)| \rightarrow 0$ deb olsak, (11.3.8) kriteriyga asosan, (10.3.10) shart kelib chiqadi. ■

11.3.11. Natija. $n \rightarrow \infty$ da $f_n(x) \xrightarrow{X} f(x)$ bo'lishi uchun $\exists x_n \in X: n \rightarrow \infty$ da

$$|f_n(x_n) - f(x_n)| \rightarrow 0 \quad (11.3.12)$$

bo'lishi zarur va yetarli.

Isboti. Zarurligi. (11.3.12) shart bajarilsin. U holda $n \rightarrow \infty$ da $\sup_{x \in X} |f_n(x) - f(x)| \geq |f_n(x_n) - f(x_n)| \rightarrow 0$, bundan $\sup_{x \in X} |f_n(x) - f(x)| \rightarrow 0$. U holda, tekis yaqinlashish sharti bajarilmaydi, ya'ni $n \rightarrow \infty$ da $f_n(x) \not\xrightarrow{X} f(x)$ bo'ladi.

Yaqinligi. $n \rightarrow \infty$ da $f_n(x) \not\rightarrow f(x)$ bo'lsin. U holda, ekstremumning mavjudligi asosan, $\forall n \in N$ lar uchun

$$\exists x_n \in X : |f_n(x_n) - f(x_n)| \geq \sup_{x \in X} |f_n(x) - f(x)| - \frac{1}{n}.$$

Bundan, $n \rightarrow \infty$ da $|f_n(x_n) - f(x_n)| \not\rightarrow 0$, chunki aks holda,

har qanday $|f_n(x) - f(x)| \leq |f_n(x_n) - f(x_n)| + \frac{1}{n}$ dan **2.8.8-xossasiga** ko'ra,

har qanday $|f_n(x) - f(x)| = 0$ bo'lib qoladi. Bu esa, tekis yaqinlashish shartiga

ziddiyat, $n \rightarrow \infty$ da $f_n(x) \not\rightarrow f(x)$ shartiga zid bo'lib qoladi.

11.3.12. Misol. Ushbu

$$\{f_n(x)\} = \left\{ \frac{\cos n\sqrt{x}}{n} \right\}$$

funktsional ketma-ketlikning $X = [0; +\infty)$ to'plamda tekis yaqinlashuvchi ekanligini ko'rsating.

Yechilishi. Bu ketma-ketlikning limit funksiyasi

$$f(x) = \lim_{n \rightarrow \infty} \frac{\cos n\sqrt{x}}{n} = 0$$

bo'lib, $X = [0; +\infty)$ da yaqinlashuvchi bo'ladi. Endi yaqinlashish

xarakterini aniqlaymiz. $\forall \varepsilon > 0$ son olinganda ham $m = \left\lceil \frac{1}{\varepsilon} \right\rceil$ deyilsa, unda

barcha $n > m$ va $\forall x \in X$ va lar uchun

$$|f_n(x) - f(x)| = \left| \frac{\cos n\sqrt{x}}{n} - 0 \right| = \left| \frac{\cos n\sqrt{x}}{n} \right| \leq \frac{1}{n} < \frac{1}{m+1} < \varepsilon$$

bo'ladi. Yuqoridagi **11.3.2**-ta'rif va **11.3.7**-teoremaga ko'ra,

$\{f_n(x)\} = \left\{ \frac{\cos n\sqrt{x}}{n} \right\}$ ketma-ketlik $f(x) = 0$ limit funksiyaga tekis

yaqinlashadi: $\frac{\cos n\sqrt{x}}{n} \xrightarrow{n \rightarrow \infty} 0$.

11.3.13. Misol. Ushbu

$$\{f_n(x)\} = \left\{ \sqrt[4]{x + \frac{1}{n}} - \sqrt[4]{x} \right\}$$

funksional ketma-ketlikni $X = [0; \infty)$ da tekis yaqinlashuvchilik tekshiring.

Yechilishi. Barcha $x \in X$ va $\forall n \in N$ lar uchun

$$x + \frac{1}{n} \leq \left(\sqrt{x} + \frac{1}{\sqrt{n}} \right)^2$$

tengsizlik o'rinli bo'ladi. U holda

$$0 \leq \sqrt[4]{x + \frac{1}{n}} - \sqrt[4]{x} = \frac{\sqrt{x + \frac{1}{n}} - \sqrt{x}}{\sqrt[4]{x + \frac{1}{n}} + \sqrt[4]{x}} \leq n^{\frac{1}{4}} \left(\sqrt{\left(\sqrt{x} + \frac{1}{\sqrt{n}} \right)^2} - \sqrt{x} \right) = \frac{n^{\frac{1}{4}}}{\sqrt{n}} = \frac{1}{\sqrt[4]{n}} = a_n$$

bo'ladi.

Demak, **11.3.9-teoremaga** asosan, berilgan funksional ketma-ketlik $[0; \infty)$ da $f(x) = 0$ limit funksiyaga tekis yaqinlashadi.

11.3.14. Misol. Ushbu

$$\{f_n(x)\} = \left\{ \frac{\sqrt{nx}}{2 + nx} \right\}$$

funksional ketma-ketlikning $X = [0; 2]$ da notekis yaqinlashuvchilik ko'rsating.

Yechilishi. Bu ketma-ketlikning limit funksiyasi

$$f(x) = \lim_{n \rightarrow \infty} \frac{\sqrt{nx}}{2 + nx} = 0$$

bo'ladi. Endi berilgan ketma-ketlikning $f(x) = 0$ limit funksiyaga note yaqinlashishini ko'rsatamiz. $\forall \varepsilon > 0$ son olinganda ham m natural :

sifatida $m = \left\lceil \frac{1}{\varepsilon^2 x} \right\rceil$, $x \neq 0$ olinsa, u holda $\forall m < n$ uchun

$$|f_n(x) - f(x)| = \left| \frac{\sqrt{nx}}{2 + nx} - 0 \right| = \frac{\sqrt{nx}}{2 + nx} < \frac{1}{\sqrt{nx}} \leq \frac{1}{\sqrt{(m+1)x}} < \varepsilon$$

bo'ladi. Ravshanki, $x = 0$ da $\forall n \in N$ lar uchun $f_n(0) = f(0) = 0$ bo'ladi.

Bu holda m ning x ga bog'liqligi evaziga, ixtiyoriy natural n son uchun

$\varepsilon_0 = \frac{1}{4}$ va $x_n = \frac{1}{n} \in (0; 2]$ deb olsak,

$$|f_n(x_n) - f(x_n)| = \frac{\sqrt{\frac{1}{n}}}{2 + \frac{1}{n}} = \frac{1}{3} > \varepsilon_0$$

bo'ladi. Bu esa berilgan $\{f_n(x)\}$ funksional ketma-ketlikning $f(x)=0$ limit funksiyaga notekis yaqinlashishini bildiradi, ya'ni $x \in [0;2]$ ning barcha qiymatlari uchun bir vaqtda yaraydigan m nomer mavjud emas. **11.3.5-**

ta'rifga ko'ra berilgan ketma-ketlik $f_n(x) = \frac{\sqrt{nx}}{2 + nx} \stackrel{[0,2]}{\neq} 0$.

11.3.15. Teorema (funktional ketma-ketlikning tekis yaqinlashishi uchun Koshi kriteriyasi). $\{f_n(x)\}$ funksional ketma-ketlik X to'plamda limit funksiyaga ega bo'lishi va unga tekis yaqinlashishi uchun, ixtiyoriy $\varepsilon > 0$ uchun x ga bog'liq bo'lmagan shunday nomer $m(\varepsilon)$ mavjud bo'lib, $n > m$ bo'lganda va istalgan $p \in N$ da x ning X dagi hamma qiymatlari uchun bir vaqtning o'zida

$$|f_{n+p}(x) - f_n(x)| < \varepsilon$$

tengsizlikning o'rinli bo'lishi zarur va yetarli.

11.3.15-teoremadagi Koshi shartini, qisqacha, kvantor belgisidan foydalanib, quyidagicha yozish mumkin:

$$\forall \varepsilon > 0 \exists m(\varepsilon) : \forall n \geq m, \forall p \in N, \forall x \in X : |f_{n+p}(x) - f_n(x)| < \varepsilon. \quad (11.3.16)$$

Isboti. Zarurligi. $n \rightarrow \infty$ da $f_n(x) \xrightarrow{x} f(x)$ bo'lsin. U holda, **11.3.2-ta'rifga** ko'ra,

$$\forall \varepsilon > 0 \exists m(\varepsilon) \in N : \forall n \geq m, \forall p \in N, \forall x \in X : |f_n(x) - f(x)| < \frac{\varepsilon}{2}.$$

Xuddi shunday,

$$\forall \varepsilon > 0 \exists m(\varepsilon) \in N, \forall p \in N, : \forall n + p > n > m, \forall x \in X : |f_{n+p}(x) - f(x)| < \frac{\varepsilon}{2}$$

tengsizlik o'rinli bo'ladi. Yuqoridagi tengsizliklarga asosan, $\forall x \in X$ uchun

$$|f_{n+p}(x) - f_n(x)| < |f_{n+p}(x) - f(x)| + |f_n(x) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Demak, (11.3.16) shart bajarildi.

Yetarliligi. (11.3.16) shart bajarilsin: $\forall \varepsilon > 0 \exists m(\varepsilon): \forall n \geq m, \forall p \in N, \forall x \in X$ uchun

$$|f_{n+p}(x) - f_n(x)| < \varepsilon$$

tengsizlik bajariladi. X to'plamdan olingan har bir $x_0 (x_0 \in A)$ $\{f_n(x)\}$ funksional ketma-ketlik $\{f_n(x_0)\}$ sonlar ketma-ketligiga aylana va (11.3.16) Koshi shart bajariladi. U holda Koshi teoremasiga asoslanib $\{f_n(x_0)\}$ sonlar ketma-ketligi yaqinlashuvchi.

Demak, X to'plamning har bir $x_0 (x_0 \in X)$ nuqtasida $\{f_n(x_0)\}$ sonlar ketma-ketligi yaqinlashuvchi. Bu $\{f_n(x)\}$ funksional ketma-ketlikning limit funksiyasini $f(x)$ deylik:

$$\lim_{n \rightarrow \infty} f_n(x) = f(x) \quad (x \in X).$$

Endi (11.3.16) tengsizlikda n va x larni belgilab $p \rightarrow \infty$ da limitga o'tamiz. Natijada,

$$|f(x) - f_n(x)| < \frac{\varepsilon}{2} < \varepsilon$$

hosil bo'ladi. Bundan esa,

$$\forall \varepsilon > 0 \exists m(\varepsilon) \in N: \forall n \geq m, \forall p \in N, \forall x \in X: |f_n(x) - f(x)| < \varepsilon, \text{ ya'ni}$$

$n \rightarrow \infty$ da $f_n(x) \xrightarrow{x} f(x)$ ekanligi kelib chiqadi. ■

Agar Koshi sharti bajarilmasa, ya'ni

$$\exists \varepsilon_0 > 0: \forall k \in N, \exists n \geq k \exists p \in N \exists \bar{x} \in X: |f_{n+p}(\bar{x}) - f_n(\bar{x})| \geq \varepsilon_0$$

bo'lsa, u holda $\{f_n(x)\}$ funksional ketma-ketlik X to'plamda $f(x)$ funksiyaga *notekis yaqinlashuvchi* deyiladi.

Xususiyl holda, agar

$$\exists \varepsilon_0 > 0: \exists m \in N \forall n \geq m \exists p \in N \exists x_n \in X: |f_{n+p}(x_n) - f(x_n)| \geq \varepsilon_0$$

bo'lsa, $\{f_n(x)\}$ ketma-ketlik X to'plamda $f(x)$ funksiyaga tekis yaqinlashuvchi bo'lmaydi.

11.3.17. Misol. Ushbu

$$\{f_n(x)\} = \left\{ \frac{\cos \sqrt{nx}}{\sqrt{n+2x}} \right\}$$

funksional ketma-ketlikning $X = [0; +\infty)$ to'plamda tekis yaqinlashuvchiligi Koshi teoremasidan foydalanib ko'rsating.

Yechilishi. $\{f_n(x)\} = \left\{\frac{\cos\sqrt{nx}}{\sqrt{n+2x}}\right\}$ ketma-ketlik uchun $[0; +\infty)$ da

(11.4.10) shartni bajarilishini ko'rsatamiz:

$$|f_{n+p}(x) - f_n(x)| = \left| \frac{\cos\sqrt{(n+p)x}}{\sqrt{n+p+2x}} - \frac{\cos\sqrt{nx}}{\sqrt{n+2x}} \right| \leq \left| \frac{\cos\sqrt{(n+p)x}}{\sqrt{n+p+2x}} \right| + \left| \frac{\cos\sqrt{nx}}{\sqrt{n+2x}} \right| \leq$$

$$\sup_{x \in X} \left| \frac{\cos\sqrt{(n+p)x}}{\sqrt{n+p+2x}} \right| + \sup_{x \in X} \left| \frac{\cos\sqrt{nx}}{\sqrt{n+2x}} \right| = \frac{1}{\sqrt{n+p}} + \frac{1}{\sqrt{n}}.$$

Ayrim $\varepsilon > 0$ songa ko'ra natural m soni $m = \left[\frac{4}{\varepsilon^2}\right] + 1$ deb olinsa, u holda

barcha $n > m$ va barcha $p \in \mathbb{N}$ lar uchun

$$\frac{1}{\sqrt{n+p}} + \frac{1}{\sqrt{n}} < \frac{2}{\sqrt{m}} < \varepsilon$$

bo'ladi.

Demak, $\forall \varepsilon > 0$ son olinganda ham shunday natural m son mavjudki, $\forall n > m, \forall p \in \mathbb{N}$ va $\forall x \in [0; +\infty)$ lar uchun

$$|f_{n+p}(x) - f_n(x)| < \varepsilon$$

tempsizlik o'rinli bo'ladi. Bu esa, $\left\{\frac{\cos\sqrt{nx}}{\sqrt{n+2x}}\right\}$ ketma-ketlikning, **11.3.15-**

ko'shli teoremasiga ko'ra, X da tekis yaqinlashuvchi ekanligini ko'rsatadi.

11.4. Funktsional qatorlarning tekis yaqinlashuvchiligi. Ushbu

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + \dots + u_n(x) + \dots \quad (11.4.1)$$

funksional qator $X (X \subseteq \mathbb{R})$ to'plamda yaqinlashuvchi va uning yig'indisi $S(x)$ bo'lsin, ya'ni

$$\lim_{n \rightarrow \infty} S_n(x) = S(x) = \sum_{n=1}^{\infty} u_n(x).$$

11.4.2. Ta'rif. Agar (11.4.1) funksional qatorning $\{S_n(x)\}$ qisman yig'indilari ketma-ketligi X to'plamda $S(x)$ ga tekis yaqinlashsa, (11.4.1) funksional qator X to'plamda $S(x)$ ga *tekis yaqinlashadi* deyiladi va u qisqacha

$$S_n(x) \xrightarrow{X} S(x) \quad (11.4.3)$$

kabi belgilanadi.

11.4.4. Eslatma. Funktsional qatorlarning tekis yaqinlashuvchil (yaqinlashmovchiligi) tushunchasi ham ularning o'z yaqinlashuvchiligi singari, funktsional ketma-ketliklarning tekis yaqinlashuvchilik (yaqinlashmovchiligi) tushunchasi orqali kiritiladi.

11.4.2-ta'rifni qisqacha, kvantor belgisidan foydalanib quyidagicha yozish mumkin:

$$\forall \varepsilon > 0 \exists m(\varepsilon) : \forall n > m \forall x \in X \rightarrow |S_n(x) - S(x)| < \varepsilon. \quad (11.4.5)$$

11.4.6. Ta'rif. (11.4.1) qatorning dastlabki n ta hadini tashlab yuborgandan so'ng, hosil bo'lgan ushbu

$$r_n(x) = u_{n+1}(x) + u_{n+2}(x) + \dots = \sum_{k=n+1}^{\infty} u_k(x)$$

qatorga (11.4.1) funktsional qatorning n ta hadidan keyingi *qoldiq* deyiladi.

Bunda

$$r_n(x) = S(x) - S_n(x)$$

bo'ladi. U holda (11.4.3) shartni quyidagi ko'rinishda ifodalash mumkin:

$$r_n(x) \xrightarrow{X} 0. \quad (11.4.7)$$

(11.4.3) va (11.4.7) shartlar teng kuchli.

11.4.8. Ta'rif. Agar X to'plamda $S_n(x)$ ketma-ketlikning limit funksiyasi mavjud bo'lsa va (11.4.5) shart bajarilmasa, ya'ni

$$\forall \varepsilon_0 > 0 : \forall k \in \mathbb{N} \exists n \geq k \forall \bar{x} \in X \rightarrow |S_n(\bar{x}) - S(\bar{x})| \geq \varepsilon_0$$

bo'lsa, $S_n(x)$ ketma-ketlik X to'plamda $S(x)$ ga *notekis yaqinlasha* deyiladi.

11.4.9. Misol. Ushbu

$$\sum_{n=0}^{\infty} (e^{-nx} - e^{-(n+1)x})$$

funktsional qatori, ta'rifga ko'ra, a) $X = (0; \infty)$; b) $X_1 = [\delta; \infty)$, $\delta > 0$ soxalarda tekis yaqinlashishga tekshiring.

Yechilishi. Berilgan qatorning qisman yig'indisini topamiz:

$$\begin{aligned} S_n(x) &= (1 - e^{-x}) \cdot (1 + e^{-x} + e^{-2x} + \dots + e^{-(n-1)x}) = \\ &= (1 - e^{-x}) \frac{(1 - e^{-nx})}{1 - e^{-x}} = 1 - e^{-nx}. \end{aligned}$$

Hundan

$$\lim_{n \rightarrow \infty} S_n(x) = 1, \quad S(x) = 1.$$

a) Funktsional qatorni ta'rifga ko'ra, tekis yaqinlashishga tekshirish uchun $r_n(x) = S(x) - S_n(x)$ ayirmani qaraymiz:

$$|r_n(x)| = |1 + e^{-nx} - 1| = e^{-nx},$$

$\varepsilon > 0$ son olganda $m = \left\lceil \frac{1}{x} \ln \frac{1}{\varepsilon} \right\rceil + 1$ ($x \neq 0$) deyilsa, u holda $n > m$ lar uchun

$$|r_n(x)| = |S_n(x) - S(x)| = e^{-nx} < \varepsilon \quad (11.4.10)$$

tengsizlik bajariladi.

Agar $x = 0$ bo'lsa, ravshanki $\forall n \in \mathbb{N}$ lar uchun $S_n(0) = 0$; $S(0) = 1$ bo'lib, $|r_n(0)| = |S_n(0) - S(0)| = 1$ bo'ladi. m natural son $\varepsilon > 0$ va $x \in (0; +\infty)$ larga bog'liq bo'lib, u barcha x lar uchun umumiy bo'la olmaydi, chunki $m = \left\lceil \frac{1}{x} \ln \frac{1}{\varepsilon} \right\rceil + 1$ ning $(0; \infty)$ da x bo'yicha maksimumi chekli son bo'lmaydi, ya'ni $\forall n \in \mathbb{N}$ son olsak ham $\exists \varepsilon_0 > 0$ ($\varepsilon_0 = \frac{1}{e^2}$) va $x_n = \frac{1}{n} \in (0, \infty)$ nuqta topiladiki,

$$\left| S\left(\frac{1}{n}\right) - S_n\left(\frac{1}{n}\right) \right| = e^{-1} > \varepsilon_0$$

bo'ladi.

Demak, berilgan funktsional qator, **11.4.8**-ta'rifga ko'ra, $\forall x \in (0; \infty)$ sohada notekis yaqinlashuvchi bo'ladi.

$$b) \quad \forall \varepsilon > 0 \text{ son olinganda ham } m_1 = \max_{x \in [\delta; \infty)} \left\{ \left\lceil \frac{1}{x} \ln \frac{1}{\varepsilon} \right\rceil + 1 \right\} =$$

$$\left\lceil \frac{1}{\delta} \ln \frac{1}{\varepsilon} \right\rceil + 1 \text{ deb olinsa, } \forall n > m_1 \text{ va } \forall x \in [\delta; \infty) \text{ lar uchun birdaniga}$$

(11.4.10) tengsizlik bajariladi. Shunday qilib, (11.4.5) shartga asosan,

berilgan funksional qator $[\delta; \infty)$ sohada x ga nisbatan tekis yaqinlashuvchi bo'ladi, ya'ni $S_n(x) \xrightarrow{[\delta; \infty)} 1$.

Funksional qator tekis yaqinlashish ta'rifidan hamda funksional ketma-ketlikning tekis yaqinlashish kriteriysidan quyidagi funksional qator tekis yaqinlashish va kriteriysi kelib chiqadi.

6.4.10. Teorema. (11.4.1) funksional qatorning x da tekis yaqinlashishi uchun

$$\lim_{n \rightarrow \infty} \sup_{x \in X} |r_n(x)| = 0 \quad (11.4.11)$$

shartning bajarilishi zarur va yetarlidir.

11.4.12. Teorema (funktional qatorning tekis yaqinlashishi uchun Koshi kriteriysi). (11.4.1) funksional qator X da tekis yaqinlashishi uchun $\forall \varepsilon > 0$ son olinganda ham $\exists m(\varepsilon)(m \in \mathbb{N})$ nom topilib, $\forall n \geq m(\varepsilon)$, barcha butun $p \geq 0$ sonlar va $\forall x \in X$ lar uchun

$$|S_{n+p} - S_n| = \left| \sum_{k=n+1}^{n+p} u_k(x) \right| < \varepsilon \quad (11.4.13)$$

shartning bajarilishi zarur va yetarli.

Teoremaning isboti funksional ketma-ketlikning tekis yaqinlashish kriteriysi, ya'ni 11.3.15-teoremani funksional qator qisman yig'indisi qo'llash natijasida kelib chiqadi.

11.4.14. Teorema (zaruriy shart). Agar (11.4.1) funksional qator X da tekis yaqinlashuvchi bo'lsa, u holda uning umumiy hadi $u_n(x) (n=1,2,\dots) \xrightarrow{X} 0$ bo'ladi.

Isboti. Shartga ko'ra, (11.4.1) funksional qator x to'plamda tekis yaqinlashuvchi bo'lgani uchun (11.4.13) shart bajariladi. (11.4.1) shartda $p=1$ deb olinsa, u holda

$$\forall \varepsilon > 0 \exists m(\varepsilon) \in \mathbb{N} : \forall n > m \forall x \in X \rightarrow |u_{n+1}(x)| < \varepsilon$$

tengsizlik bajariladi, ya'ni $n \rightarrow \infty$ da $u_n(x) \xrightarrow{X} 0$. ■

11.4.16. Eslatma. Koshi kriteriysidan, ya'ni 11.4.12-teoremad xususiy holda, $p=0$ bo'lganda, 11.4.14-teorema kelib chiqadi.

Agar 11.4.12-teoremaning shartlari bajarilmasa, ya'ni

$$\exists \varepsilon_0 > 0 \quad m \in \mathbb{N} \quad \exists n \geq m \quad \exists p \in \mathbb{N} \quad \exists \bar{x} \in X \rightarrow \left| \sum_{k=n+1}^{n+p} u_k(x) \right| \geq \varepsilon_0$$

bo'lsa, (11.4.1) funksional qator X da tekis yaqinlashuvchi bo'lmaydi.

Xususiylashtirishda, agar

$$\exists \varepsilon_0 > 0 \quad \exists n_0 \in \mathbb{N} \quad \forall n \geq n_0 \quad \exists x_n \in X \rightarrow |u_n(x_n)| \geq \varepsilon_0$$

bajarilsa, u holda (11.4.1) funksional qator X da tekis yaqinlashuvchi bo'lmaydi.

11.4.17. Misol. Ushbu

$$\sum_n \frac{\sin nx}{3^n}$$

funksional qatorning Koshi kriteriysiga ko'ra, $X = [0, \infty)$ da tekis yaqinlashuvchi ekanligini ko'rsating.

Yechilishi. $S_{n+p} - S_n$ ayirmani tuzamiz va uni baholaymiz:

$$\begin{aligned} |S_{n+p} - S_n| &= \left| \frac{\sin(n+1)x}{3^{n+1}} + \frac{\sin(n+2)x}{3^{n+2}} + \dots + \frac{\sin(n+p)x}{3^{n+p}} \right| \leq \\ &\leq \frac{1}{3^{n+1}} + \frac{1}{3^{n+2}} + \dots + \frac{1}{3^{n+p}} = \\ &= \frac{1}{3^{n+1}} \left(1 + \frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^{p-1}} \right) = \frac{1}{3^{n+1}} \cdot \frac{1 - \left(\frac{1}{3}\right)^p}{1 - \frac{1}{3}} = \frac{1}{3^n \cdot 2} \left(1 - \left(\frac{1}{3}\right)^p \right) < \frac{1}{3^n}. \end{aligned}$$

$\forall \varepsilon > 0$ son olinganda ham $m = [\log_3 \frac{1}{\varepsilon}] + 1$ deb olinsa, u holda

$\forall x \in [0, \infty)$, $\forall n > m$ va $\forall p$ lar uchun $|S_{n+p} - S_n| < \varepsilon$ tengsizlik bajariladi.

Demak, Koshi kriteriysiga asosan, berilgan funksional qator $X = [0, \infty)$ da tekis yaqinlashuvchi bo'ladi.

11.4.18. Teorema (taqqoslash alomati). $u_k(x)$ va $v_k(x)$ funksiyalar X to'plamda aniqlangan bo'lib, $\forall x \in X$ va $\forall k \in \mathbb{N}$ uchun

$$|u_k(x)| \leq v_k(x)$$

tengsizlik o'rinli bo'lib, $\sum_{k=1}^{\infty} v_k(x)$ funksional qator X to'plamda tekis

yaqinlashuvchi bo'lsa, u holda $\sum_{k=1}^{\infty} u_k(x)$ funksional qator ham X

to'plamda absolyut va tekis yaqinlashuvchi bo'ladi.

Isboti. $\forall x \in X$ va $\forall n \in N, \forall p \in N$ uchun

$$\left| \sum_{k=n+1}^{n+p} u_k(x) \right| \leq \sum_{k=n+1}^{n+p} |u_k(x)| \leq \sum_{k=n+1}^{n+p} v_k(x)$$

tengsizlik o'rinli bo'ladi. Shartga ko'ra, $\sum_{k=1}^{\infty} v_k(x)$ funksional qator to'plamda tekis yaqinlashuvchi bo'lganligi uchun **11.4.12-teorema** asosan, $\forall \varepsilon > 0 \exists n_0(\varepsilon) \in N: \forall n \geq n_0(\varepsilon), \forall p \in N, \forall x \in X$ uchun

$$\sup_{x \in X} \sum_{k=n+1}^{n+p} v_k(x) < \varepsilon$$

tengsizlik o'rinli bo'ladi. U holda, $\forall \varepsilon > 0 \exists n_0(\varepsilon) \in N: \forall n \geq n_0(\varepsilon), \forall p \in N, \forall x \in X$ uchun

$$\sup_{x \in X} \left| \sum_{k=n+1}^{n+p} u_k(x) \right| \leq \sup_{x \in X} \sum_{k=n+1}^{n+p} |u_k(x)| < \varepsilon.$$

Demak, Koshi kriteriysiga asosan, $\sum_{k=1}^{\infty} u_k(x)$ va $\sum_{k=1}^{\infty} |u_k(x)|$ funksion qatorlar ham X to'plamda tekis yaqinlashuvchi bo'ladi. ■

Bu teoremadan xususiylashtirilgan holda quyidagi teorema kelib chiqadi.

11.4.19. Teorema (Veyershtrass alomati). Agar (11.4.1) funksional qatorning har bir hadi X da aniqlangan bo'lib, $\forall x \in X, \forall n > n_0$ uchun

$$|u_n(x)| \leq c_n \quad (11.4.20)$$

tengsizlikni qanoatlantirsa va

$$\sum_{n=1}^{\infty} c_n = c_1 + c_2 + \dots + c_n + \dots \quad (11.4.21)$$

sonli qator yaqinlashuvchi bo'lsa, u holda (11.4.1) funksional qator X da absolyut va tekis yaqinlashuvchi bo'ladi.

Isboti. (11.4.20) shartga asosan, $\forall x \in X, \forall n > n_0$ va barcha butun $p \geq 0$ uchun

$$|u_{n+1}(x) + u_{n+2}(x) + \dots + u_{n+p}(x)| \leq c_{n+1} + c_{n+2} + \dots + c_{n+p}$$

tengsizlik o'rinli bo'ladi. (11.4.21) sonli qator yaqinlashuvchi bo'lganligi uchun $\exists n_0 \in N$ va $\forall n > n_0$ boshlab

$$c_{n+1} + c_{n+2} + \dots + c_{n+m} < \varepsilon$$

bo'ladi.

Demak, $\forall \varepsilon > 0$ son olinganda ham $\exists n_0(\varepsilon)(n_0 \in \mathbb{N})$ nomer topilib, $n \geq n_0(\varepsilon)$, barcha butun $p \geq 0$ sonlar va $\forall x \in X$ lar uchun

$$|S_{n+p} - S_n| = \left| \sum_{k=n+1}^{n+p} u_k(x) \right| < \varepsilon$$

tengsizlik o'rinli bo'ladi. Bundan **11.4.12-teoremaga** asosan, (11.4.1) funksional qator tekis yaqinlashuvchi bo'ladi. ■

11.4.22. Natija. Agar $\sum_{n=1}^{\infty} a_n$ sonli qator yaqinlashuvchi bo'lsa, bunda $u_n = \sup_{x \in E} |u_n(x)|$, (11.4.1) funksional qator tekis yaqinlashuvchi bo'ladi.

11.4.23. Eslatma. 11.4.19-Veyershtrass alomati funksional qatorning tekis yaqinlashishi uchun faqat yetarli shart bo'lib, lekin zaruriy shart bo'la olmaydi.

11.4.24. Misol. Ushbu

$$\sum_{n=1}^{\infty} \frac{1}{x^4 + n^4}$$

funksional qatorni $X = (-\infty; +\infty)$ da tekis yaqinlashuvchilikka tekshiring.

Yechilishi. Bu funksional qatorning umumiy hadi $u_n(x) = \frac{1}{x^4 + n^4}$ funksiyadan iborat. X dagi barcha x va $n \in \mathbb{N}$ lar uchun

$$|u_n(x)| = \left| \frac{1}{x^4 + n^4} \right| \leq \frac{1}{n^4}$$

tengsizlik o'rinli bo'ladi. Bunda

$$\sum_{n=1}^{\infty} \frac{1}{n^4}$$

sonli qator yaqinlashuvchi.

Demak, **11.4.19-Veyershtrass** alomatiga ko'ra, berilgan funksional qator X da absolyut va tekis yaqinlashuvchi bo'ladi.

Quyidagi

$$\sum_{n=1}^{\infty} a_n(x)b_n(x) = a_1(x)b_1(x) + \dots + a_n(x)b_n(x) + \dots \quad (11.4.25)$$

funksional qatorni qaraymiz, bunda $a_n(x), b_n(x), (n=1,2,\dots)$ funksiyalar $X (X \subset R)$ da aniqlangan.

11.4.26. Teorema (Dirixle alomati). Agar: 1) $\{a_n(x)\}$ funksionlar ketma-ketlik $\forall x \in X$ lar va $\forall n \in N$ lar uchun monoton bo'lib, $a_{n+1}(x) \cdot a_{n+1}(x)$ yoki $a_{n+1}(x) \geq a_n(x)$ va $a_n(x) \xrightarrow{N} 0$ bo'lsa;

2) $\sum_{k=1}^n b_k(x)$ qatorning qisman yig'indilari ketma-ketligi $B_n(x) = \sum_{k=1}^n b_k(x)$ $\forall n \in N, \forall x \in X$ larda chegaralangan bo'lsa, ya'ni $\exists M > 0: \forall n \in N, \forall x \in X$ uchun

$$|B_n(x)| \leq M$$

bo'lsa, u holda (11.4. 25) qator X to'plamda tekis yaqinlashuvchi bo'ladi.

Isboti. (11.4.25) qator uchun Koshi shartining bajarilishini ko'rsatasatamiz. Buning uchun quyidagi belgilashlarni kiritamiz:

$$B_n(x) = \sum_{k=1}^n b_k(x). \quad (11.4.27)$$

$$\sigma = \sum_{k=n+1}^{n+p} a_k(x) b_k(x), \quad n \in N, p \in N \quad (11.4.28)$$

Ravshanki, $k > 1$ da $b_k(x) = B_k(x) - B_{k-1}(x)$. Buni va (11.4.27) e'tiborga olgan holda σ ni quyidagicha ifodalaymiz:

$$\sigma = \sum_{k=n+1}^{n+p} a_k(x) B_k(x) - \sum_{k=n+1}^{n+p} a_k(x) B_{k-1}(x),$$

$$\text{bunda } \sum_{k=n+1}^{n+p} a_k(x) B_{k-1}(x) = \sum_{k=n+1}^{n+p-1} a_{k+1}(x) B_k(x) + a_{n+1}(x) B_n(x).$$

Demak,

$$\sigma = a_{n+p}(x) B_{n+p}(x) - a_{n+1}(x) B_n(x) + \sum_{k=n+1}^{n+p-1} (a_k(x) - a_{k+1}(x)) B_k(x) \quad (11.4.29)$$

Shartga ko'ra, $\forall n \in N$ uchun $a_{n+1}(x) \leq a_n(x)$ bo'lsin deb faraz qilaylik. holda $|B_n(x)| \leq M$ ekanligini e'tiborga olgan holda (11.4.29) da quyidagiga ega bo'lamiz:

$$|\sigma| \leq M(|a_{n+p}(x)| + |a_{n+1}(x)| + M \sum_{k=n+1}^{n+p-1} (a_k(x) - a_{k+1}(x)))$$

$$\text{bundan } \sum_{k=n+1}^{n+p-1} (a_k(x) - a_{k+1}(x)) = a_{n+1}(x) - a_{n+p}(x) \leq |a_{n+1}(x)| + |a_{n+p}(x)|$$

Shunday qilib, $\forall n \in N, \forall p \in N$ uchun

$$|\sigma| \leq 2M(|a_{n+1}(x)| + |a_{n+p}(x)|) \quad (11.4.30)$$

$n \in N$ uchun $a_{n+1}(x) \geq a_n(x)$ bo'lgan holda ham (11.4.30) o'rinli bo'lishini

ko'rsatish qiyin emas. $a_n(x) \xrightarrow{X} 0$ shartga asosan, $\forall \varepsilon > 0 \exists n_0(\varepsilon): \forall n \geq n_0(\varepsilon)$

bo'lin

$$|a_n(x)| < \frac{\varepsilon}{4M}. \quad (11.4.31)$$

(11.4.28), (11.4.30) va (11.4.31) larni e'tiborga olsak, u holda $\forall \varepsilon > 0 \exists n_0(\varepsilon): \forall n \geq n_0(\varepsilon) \forall p \in N$ uchun Koshi sharti

$$\left| \sum_{k=n+1}^{n+p} a_k(x) b_k(x) \right| < \varepsilon$$

o'rinli bo'ladi. Bundan esa, (11.4.25) qatorning X da tekis yaqinlashuvchiligi kelib chiqadi. ■

11.4.32. Teorema (Abel alomati). Agar: 1) $\sum_{n=1}^{\infty} b_n(x)$ qator X

to'plamda tekis yaqinlashuvchi bo'lsa, ya'ni $B_n(x) \xrightarrow{X} B(x)$;

2) $\{a_n(x)\}$ ketma-ketlik X to'plamda monoton bo'lsa, ya'ni $\forall n \in N$, $x \in X$ uchun $a_{n+1}(x) \leq a_n(x)$ yoki $a_{n+1}(x) \geq a_n(x)$ va $\exists M > 0: \forall n \in N$, $x \in X$ lar uchun $|a_n(x)| \leq M$ bo'lsa, u holda (6.15) funksional qator X da yaqinlashuvchi bo'ladi.

Isboti. Monoton ketma-ketlikning limiti haqidagi teorema asosan, $\lim_{n \rightarrow \infty} a_n(x) = a$, bundan $\{a_n(x) - a\}$ ketma-ketlik monoton bo'lib nolga intiladi. $\sum_{n=1}^{\infty} b_n(x)$ qatorning yaqinlashuvchiligidan, uning qisman yig'indisi $B_n(x)$ chegaralangan bo'ladi. U holda, Dirixli alomatiga ko'ra, $\sum_{n=1}^{\infty} (a_n(x) - a)$ qator yaqinlashuvchi. Bundan hamda $\sum_{n=1}^{\infty} b_n(x)$ qatorning yaqinlashuvchiligidan (11.4.25) yaqinlashuvchiligi kelib chiqadi, chunki $a_n(x) b_n(x) = (a_n(x) - a) b_n(x) + a b_n(x)$. ■

11.4.33. Eslatma. Dirixle alomatidan xususiylar haqida
 $b_n(x) = (-1)^{n+1}$ Leybnits alomati kelib chiqadi.

11.4.34. Misol. Ushbu

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^{\alpha}}$$

funksional qatorni $X = [\varepsilon, 2\pi - \varepsilon]$, $\varepsilon > 0$ da tekis yaqinlashuvchilik tekshiring.

Yechilishi. Agar $\alpha > 1$ bo'lsa, Veyershtrass alomatiga ko'ra berilgan funksional qator X da tekis yaqinlashuvchi bo'ladi.

Haqiqatan ham, $\forall x \in X$ va $\forall n \in N$ lar uchun

$$|\cos nx| \leq 1, \quad |u_n(x)| = \left| \frac{\cos nx}{n^{\alpha}} \right| \leq \frac{1}{n^{\alpha}}$$

bo'lib, $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$ sonli qator yaqinlashuvchi bo'ladi.

Agar $0 < \alpha < 1$ bo'lsa, $\{a_n\} = \left\{ \frac{1}{n^{\alpha}} \right\}$ ketma-ketlik mono-

kamayuvchi va $a_n \xrightarrow{n \rightarrow \infty} 0$. Ushbu $\sum_{n=1}^{\infty} \cos nx$ funksional qatorning qisqartirilgan yig'indilari ketma-ketligi uchun

$$\{B_n(x)\} = \left\{ \sum_{k=1}^n \cos kx \right\} = \left\{ \frac{\sin \frac{nx}{2} \cos \frac{(n+1)x}{2}}{\sin \frac{x}{2}} \right\}, \quad x \neq 2\pi m, \quad m \in Z$$

formula o'rinli. Bundan $\forall x \in X$ va $\forall n \in N$ lar uchun

$$\left| \sum_{k=1}^n \cos kx \right| \leq \frac{1}{\left| \sin \frac{x}{2} \right|} \leq \frac{1}{\sin \frac{\varepsilon}{2}}, \quad x \neq 2\pi m, \quad m \in Z$$

tengsizlik o'rinli bo'ladi.

Demak, 11.4.26- teoremgacha (Dirixle alomati) ko'ra, berilgan qator X da tekis yaqinlashuvchi bo'ladi.

Agar $\alpha \leq 0$ bo'lsa, u holda berilgan funksional qator X uzoqlashuvchi bo'ladi. Haqiqatan ham, har bir berilgan $x \in X$ uchun qator yaqinlashuvchiligidan zaruriy sharti bajarilmaydi, ya'ni $u_n(x)$ to'plamda 0 ga tekis intilmaydi: $\lim_{n \rightarrow \infty} u_n(x) \neq 0$.

11.4.35. Misol. Ushbu

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^3} + \sqrt{x}} \left(1 + \frac{x}{n}\right)^n$$

funksional qatorni $X=[0; 1]$ da tekis yaqinlashuvchilikka tekshiring.

Yechilishi. Quyidagi belgilashlarni kiritamiz:

$$b_n(x) = \frac{(-1)^n}{\sqrt{n^3} + \sqrt{x}}, \quad a_n(x) = \left(1 + \frac{x}{n}\right)^n.$$

Ayersstrass alomatiga ko'ra, $\sum_{n=1}^{\infty} b_n(x)$ qator X da tekis yaqinlashuvchi, ya'ni $\forall x \in X$ va $\forall n \in N$ lar uchun

$$|b_n(x)| \leq \frac{1}{n^{\frac{3}{2}}}$$

ompisizlik o'rinli va $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}}$ sonli qator yaqinlashuvchi.

$\{a_n(x)\} = \left\{ \left(1 + \frac{x}{n}\right)^n \right\}$ ketma-ketlik $[0; 1]$ da chegaralangan,

$$\left(1 + \frac{x}{n}\right)^n \leq \left(1 + \frac{1}{n}\right)^n < e$$

va $\varphi(t) = \left(1 + \frac{x}{t}\right)^t$ funksiya barcha $t \geq 1$ va $\forall x \in X$ lar uchun o'suvchi funksiyadir.

Demak, berilgan funksional qator, **11.4.32-teoremaga** (Abel alomati) ko'ra, $[0; 1]$ da tekis yaqinlashuvchi bo'ladi.

51-§. Funksional qatorlar yig'indisining hamda funksional ketma-ketlik limit funksiyasining funksional xossalari

11.5. Funksional qator yig'indisining uzluksizligi. Funksional qatorning tekis yaqinlashish haqidagi 11.4.2-ta'rifni e'tiborga olganda funksional qatorning barcha xossalarini funksional ketma-ketlikning xossalaridan va aksincha ketma-ketlikning barcha xossalarini funksional qatorning xossalaridan keltirib chiqarish qiyinchilik tug'dirmaydi. Lekin, bu nazariyaning barcha tadbirlarida ham amaliyotda ham deyarli

xamma vaqt funksional qator uchraydi. Shu sababli biz kelgi funksional qator yig'indisining xossalarini ifodalovchi ham teoremlarni isboti bilan, funksional ketma-ketliklar limit funksiyasini xossalarini ifodalovchi teoremlarni isbotsiz keltirishni lozim topdik.

X to'plamda yaqinlashuvchi

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + \dots + u_n(x) + \dots \quad (11.5.1)$$

funksional qator berilgan bo'lib, uning yig'indisi $S(x)$ bo'lsin.

11.5.2. Teorema. Agar (11.5.1) funksional qatorning har 1 hadi $u_n(x)$ ($n=1,2,\dots$) X to'plamda uzluksiz bo'lib, funksional qator X tekis yaqinlashuvchi bo'lsa, u holda funksional qatorning yig'indi $S(x)$ ham X to'plamda uzluksiz bo'ladi.

Isboti. X To'plamda biror x_0 ($x_0 \in X$) nuqtasi olib, $S(x)$ funksiyaning x_0 nuqtada uzluksizligini ko'rsatamiz. (11.5.1) funksional qatorning qisman yig'indisi qaraymiz:

$$S_n(x) = u_1(x) + u_2(x) + \dots + u_n(x).$$

Teoremaning shartiga ko'ra, $S_n(x) \xrightarrow{x} S(x)$. Tekis yaqinlashish ta'rifi

ko'ra, $\forall \varepsilon > 0 \exists n_0 = n_0(\varepsilon) \in \mathbb{N} : n > n_0(\varepsilon), x \in X$ uchun

$$|S_n(x) - S(x)| < \frac{\varepsilon}{3} \quad (11.5.3)$$

jumladan,

$$|S_n(x_0) - S(x_0)| < \frac{\varepsilon}{3} \quad (11.5.4)$$

tengsizlik ham o'rinli bo'ladi. (11.5.3) bilan (11.5.4) tengsizlik n ni n_0 dan katta biror n_1 qiymatida ham o'rinli bo'ladi :

$$|S_n(x) - S(x)| < \frac{\varepsilon}{3}, |S_n(x_0) - S(x_0)| < \frac{\varepsilon}{3}$$

Teoremaning sharti va chekli sondagi uzluksiz funksiyalarning yig'indisi ham uzluksiz bo'lishidan :

$$S_{n_1}(x) = u_1(x) + u_2(x) + \dots + u_{n_1}(x)$$

funksiya ham X tuplamda uzluksiz bo'ladi. Demak, $S_{n_1}(x)$ funksiya $x_0 \in X$ nuqtada uzluksiz. U holda, $\forall \varepsilon > 0 \exists \delta = \delta(\varepsilon) > 0, |x - x_0| < \delta$ bo'lganlikni qanoatlantiruvchi barcha $x \in X$ da

$$|S_{n_1}(x) - S_{n_1}(x_0)| < \frac{\varepsilon}{3} \quad (11.5.5)$$

bo'linadi. (11.5.3), (11.5.4), (11.5.5) tengsizliklardan

$$|S_n(x) - S(x_0)| = |S_n(x) - S_{n_1}(x) + (S_{n_1}(x) - S_{n_1}(x_0)) + (S_{n_1}(x_0) - S(x_0))| < \\ |S_n(x) - S_{n_1}(x)| + |S_{n_1}(x) - S_{n_1}(x_0)| + |S_{n_1}(x_0) - S(x_0)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Shundan $S(x)$ funksiyaning $x = x_0$ nuqtada uzluksiz ekanligi kelib chiqadi.

Bu teoremaning tasdiq'ini quyidagicha ifodalash mumkin :

$$\lim_{x \rightarrow x_0} \sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \lim_{x \rightarrow x_0} u_n(x).$$

11.5.6. Eslatma. 11.5.2-teoremadagi (11.5.1) qatorning X da tekis yaqinlashuvchilik sharti funksional qator yig'indisi $S(x)$ ning uzluksiz bo'lishi uchun yetarli shart bo'ladi, lekin zaruriy shart bo'la olmaydi.

11.5.7. Misol. Ushbu

$$\sum_{n=1}^{\infty} [nxe^{-nx} - (n-1)xe^{-(n-1)x}]$$

funksional qatorning $X = [0; 1]$ da notekis yaqinlashishini, lekin uning yig'indisi X da uzluksiz funksiya bo'lishini ko'rsating.

Yechilishi. Berilgan funksional qatorning qisman yig'indisini topamiz:

$$S_n(x) = \sum_{k=1}^n [kxe^{-kx} - (k-1)xe^{-(k-1)x}] = xe^{-x} + 2xe^{-2x} - xe^{-x} + \dots + \\ + nxe^{-nx} - (n-1)xe^{-(n-1)x} = nxe^{-nx}.$$

Endi $n \rightarrow \infty$ da $S_n(x)$ funksiyaning limitini hisoblaymiz:

$$S(x) = \lim_{n \rightarrow \infty} S_n(x) = \lim_{n \rightarrow \infty} \frac{nx}{e^{nx}} = 0.$$

Demak, berilgan funksional qatorning yig'indisi $S(x)=0$ X da uzluksiz funksiya ekan. Ammo, berilgan qator X da tek yaqinlashuvchi emas. Haqiqatan ham,

$$\sup_{x \in [0;1]} |S_n(X) - S(x)| = \sup_{x \in [0;1]} |nxe^{-nx}| = \frac{1}{e} \Rightarrow \lim_{n \rightarrow \infty} \sup_{x \in [0;1]} |f_n(x)| = e^{-1} \neq 0.$$

Shuning uchun, qator X da o'zining $S(x)$ yig'indisiga tek yaqinlashmaydi.

11.5.8. Misol. Ushbu $\sum_{n=1}^{\infty} x^2 e^{-nx}$ funksional qatorning yig'indisi $X=[0;1]$ da uzluksizlikka tekshiring.

Yechilishi. Berilgan funksional qatorning har 1 $u_n(x) = x^2 e^{-nx}$ ($n=1,2,\dots$) hadi X da uzluksiz bo'lib, bu funksional qat X da, Veyershtrass alomatiga ko'ra, tekis yaqinlashuvchi bo'la Haqiqatan ham, $\forall n \in \mathbb{N}$ va $\forall x \in [0;1]$ lar uchun

$$|u_n(x)| = |x^2 e^{-nx}| \leq \frac{1}{e^n} = a_n$$

va $\sum_{n=1}^{\infty} \frac{1}{e^n}$ sonli qator yaqinlashuvchi. Haqiqatan ham

$$S(x) = \frac{x^2}{e^x - 1}, x > 0. \quad x=0 \text{ bo'lganda } S(0)=0.$$

Demak, 11.5.2-teorema asosan, berilgan funksional qatorni yig'indisi X da uzluksiz funksiya bo'ladi.

11.6. Funksional ketma-ketlik limit funksiyasining uzluksizli X ($X \subset R$) to'plamda $\{f_n(x)\}$:

$$f_1(x), f_2(x), \dots, f_n(x) \dots \quad (11.6.1)$$

funksional ketma-ketlik berilgan bo'lib, uning limit funksiyasi $f(x)$ bo'lsin, ya'ni

$$\lim_{x \rightarrow x_0} f_n(x) = f(x).$$

11.6.2. Teorema. Agar $\{f_n(x)\}$ funksional ketma-ketlikning 1 bir $f_n(x)$ ($n=1,2,3,\dots$) hadi X to'plamda uzluksiz bo'lib, bu (11.6) funksional ketma-ketlik X to'plamda $f(x)$ ga tekis yaqinlashuvchi bo'lsa, u holda $f(x)$ limit funksiya ham X da uzluksiz bo'ladi.

11.6.3. Eslatma. 11.6.2-teoremaning shartlari bajarilganda

$$f(x_0) = \lim_{x \rightarrow x_0} (\lim_{n \rightarrow \infty} f_n(x)) = \lim_{n \rightarrow \infty} (\lim_{x \rightarrow x_0} f_n(x)).$$

tenglik o'rinli bo'ladi ($x_0 \in X$).

11.6.4. Misol. Ushbu

$$\{f_n(x)\} = \left\{ \frac{n^2 x^2}{2 + n^2 + x^2} \right\}$$

funksional ketma-ketlikning limit funksiyasini $X = [0; 1]$ da uzluksizlikka tekshiring.

Yechilishi. $f_n(x) (n=1, 2, \dots)$ ketma-ketlikning har bir hadi $X = [0; 1]$ da uzluksiz. Berilgan ketma-ketlikning limit funksiyasini topamiz:

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \frac{n^2 x^2}{2 + n^2 + x^2} = x^2; \quad f(x) = x^2.$$

$$\sup_{x \in [0; 1]} |f_n(x) - f(x)| = \sup_{x \in [0; 1]} \left| \frac{n^2 x^2}{2 + n^2 + x^2} - x^2 \right| = \sup_{x \in [0; 1]} \left| \frac{2x^2 + x^4}{2 + n^2 + x^2} \right| \leq \frac{3}{n^2}.$$

Bundan $\lim_{n \rightarrow \infty} \sup_{x \in [0; 1]} |f_n(x) - f(x)| = 0$ ekanligi kelib chiqadi, ya'ni $f_n(x) \xrightarrow{[0; 1]} x^2$.

Demak, 11.6.2-teorema asosan, berilgan ketma-ketlikning limit funksiyasi $f(x) = x^2$ ham $[0; 1]$ da uzluksiz funksiya bo'ladi. Bu yerda 11.6.2-eslatmadagi tenglikning o'rinli ekanligiga ishonch hosil qilish qiyin emas, ya'ni

$$\lim_{x \rightarrow x_0} (\lim_{n \rightarrow \infty} f_n(x)) = \lim_{n \rightarrow \infty} (\lim_{x \rightarrow x_0} f_n(x)) = x_0^2, \quad x_0 \in [0; 1].$$

11.7. Funktsional qatorlarda hadma-had limitga o'tish. Yaqinlashuvchi (11.5.1) funksional qator berilgan bo'lib, uning yig'indisi $S(x)$, x_0 nuqta esa, X to'plamning limit nuqtasi bo'lsin.

11.7.1. Teorema. Agar $x \rightarrow x_0$ da (11.5.1) funksional qatorning har bir $u_n(x)$ ($n=1, 2, \dots$) hadi chekli

$$\lim_{x \rightarrow x_0} u_n(x) = c_n \quad (n=1, 2, \dots) \quad (11.7.2)$$

limitga ega bo'lib, berilgan qator X to'plamda tekis yaqinlashuvchi bo'lsa,

$$\sum_{n=1}^{\infty} c_n = c_1 + c_2 + \dots + c_n + \dots \quad (11.7.3)$$

qator ham yaqinlashuvchi, uning yig'indisi C esa, $S(x)$ ning $x \rightarrow x_0$ dagi limitiga

$$\lim_{x \rightarrow x_0} S(x) = C \quad (11.7)$$

teng bo'ladi.

Isboti. Teoremaning shartiga ko'ra, (11.5.1) funksional qatorda tekis yaqinlashuvchi bo'lgani uchun $\forall \varepsilon > 0$ son olinganda $\exists n_0(\varepsilon) \in N$ nomer topilib, $n > n_0(\varepsilon) \in N$, barcha butun $p > 0$ va $\forall x \in X$ uchun

$$\left| \sum_{k=n+1}^{n+p} u_k(x) \right| = |u_{n+1}(x) + u_{n+2}(x) + \dots + u_{n+p}(x)| < \varepsilon$$

tengsizlik o'rinli bo'ladi. Bunda $x \rightarrow x_0$ da limitga o'tsak, natijada sonli (11.7.3) qatorning yaqinlashish shartini

$$|C_{n+1} + C_{n+2} + \dots + C_{n+p}| \leq \varepsilon$$

topamiz.

Agar (11.7.3) qatorning yig'indisini, qisman yig'indisini, qoldig'ini mos ravishda C, C_n, r_n deb belgilasak, (11.5.1) funksional qatorning yig'indisini, qisman yig'indisini, qoldig'ini esa, mos ravishda $S(x), S_n(x), r_n(x)$ deb belgilasak, u holda

$$C = C_n + r_n, \quad S(x) = S_n(x) + r_n(x)$$

ifodalarni e'tiborga olgan holda,

$$|S(x) - C| \leq |S_n(x) - C| + |r_n(x)| = |r_n(x)| \quad (11.7.5)$$

tengsizlikka ega bo'lamiz. (11.5.1) tekis yaqinlashuvchi, (11.7.5) qatorning yaqinlashuvchi ekanligidan $\forall \varepsilon > 0 \exists n_0 = n_0(\varepsilon) \in N$:

$n > n_0(\varepsilon), \forall x \in X$ uchun

$$|r_n(x)| < \frac{\varepsilon}{3}, \quad |r_n(x)| < \frac{\varepsilon}{3} \quad (11.7.6)$$

bajariladi. Ravshanki,

$$\lim_{x \rightarrow x_0} S_n(x) = \lim_{x \rightarrow x_0} \sum_{k=1}^n C_k = C_n,$$

bundan $\forall \varepsilon > 0, \exists \delta > 0, |x - x_0| < \delta$ ni qanoatlantiruvchi x ni qiymatlarida

$$|S_n(x) - C_n| < \frac{\varepsilon}{5} \quad (11.7.7)$$

limitlik bajariladi. (11.7.5), (11.7.6) va (11.7.7) tengsizliklardan (11.7.4) tengsizlik o'rinli bo'ladi (11.7.2) ni e'tiborga olsak, u xolda (11.7.4) ni,

$$\lim_{x \rightarrow x_0} \sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \lim_{x \rightarrow x_0} u_n(x)$$

to'g'ri yozish mumkin. ■

11.7.8. Eslatma. 11.7.1-teoremaning shartlari bajarilganda

$$\lim_{x \rightarrow x_0} \sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \lim_{x \rightarrow x_0} u_n(x)$$

to'g'ri bo'ladi.

11.7.9. Misol. Ushbu

$$\lim_{x \rightarrow 1-0} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{x+n-1} \cdot \frac{x^n}{x^n+1}$$

limitni toping.

Yechilishi. Berilgan funksional qator $X=[1; \infty)$ to'plamda Abel alomatiga ko'ra, tekis yaqinlashuvchi bo'ladi, ya'ni

$$a) \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{x+n-1} \xrightarrow{x \rightarrow 1-0} S(x);$$

$$b) \text{ barcha } x \in X \text{ va } \forall n \in N \text{ lar uchun } a_n(x) = 1 - \frac{1}{x^n+1} \leq 1 \text{ va}$$

$a_{n+1}(x) \leq a_n(x)$ bo'ladi. Bundan tashqari,

$$\lim_{x \rightarrow 1-0} u_n(x) = \lim_{x \rightarrow 1-0} \frac{(-1)^{n+1}}{x+n-1} \cdot \frac{x^n}{x^n+1} = \frac{(-1)^{n+1}}{2n} = c_n, \quad n \in N.$$

$\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n}$ yaqinlashuvchi va uning yig'indisi $\frac{1}{2} \ln 2$ ga teng.

11.7.1-teoremaning shartlari bajarilayapti. Endi funksional qatorda hadma-had limitga o'tish mumkin:

$$\lim_{x \rightarrow 1-0} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{x+n-1} \cdot \frac{x^n}{x^n+1} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{2} \ln 2.$$

11.8. Funksional ketma-ketlikda hadma-had limitga o'tish. (11.6.1) funksional ketma-ketlik X to'plamda berilgan bo'lib, uning limit funksiyasi $f(x)$, x_0 nuqta esa X to'plamning limit nuqtasi bo'lsin.

11.8.1. Teorema. Agar $x \rightarrow x_0$ da $\{f_n(x)\}$ funksional ketma-ketlikning har bir $f_n(x)$ ($n=1,2,\dots$) hadi chekli

$$\lim_{x \rightarrow x_0} f_n(x) = a_n$$

limitga ega bo'lib, bu ketma-ketlik x da tekis yaqinlashuvchi bo'lsa, holda $\{a_n\}$ ketma-ketlik ham yaqinlashuvchi bo'ladi, uning

$$\lim_{n \rightarrow \infty} a_n = a$$

limiti esa, $f(x)$ ning $x \rightarrow x_0$ dagi limitiga teng, $\lim_{x \rightarrow x_0} f(x) = a$.

11.8.2. Misol. Ushbu

$$\{f_n(x)\} = \left\{ \frac{1}{1 + \left(1 + \frac{x}{n}\right)^n} \right\}$$

funksional ketma-ketlik $[0; 1]$ da 11.8.1-teoremaning shartlaru qanoatlantirishini ko'rsating.

Yechilishi. $\lim_{x \rightarrow 1} f_n(x)$ ni topamiz:

$$\lim_{x \rightarrow 1} f_n(x) = \lim_{x \rightarrow 1} \frac{1}{1 + \left(1 + \frac{x}{n}\right)^n} = \frac{1}{1 + \left(1 + \frac{1}{n}\right)^n} = a_n, \quad \forall n \in \mathbb{N}.$$

Berilgan funksional ketma-ketlikning $[0; 1]$ da tek yaqinlashuvchiligini ko'rsatamiz:

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \frac{1}{1 + \left(1 + \frac{x}{n}\right)^n} = \frac{1}{1 + e^x} = f(x),$$

$$|f_n(x) - f(x)| = \left| \frac{1}{1 + \left(1 + \frac{x}{n}\right)^n} - \frac{1}{1 + e^x} \right| = \frac{\left| e^x - \left(1 + \frac{x}{n}\right)^n \right|}{(1 + e^x)(1 + \left(1 + \frac{x}{n}\right)^n)} \leq \left| e^x - \left(1 + \frac{x}{n}\right)^n \right| <$$

$$< \sup_{x \in [0; 1]} \left| e^x - \left(1 + \frac{x}{n}\right)^n \right| \leq \max \left\{ \left(e - \left(1 + \frac{1}{n}\right)^n \right), \frac{1}{n} \left(1 + \frac{1}{n}\right)^{n-1} \right\} = \frac{1}{n} \left(1 + \frac{1}{n}\right)^{n-1} < \frac{3}{n} \rightarrow 0$$

$\forall x \in [0; 1]$.

Demak, berilgan funksional ketma-ketlik $[0; 1]$ da tek yaqinlashuvchi.

$$\{a_n\} = \left\{ \frac{1}{1 + \left(1 + \frac{1}{n}\right)^n} \right\}$$

yaqinlashuvchi va

$$\lim_{n \rightarrow \infty} a_n = \frac{1}{1+e}, \quad \lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} \frac{1}{1+e^x} = \frac{1}{1+e} = a.$$

Shunday qilib, funksional kema-ketlik 11.8.1-teoremaning hamma shartlarini qanoatlantirar ekan.

11.9. Funksional qatorni hadma-had integrallash. Yaqinlashuvchi (11.5.1) funksional qator $X=[a,b]$ segmentda berilgan bo'lib, uning yig'indisi $S(x)$ bo'lsin.

11.9.1. Teorema. Agar (11.5.1) funksional qatorning har bir $u_n(x)$ hadi $X=[a,b]$ segmentda uzluksiz bo'lib, qatorning o'zi shu segmentda tekis yaqinlashuvchi bo'lsa, u holda

$$\int_a^b u_1(x)dx + \int_a^b u_2(x)dx + \dots + \int_a^b u_n(x)dx + \dots$$

qator ham yaqinlashuvchi bo'ladi va

$$\int_a^b \sum_{n=1}^{\infty} u_n(x)dx = \int_a^b S(x)dx \quad (11.9.2)$$

tenglik o'rinli bo'ladi.

Isboti. Teorema shartiga ko'ra, $u_n(x)$ ($n=1,2,3,\dots$) uzluksiz bo'lgani uchun hamda 11.5.2-teoremaga asosan, (11.5.1) funksional qatorning yig'indisi $S(x)$ ham uzluksiz bo'lgani sababli (11.9.2) dagi integrallarning hammasi mavjud.

Ushbu

$$S(x) = u_1(x) + u_2(x) + \dots + u_n(x) + r_n(x),$$

(bunda $r_n(x)$ qatorning qodig'i) tenglikni $[a,b]$ da hadma-had integrallab topamiz:

$$\int_a^b S(x)dx = \int_a^b u_1(x)dx + \int_a^b u_2(x)dx + \dots + \int_a^b u_n(x)dx + \int_a^b r_n(x)dx.$$

Bundan

$$\int_a^b S(x)dx - \int_a^b S_n(x)dx = \int_a^b r_n(x)dx.$$

(11.9.2) ni isbotlash uchun

$$\lim_{n \rightarrow \infty} \int_a^b r_n(x)dx = 0$$

ekanligi ko'rsatish yetarli. Shartga ko'ra $S_n(x) \xrightarrow{b} S(x) (x \in [a, b])$.

xolda (11.5.1) funksional qatorning tekis yaqinlashish ta'rifiga ko'ra $\forall \varepsilon > 0 \forall \varepsilon > 0$ son olinganda ham $\exists n_0 = n_0(\varepsilon) \in \mathbb{N}$ topilib $\forall n > n_0(\varepsilon), \forall x \in [a, b]$ uchun $|r_n(x)| < \varepsilon$ tengsizlik bajariladi. Bundan, esa

$$\left| \int_a^b r_n(x) dx \right| \leq \int_a^b |r_n(x)| dx < (b-a)\varepsilon$$

tengsizlik bajariladi. ■

11.9.3. Eslatma. 11.9.1-teoremda qatorning tekis yaqinlashuvchiligi yetarli shart bo'lib, lekin zaruriy shart bo'lmaydi, ya'ni ba'zan tekis yaqinlashuvchilik sharti bajarilmagan funksional qatorni ham hadma-had integrallash mumkin.

11.9.4. Misol. Ushbu

$$\sum_{n=1}^{\infty} (x^{\frac{1}{2n+1}} - x^{\frac{1}{2n-1}})$$

funksional qatorni $[0; 1]$ da hadma-had integrallash mumkinmi?

Yechilishi. Berilgan funksional qatorning qisman yig'indisini topamiz:

$$S_n(x) = (x^{\frac{1}{3}} - x) + (x^{\frac{1}{5}} - x^{\frac{1}{3}}) + \dots + (x^{\frac{1}{2n-1}} - x^{\frac{1}{2n-3}}) + (x^{\frac{1}{2n+1}} - x^{\frac{1}{2n-1}}) = x^{\frac{1}{2n+1}} - x$$

Endi $n \rightarrow \infty$ da limitga o'tamiz:

$$S(x) = \lim_{n \rightarrow \infty} (x^{\frac{1}{2n+1}} - x) = \begin{cases} 0, & x = 0, x = 1 \text{ bo'lganda,} \\ 1 - x, & 0 < x < 1 \text{ bo'lganda.} \end{cases}$$

Demak, berilgan funksional qatorning yig'indisi uzilishga ega bo'lgan funksiyadir. Shuning uchun berilgan funksional qator uchun $[0; 1]$ da tekis yaqinlashuvchilik sharti bajarilmaydi.

Demak, 11.9.1-teoremni qo'llash huquqiga ega emasmiz. Lekin, qator yig'indisini va qatorni hadma-had integrallash mumkin:

$$\begin{aligned} \int_0^1 S(x) dx &= \frac{1}{2}, \quad \sum_{n=1}^{\infty} \int_0^1 (x^{\frac{1}{2n+1}} - x^{\frac{1}{2n-1}}) dx = \sum_{n=1}^{\infty} \left(\frac{2n+1}{2(n+1)} - \frac{2n-1}{2n} \right) = \\ &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1} \right) = \frac{1}{2}. \end{aligned}$$

Buni e'tiborga olsak,

$$\int_0^1 S(x) dx = \int_0^1 \left[\sum_{n=1}^{\infty} \left(x^{\frac{1}{2n+1}} - x^{\frac{1}{2n-1}} \right) \right] dx = \sum_{n=1}^{\infty} \int_0^1 \left(x^{\frac{1}{2n+1}} - x^{\frac{1}{2n-1}} \right) dx$$

tenglik o'rinli bo'ladi.

11.9.5. Misol. Ushbu

$$f(x) = \sum_{n=1}^{\infty} \frac{\cos^2 nx}{n(n+1)}$$

funksiyaning R da uzluksizligini isbotlang va $\int_0^{2\pi} f(x) dx$ integralni hisoblang.

Yechilishi. $u_n(x) = \frac{\cos^2 nx}{n(n+1)}$ ($n=1,2,\dots$) funksiyalar R da uzluksiz.

$\forall x \in R$ va $\forall n \in N$ lar uchun

$$|u_n(x)| \leq \frac{1}{n(n+1)}$$

tengsizlik o'rinli bo'ladi.

Veyersstrass alomatiga ko'ra, berilgan funksional qator tekis yaqinlashuvchi. **11.5.2-teorema**ga ko'ra, $f(x)$ funksiya R da uzluksiz. Bu yerda **11.9.1-teoremaning** ham hamma shartlari bajariladi. Shuning uchun berilgan funksional qatorni hadma-had integrallash mumkin:

$$\begin{aligned} \int_0^{2\pi} \sum_{n=1}^{\infty} \frac{\cos^2 nx}{n(n+1)} dx &= \int_0^{2\pi} \frac{\cos^2 x}{1 \cdot 2} dx + \int_0^{2\pi} \frac{\cos^2 2x}{2 \cdot 3} dx + \dots + \int_0^{2\pi} \frac{\cos^2 nx}{n(n+1)} dx + \dots \\ &+ \dots = \pi \left(\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n(n+1)} + \dots \right) = \pi \end{aligned}$$

Demak,
$$\int_0^{2\pi} f(x) dx = \pi.$$

11.10. Funksional ketma-ketliklarni hadma-had integrallash.

(11.6.1) yaqinlashuvchi funksional ketma-ketlik $\{a, b\}$ da berilgan bo'lib, $f(x)$ uning limit funksiyasi bo'lsin.

11.10.1. Teorema. Agar $\{f_n(x)\}$ funksional ketma-ketlikning har bir $f_n(x)$ ($n=1,2,\dots$) hadi $[a, b]$ segmentda uzluksiz bo'lib, funksional ketma-ketlik $[a, b]$ segmentda tekis yaqinlashuvchi bo'lsa, u holda

$$\int_a^b f_1(x)dx, \int_a^b f_2(x)dx, \dots, \int_a^b f_n(x)dx, \dots$$

ketma-ketlik yaqinlashuvchi bo'ladi, uning limiti esa $\int_a^b f(x)dx$ bo'lib, ya'ni

$$\lim_{n \rightarrow \infty} \int_a^b f_n(x)dx = \int_a^b \lim_{n \rightarrow \infty} f_n(x)dx = \int_a^b f(x)dx$$

tenglik o'rinli bo'ladi.

11.10.2. Misol. Ushbu

$$\{f_n(x)\} = \{n^{\frac{1}{2}}xe^{-nx}\}$$

ketma-ketlik 11.10.1 -teoremaning hamma shartlarini qanoatlantirish ko'rsating, ya'ni

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x)dx = \int_0^1 (\lim_{n \rightarrow \infty} f_n(x))dx \quad (11.10.3)$$

tenglik o'rinli ekanligini isbotlang.

Yechilishi. $f_n(x) = n^{\frac{1}{2}}xe^{-nx}$ ($n = 1, 2, \dots$) funksiyalar $[0; 1]$ segmenti uzluksiz, hamda

$$\lim_{n \rightarrow \infty} f_n(x) = 0, \quad f(x) = 0,$$

$$\sup_{x \in [0; 1]} |f_n(x) - f(x)| = \sup_{x \in [0; 1]} n^{\frac{1}{2}}xe^{-nx} \leq n^{\frac{1}{2}} \cdot \frac{1}{n} e^{-1} = \frac{e^{-1}}{\sqrt{n}} \rightarrow 0.$$

Demak, $\{f_n(x)\} = \{n^{\frac{1}{2}}xe^{-nx}\}$ ketma-ketlik $[0; 1]$ segmentda te yaqinlashuvchi. U holda, 11.10.1-teoreмага asosan, berilgan funktsiya ketma-ketlikni hadma-had integrallash mumkin:

$$\int_0^1 xe^{-x} dx, \int_0^1 xe^{-2x} dx, \int_0^1 xe^{-3x} dx, \dots, \int_0^1 xe^{-nx} dx, \dots$$

Bu ketma-ketlik yaqinlashuvchi va uning limiti 0 ga teng.

Shunday qilib, (11.10.3) tenglikning o'rinli ekanligiga ishon hosil qilamiz.

11.11. Funksional qatorni hadma-had differensialla
Yaqinlashuvchi (11.5.1) funksional qator $[a, b]$ segmentda berilgan bo'lib, $S(x)$ uning yig'indisi bo'lsin.

11.11.1. Teorema. Agar (11.5.1) funksional qatorning har bir $u_n(x)$ hadi $[a, b]$ segmentda uzluksiz $u'_n(x)$ hosilaga ega bo'lib,

$$\sum_{n=1}^{\infty} u'_n(x) = u'_1(x) + u'_2(x) + \dots + u'_n(x) + \dots \quad (11.11.2)$$

qator $[a, b]$ segmentda tekis yaqinlashuvchi bo'lsa, u holda (11.5.1) qatorning $S(x)$ yig'indisi $[a, b]$ segmentda $S'(x)$ hosilaga ega va

$$S'(x) = \left(\sum_{n=1}^{\infty} u_n(x) \right)' = \sum_{n=1}^{\infty} u'_n(x)$$

bo'ladi.

Isboti. (11.11.2) qatorning yig'indisinin $S^*(x)$ bilan belgilaymiz. 11.5.2 teoreмага asosan, bu funksiya uzluksiz bo'ladi. 11.9.1- teoreмага asosan, (11.11.2) qatorni "a" dan "x" gacha hadma-had integrallaymiz :

$$\int_a^x S^*(t) dt = \sum_{n=1}^{\infty} \int_a^x u'_n(t) dt.$$

bunda $\int_a^x u'_n(t) dt = u_n(x) - u_n(a)$ ekanligini e'tiborga olgan holda

$$\int_a^x S^*(t) dt = \sum_{n=1}^{\infty} [u_n(x) - u_n(a)] = \sum_{n=1}^{\infty} u_n(x) - \sum_{n=1}^{\infty} u_n(a) = S(x) - S(a).$$

Fenglikning chap tomonidagi integralda, integral ostidagi funksiya uzluksiz bo'lgani uchun, uning hosilasi $S^*(x)$ ga teng bo'ladi . Chap tomonning hosilasi $S'(x)$ ga teng.

Demak,

$$S^*(x) = S'(x), \quad \left(\sum_{n=1}^{\infty} u_n(x) \right)' = \sum_{n=1}^{\infty} u'_n(x)$$

bo'ladi.

11.11.3. Eslatma. 11.11.1-teoremadagi funksional qatorning tekis yaqinlashuvchilik sharti yetarli bo'lib, u zaruriy shart emas.

11.11.4. Misol. Ushbu

$$\sum_{n=1}^{\infty} \arctg \frac{x}{n^4}$$

funksional qatorni $(-\infty; +\infty)$ da hadma-had differensiallash mumkinmi?

Yechilishi. Berilgan qatorning umumiy hadi $u_n(x) = \arcc \frac{x}{n^4}$ ($n \in \mathbb{N}$) ($-\infty; +\infty$) da uzluksiz $u'_n(x) = \frac{n^4}{n^8 + x^2}$ hosilaga ega bo'ladi. $(-\infty; \infty)$ da berilgan funksional qator taqqoslash alomatiga ko'ra ($n \rightarrow \infty$ da $\arcc \frac{x}{n^4} \sim \frac{x}{n^4}$) yaqinlashuvchi va $S(x)$ yig'indiga ega. Bundan tashqari, hosilalardan tuzilgan

$$\sum_{n=1}^{\infty} u'_n(x) = \sum_{n=1}^{\infty} \frac{n^4}{n^8 + x^2}$$

funksional qator $(-\infty; \infty)$ da Veayershtrass alomatiga ko'ra, tekis yaqinlashuvchi bo'ladi.

Demak, berilgan funksional qatorning yig'indisi $(-\infty; +\infty)$ da $S'(x)$ hosilaga ega va

$$S'(x) = \left(\sum_{n=1}^{\infty} \arcc \frac{x}{n^4} \right)' = \sum_{n=1}^{\infty} \frac{n^4}{n^8 + x^2}.$$

11.12. Funksional ketma-ketliklarni hadma-had differensiallash. $[a, b]$ kesmada yaqinlashuvchi (11.6.1) funksional ketma-ketlik berilgan bo'lib, uning limit funksiyasi $f(x)$ bo'lsin.

11.12.1. Teorema. Agar $\{f_n(x)\}$ funksional ketma-ketlikning har bir $f_n(x)$ hadi $[a, b]$ kesmada uzluksiz $f'_n(x)$ hosilaga ega bo'lib, bu hosilalardan tuzilgan

$$f'_1(x), f'_2(x), \dots, f'_n(x), \dots$$

funksional ketma-ketlik $[a, b]$ kesmada tekis yaqinlashuvchi bo'lsa, u holda $f(x)$ limit funksiya shu $[a, b]$ kesmada $f'(x)$ hosilaga ega bo'lib, bu hosila $\{f'_n(x)\}$ ketma-ketlikning limitiga teng bo'ladi.

11.12.2. Misol. Ushbu

$$\{f_n(x)\} = \left\{ \frac{1}{n} \arcc x^n \right\}$$

funksional ketma-ketlik $(-\infty; +\infty)$ da $f(x)$ limit funksiya tekis yaqinlashishini ham

$$\left[\lim_{n \rightarrow \infty} f_n(x) \right]' \Big|_{x=1} \neq \lim_{n \rightarrow \infty} f'_n(1)$$

emasligini ko'rsating.

Yechilishi. Barcha $x \in (-\infty; \infty)$ va $\forall_n \in \mathbb{N}$ lar uchun

$$f(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \arctg x^n = 0$$

bo'ladi. $\{f_n'(x)\}$ funksional ketma-ketlikning $(-\infty; \infty)$ da tekis yaqinlashuvchi ekanligini ko'rsatamiz:

$$\sup_{x \in (-\infty; \infty)} |f_n(x) - f(x)| = \sup_{x \in R} \left| \frac{1}{n} \arctg x^n \right| = \frac{\pi}{2n} = a_n,$$

$$\lim_{n \rightarrow \infty} \sup_{x \in R} |f_n(x) - f(x)| = \lim_{n \rightarrow \infty} \frac{\pi}{2n} = 0, \quad f_n' \xrightarrow{n} 0.$$

Endi $f_n'(x) = \left(\frac{1}{n} (\arctg x^n) \right)' = \frac{nx^{n-1}}{n(1+x^{2n})}$, $f_n'(1) = \frac{1}{2}$ bo'ladi. Bu yerdan

$$\left[\lim_{n \rightarrow \infty} f_n(x) \right]' \Big|_{x=1} = [0]' \Big|_{x=1} = 0, \quad \lim_{n \rightarrow \infty} f_n'(1) = \frac{1}{2}; \quad 0 \neq \frac{1}{2}.$$

52-§. Darajali qatorlar

11.13. Darajali qator, uning yaqinlashish radiusi va intervali.

Ushbu

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \dots + a_n (x - x_0)^n + \dots \quad (11.13.1)$$

qatorga darajali qator deyiladi. Bunda $a_0, a_1, a_2, \dots, a_n, \dots$ o'zgarmas haqiqiy sonlar darajali qatorning koeffitsiyentlari deyiladi, x_0 esa, ixtiyoriy o'zgarmas son. (11.13.1) darajali qator ushbu

$$\sum_{n=0}^{\infty} u_n(x)$$

funksional qatorning xususiy holi bo'lib hisoblanadi:

$$u_n(x) = a_n (x - x_0)^n, \quad n = 0, 1, 2, \dots$$

$x - x_0 = t$ belgilash yordamida (11.13.1) darajali qatorni

$$\sum_{n=0}^{\infty} a_n t^n = a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n + \dots$$

ko'rinishga keltirish mumkin. Shuning uchun biz bundan keyin ushbu

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots \quad (11.13.2)$$

ko'rinishdagi qatorni o'rganish bilan kifoyalanamiz.

11.13.3. Teorema (Abel teoremasi). Agar (11.13.2) darajali qator x ning $x = x_0$ ($x_0 \neq 0$) qiymatiga yaqinlashuvchi bo'lsa, u holda x ning $|x| < |x_0|$ tengsizlikni qanoatlantiruvchi barcha qiymatlarida (11.13.2) darajali qator absolyut yaqinlashuvchi bo'ladi.

Isboti. Shartga ko'ra, ushbu

$$\sum_{n=0}^{\infty} a_n x_0^n = a_0 + a_1 x_0 + a_2 x_0^2 + \dots + a_n x_0^n + \dots \quad (11.13.4)$$

qatorning yaqinlashuvchiligidan uning umumiy hadi $\lim_{n \rightarrow \infty} a_n x_0^n = 0$ bo'ladi.

Demak, $\{a_n x_0^n\}$ ketma-ketlik chegaralangan, ya'ni $\exists M > 0, \forall n \in N$ da

$$|a_n x_0^n| \leq M \quad (n = 0, 1, 2, \dots) \quad (11.13.5)$$

x ning $|x| < |x_0|$ shartni qanoatlantiruvchi ixtiyoriy qiymatini olib ushbu

$$\sum_{n=0}^{\infty} |a_n x^n| = |a_0| + |a_1 x| + |a_2 x^2| + \dots + |a_n x^n| + \dots$$

qatorni tuzamiz. Bunda (11.13.5) ni e'tiborga olgan holda :

$$|a_n x^n| = |a_n x_0^n| \left| \frac{x}{x_0} \right|^n \leq M \left| \frac{x}{x_0} \right|^n, \quad (11.13.6)$$

$$|x| < |x_0| \quad \text{dan} \quad \left| \frac{x}{x_0} \right| = q < 1.$$

Demak,

$$\sum_{n=0}^{\infty} \left| \frac{x}{x_0} \right|^n = \sum_{n=0}^{\infty} q^n$$

geometrik qator yaqinlashuvchi. U holda ushbu $\sum_{n=0}^{\infty} M \left| \frac{x}{x_0} \right|^n$ qator ham yaqinlashuvchi bo'ladi. (11.13.6) ni e'tiborga olgan holda, sonli qatorni solishtirish haqidagi 8.4.1-teoremaga asosan, (11.13.4) qatorning absolyut yaqinlashuvchi ekanini topamiz. ■

11.13.6. Natija. Agar (11.13.2) qator x ning $x = x_0$ qiymatida uzoqlashuvchi bo'lsa, u x ning $|x| > |x_0|$ tengsizlikni qanoatlantiruvchi barcha qiymatlarida uzoqlashuvchi bo'ladi.

Isboti. Teskarisini faraz qilaylik, ya'ni (11.3.2) qator $|x| > |x_0|$ tengsizlikni qanoatlantiruvchi biror $x = x^*$ nuqtada $(|x^*| > |x_0|)$ yaqinlashuvchi bo'lsin. U holda Abel teoremasiga asosan, $|x| < |x^*|$ tengsizlikni qanoatlantiruvchi barcha x larda yaqinlashuvchi, jumladan x_0 nuqtada ham yaqinlashuvchi bo'lib qoladi. Bu esa, shartga zid bo'lib qoladi. ■

$\sum_{n=0}^{\infty} a_n x^n$ qator berilgan bo'lsin. Agar R -manfiy bo'lmagan son yoki $+\infty$ bo'lib, u quyidagi shartni qanoatlantirsa: $|x| < R$ tengsizlikni qanoatlantiruvchi barcha x larda qator yaqinlashuvchi, $|x| > R$ tengsizlikni qanoatlantiruvchi barcha x larda qator uzoqlashuvchi bo'lsa, u holda R ga qatorning *yaqinlashish radiusi* deyiladi. $|x| < R$ tengsizlikni qanoatlantiruvchi barcha x lar to'plamiga, ya'ni $(-R, R)$ yoki $K = \{x \in R : |x| < R\}$ ga qatorning *yaqinlashish oralig'i* deyiladi. Yaqinlashish oralig'ining chegaralarida, ya'ni $x = \pm R$ da darajali qator yaqinlashuvchi ham, uzoqlashuvchi ham bo'lishi mumkin.

11.14. Darajali qatorning yaqinlashish radiusini uning koefitsiyentlari orqali ifodalash. (11.13.2) qator berilgan bo'lsin . Ushbu

$$\rho_1 = |a_1|, \rho_2 = \sqrt{|a_2|}, \rho_3 = \sqrt[3]{|a_3|}, \dots, \rho_n = \sqrt[n]{|a_n|}, \dots$$

ketma-ketlikni qaraymiz. Bu ketma-ketlikning aniq yuqori ([15], 1-tom, 42- bandga q.) limiti mavjud va uni ρ bilan belgilaymiz :

$$\rho = \overline{\lim}_{n \rightarrow \infty} \rho_n = \overline{\lim}_{n \rightarrow \infty} \sqrt[n]{|a_n|}.$$

11.14.1. Teorema (Koshi-Adamar). Har qanday (11.13.2) darajali qatorning yaqinlashish radiusi R mavjud bo'lib, u $R = \frac{1}{\rho}$ formula orqali topiladi (bunda, $\rho = 0$ bo'lsa, $R = +\infty$, agar $\rho = +\infty$ bulsa, $R = 0$ bo'ladi).

Isboti. a) $\rho = 0$ bo'lsin . Bu holda $R = +\infty$ bo'lib, (11.13.2) qatorning $\forall x$ larda absolyut yaqinlashuvchi ekanini isbotlaymiz.

$\{\sqrt[n]{|a_n|}\}$ ketma-ketlikning elementlari musbat bo'lgani hamul
 $\rho = \overline{\lim}_{n \rightarrow \infty} \rho_n = \overline{\lim}_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 0$ bo'lganidan $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 0$ kelib chiqadi. Bunda
 Koshi alomatiga asosan, $\forall x$ uchun $n \rightarrow \infty$ da $K_n = \sqrt[n]{|a_n|} \cdot |x|^n = |x| \sqrt[n]{|a_n|}$ 0 ga
 bo'ladi. Bundan, (11.13.2) qator hadlarining absolyut qiymatlaridan
 tuzilgan $\sum_{n=1}^{\infty} |a_n| \cdot |x|^n$ qatorning yaqinlashuvchiligi, $\sum_{n=1}^{\infty} a_n x^n$ qatorning esa
 absolyut yaqinlashuvchiligi kelib chiqadi.

b) $\rho = +\infty$ bulsin. Bu holda $R = 0$, ya'ni $\forall x \neq 0$ lar uchun (11.13.2)
 qatorning uzoqlashuvchi ekanini ko'rsatamiz. $\rho = \overline{\lim}_{n \rightarrow \infty} \rho_n = \overline{\lim}_{n \rightarrow \infty} \sqrt[n]{|a_n|} = +\infty$
 bo'lgani uchun shundan $\{n_k\}$ qisman ketma-ketlik topiladi
 $\lim_{k \rightarrow \infty} \sqrt[n_k]{|a_{n_k}|} = +\infty$.

Shunday qilib, har bir $x \neq 0$ uchun $\exists k_0$ nomer topiladiki $k < k_0$ da
 boshlab $\sqrt[n_k]{|a_{n_k}|} > \frac{1}{|x|}$ yoki $|a_{n_k}| |x|^{n_k} > 1$ bo'ladi. Bundan esa, qator
 yaqinlashishining zaruriy sharti bajarilmasligi kelib chiqadi.

Demak, berilgan qator uzoqlashuvchi.

c) $0 < \rho < +\infty$ bo'lsin. Bu holda $R = \frac{1}{\rho}$ bo'lib, $|x| < \frac{1}{\rho}$ ni
 qanoatlantiruvchi x ning barcha qiymatlari qatorning absolyut
 yaqinlashuvchi, $|x| > \frac{1}{\rho}$ ni qanoatlantiruvchi x ning barcha qiymatlarid
 uzoqlashuvchi ekanini isbotlaymiz.

$|x| < \frac{1}{\rho}$ tengsizlikni qanoatlantiruvchi $\forall x$ ni olamiz. $\varepsilon > 0$ soni
 shunday kichik qilib tayinlaylikki, $|x| < \frac{1}{\rho + \varepsilon}$ tengsizlik bajarilsin. Sh
 ε bo'yicha shunday $n_0(\varepsilon)$ ni topish mumkinki, $n > n_0$ dan boshlab
 aniq yuqori limitning 1^o-xossasiga asosan ([15], 1-tom, 42-bandga q.
 $\sqrt[n]{|a_n|} < \rho + \varepsilon$ tengsizlik bajariladi. Bundan Koshi alomatiga asosan $n > n_0$
 dan boshlab $K_n = \sqrt[n]{|a_n|} \cdot |x|^n = |x| \sqrt[n]{|a_n|} < |x|(\rho + \varepsilon) < 1$ bajariladi. U holda
 Koshi belgisiga asosan (11.13.2) qator hadlarining absolyut

qiyomatlaridan tuzilgan qator yaqinlashuvi, (11.3.2) qator esa, absolyut yaqinlashuvchi bo'ladi.

Endi $|x| > \frac{1}{\rho}$ ni qanoatlantiradigan $\forall x$ ni olamiz. ε ni shunday

kichik qilib tayinlaylikki $|x| > \frac{1}{\rho - \varepsilon}$ o'rinli bo'ladi.

Yuqori limitning 2-xossasiga asosan ([15], 1-tom, 42-bandga q.), istalgancha n lar uchun $\sqrt[n]{|a_n|} > \rho - \varepsilon$ bajariladi. Bundan $\sqrt[n]{|a_n x^n|} > |x|(\rho - \varepsilon) > 1$.

Demak, istalgancha katta n lar uchun berilgan qatorning umumiy hadi $|a_n x^n| > 1$. Bundan esa, qatorning uzoqlashuvchiligi kelib chiqadi. ■

11.14.2. Eslatma. K intervalning chegarasida, ya'ni $x = \pm R$ da (11.13.2) darajali qator yaqinlashuvchi ham, uzoqlashuvchi bo'lishi ham mumkin. K ga nisbatan kichik istalgan $K_1 = \{x : |x| \leq \rho_1 < R\}$ intervalda (11.13.2) qator absolyut va tekis yaqinlashuvchi bo'ladi.

11.14.3. Eslatma. Xususiyl holda, agar 1) chekli yoki cheksiz

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}$$

mavjud bo'lsa, u holda (11.13.2) qatorning yaqinlashish radiusi R uchun

$$\frac{1}{R} = \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} \quad (11.14.4)$$

formula o'rinli:

2) chekli yoki cheksiz

$$\lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

mavjud bo'lsa, u holda (11.13.2) darajali qatorning yaqinlashish radiusi R uchun

$$R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| \quad (11.14.5)$$

formula o'rinli.

11.14.6. Eslatma. $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ darajali qatorning yaqinlashishi

intervali $(x_0 - R; x_0 + R)$ bo'ladi. Bunda R ushbu $\sum_{n=0}^{\infty} a_n x^n$ qatorning yaqinlashish radiusi.

11.14.7. Eslatma. (11.14.4)-(11.14.5) limitlar mavjud bo'lmashligi ham mumkin. Ammo, (11.13.2) darajali qatorning yaqinlashish radiusini hisoblash uchun umumiy formulaga egamiz, ya'ni

$$R = \frac{1}{\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}}. \quad (11.14.8)$$

(11.14.8) formula *Koshi-Adamar formula*si deyiladi.

11.14.9. Misol. Ushbu

$$\sum_{n=0}^{\infty} \left(\frac{2 + (-1)^n}{3} \right)^n x^n$$

darajali qatorning yaqinlashish radiusini toping.

Yechilishi. Berilgan darajali qator umumiy hadining koeffitsiyenti

$$a_n = \left(\frac{2 + (-1)^n}{3} \right)^n. \text{ Endi } R \text{ radiusni topish uchun } \{K_n\} = \{\sqrt[n]{|a_n|}\} \text{ ketma-}$$

ketlikning limitini topamiz:

$$\lim_{n \rightarrow \infty} K_n = \lim_{n \rightarrow \infty} \sqrt[n]{\left(\frac{2 + (-1)^n}{3} \right)^n} = \lim_{n \rightarrow \infty} \frac{2 + (-1)^n}{3} = \begin{cases} 1, & n = 2k, \quad k \in N, \\ \frac{1}{3}, & n = 2k - 1, \quad k \in N. \end{cases}$$

Ravshanki, $\{K_n\}$ ketma-ketlikning limiti mavjud emas. Berilgan $\{K_n\}$ ketma-ketlikning qisman limitlari mavjud, ya'ni

$$\lim_{k \rightarrow \infty} K_{2k} = 1, \quad \lim_{k \rightarrow \infty} K_{2k-1} = \frac{1}{3}.$$

$\{K_n\}$ ketma-ketlikning qisman ketma-ketliklarining limitlari orasida eng kattasi 1, eng kichigi esa $\frac{1}{3}$ ga teng ekanligini ko'rish qiyin emas.

Demak,

$$\overline{\lim}_{n \rightarrow \infty} K_n = \overline{\lim}_{n \rightarrow \infty} \frac{2 + (-1)^n}{3} = 1, \quad \underline{\lim}_{n \rightarrow \infty} K_n = \underline{\lim}_{n \rightarrow \infty} \frac{2 + (-1)^n}{3} = \frac{1}{3}.$$

11.14.7-eslatmaga ko'ra, berilgan darajali qatorning yaqinlashish radiusi

$$R = \overline{\lim}_{n \rightarrow \infty} \left[\left(\frac{2 + (-1)^n}{3} \right)^n \right]^{\frac{1}{n}} = 1$$

bo'ladi.

11.14.10. Misol. Ushbu

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n3^n}$$

darajali qatorning yaqinlashish radiusi, yaqinlashish intervali hamda yaqinlashish sohasini toping.

Yechilishi. Berilgan darajali qator n - hadining koeffitsiyenti $a_n = \frac{1}{3^n n}$. Qatorning yaqinlashish radiusini (11.13.4) formulaga asosan topamiz:

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)3^{n+1}}{n3^n} = \lim_{n \rightarrow \infty} 3\left(1 + \frac{1}{n}\right) = 3, \quad \rho = 3.$$

11.14.6-eslatmaga ko'ra, $x_0 = 1$, $\rho = 3$, berilgan qatorning yaqinlashish intervali esa $(x_0 - \rho; x_0 + \rho) = (-2; 4)$ bo'ladi. Endi qatorning yaqinlashishini yaqinlashish intervalining chegaralarida tekshirib ko'ramiz.

Agar :

a) $x = -2$ bo'lsa, $\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{3^n n}$ qator hosil bo'ladi. Bu qator Leybnits alomatiga ko'ra yaqinlashuvchi.

b) $x = 4$ bo'lsa, $\sum_{n=1}^{\infty} \frac{1}{n}$ uzoqlashuvchi garmonik qator hosil bo'ladi.

Demak, berilgan darajali qatorning yaqinlashish sohasi $[-2; 4)$ yarim intervaldan iborat.

11.15. Darajali qatorlarning xossalari

1^o. Agar (11.13.2) qatorning yaqinlashish radiusi R ($R > 0$) bo'lsa, $0 < r < R$ tengsizlikni qanoatlantiruvchi shunday r soni topiladiki, (11.13.2) qator $[-r; r]$ da x ga nisbatan tekis yaqinlashuvchi bo'ladi.

Isboti. $r < R$ bo'lgani uchun $x = r$ da (11.13.2) qator absolyut yaqinlashuvchi bo'ladi, ya'ni

$$\sum_{n=0}^{\infty} |a_n| r^n \quad (11.15.1)$$

musbat qator yaqinlashuvchi bo'ladi.

$|x| \leq r$ da (11.13.2) qator uchun (11.15.1) majorant qator bo'ladi, ya'ni $|a_n x^n| \leq |a_n| r^n$. U holda Veyershtrass alomatiga asosan, (11.13.2) qator $|x| \leq r$ ni qanoatlantiruvchi x ning barcha qiymatlarida tekis yaqinlashuvchi bo'ladi.

Demak, (11.13.2) qatorning yaqinlashish radiusi $R > 0$ bo'lsa, yuqoridagi keltirilgan xossaga asosan, bu qator $[-r, r] \subset (-R, R)$ da tekis yaqinlashuvchi bo'lar ekan. Bunda r sonini R songa harqancha yaqinlashib olganda ham, qator $(-R, R)$ da tekis yaqinlashuvchi bo'lmashligi mumkin. Masalan, $\sum_{n=0}^{\infty} x^n$ darajali qatorning yaqinlashish radiusi $R = 1$, lekin qator $(-1, 1)$ da tekis yaqinlashuvchi emas. ■

Ravshanki, darajali qator funksional qatorning xususiy holi. Shu uchun darajali qator ham tekis yaqinlashuvchi funksional qatorning xossalari kabi xossalarga ega.

Yuqorida keltirilgan 1-xossadan natija sifatida quyidagi xossa kelib chiqadi:

2^o. Agar (11.13.2) qatorning yaqinlashish radiusi R bo'lsa, bu qatorning $S(x) = \sum_{n=0}^{\infty} a_n x^n$ yig'indisi $(-R, R)$ da uzluksiz funksiya bo'ladi.

Isboti. Haqiqatan ham, 1^o-xossaga asosan, darajali qator yig'indisi yaqinlashish oralig'i $(-R, R)$ ning ichidagi har qanday kesmada uzluksiz bo'ladi. Demak, u bu oraliqning istalgan nuqtasida uzluksizdir. ■

3^o. Agar ushbu $\sum_{n=0}^{\infty} a_n x^n$, $\sum_{n=0}^{\infty} b_n x^n$ darajali qatorlar $x = 0$ nuqta atrofida bir xil yig'indilarga ega bo'lsa, u holda bu darajali qatorlar aynan bir-biriga teng, ya'ni $a_0 = b_0, a_1 = b_1, \dots, a_n = b_n, \dots$ bo'ladi.

Isboti. Ushbu $\sum_{n=0}^{\infty} a_n x^n \equiv \sum_{n=1}^{\infty} b_n x^{n-1}$ ayniyatda $x=0$ deb olinsa, u holda, $a_0 = b_0$ ekanligini topamiz. Yuqoridagi ayniyatning ikkala tomonidagi bu hadlarni olib tashlasak va ayniyatning ikkala tomonini x ga ($x \neq 0$ deb hisoblab) bo'lish natijasida yangi $\sum_{n=1}^{\infty} a_n x^{n-1} \equiv \sum_{n=0}^{\infty} b_n x^n$ ayniyatni hosil qilamiz. Ikkala yig'indining $x=0$ da uzluksizligini e'tiborga olgan holda bu ayniyatda $x=0$ deb olib, $a_1 = b_1$ ekanini topamiz. So'ngra, bu hadlarni olib tashlab, $x \neq 0$ ga bo'lish bilan, yana yig'indilarning uzluksizliklarini e'tiborga olgan holda $a_2 = b_2$ ekanligini topamiz. Bu mulohazali davom ettirish natijasida matematik induksiya usuliga asosan, $a_n = b_n$ ekanligiga ishonch hosil qilamiz. ■

4⁰. Agar (11.13.2) qatorning yaqinlashish radiusi R bo'lsa, bu qatorni $[a, b]$ ($[a, b] \subset (-R, R)$) segmentda hadma-had integrallash mumkin, ya'ni

$$\int_a^b S(x) dx = \int_a^b \sum_{n=0}^{\infty} a_n x^n dx = \sum_{n=0}^{\infty} \int_a^b a_n x^n dx.$$

Isboti. Ravshanki, berilgan darajali qator $[a, b]$ da ($[a, b] \subset (-R, R)$) tekis yaqinlashuvchi. Tekis yaqinlashuvchi funksional qatorning xossasiga asosan uni hadma-had integrallash mumkin bo'ladi. Xususiyl holda $\forall x \in (-R, R)$ uchun

$$\int_a^x S(t) dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} \quad (11.15.2)$$

(11.15.2) qatorning yaqinlashish radiusi R bo'ladi.

Hakikatdan ham, Koshi-Adamar teoremasiga asosan,

$$\overline{\lim}_{n \rightarrow \infty} \sqrt[n]{\frac{|a_n|}{n+1}} = \overline{\lim}_{n \rightarrow \infty} \frac{\sqrt[n]{|a_n|}}{\sqrt[n]{n+1}} = \overline{\lim}_{n \rightarrow \infty} \sqrt[n]{|a_n|} = R$$

bo'ladi.

5⁰. Agar (11.13.2) qatorning yaqinlashish radiusi R bo'lsa, bu qatorni $(-R, R)$ da hadma-had differensiallash mumkin, ya'ni

$$\left(\sum_{n=0}^{\infty} a_n x^n \right)' = \sum_{n=1}^{\infty} n a_n x^{n-1}. \quad (11.15.3)$$

Isboti. (11.15.3) qatorning yaqinlashish radiusi R va uning yig'indisi $S(x)$ bo'lsin. Berilgan darajali qator $[-c, c] \subset (-R, R)$ da tekis yaqinlashuvchi. U holda tekis yaqinlashuvchi funksional qatorning xossasiga asosan, berilgan darajali qatorni hadma-had diffirensiallash mumkin.

Demak, $\forall x_0 \in (-R, R)$ da

$$S'(x) = \left(\sum_{n=0}^{\infty} a_n x^n \right)' = \sum_{n=1}^{\infty} n a_n x^{n-1}. \quad (11.15.4)$$

(11.15.4) darajadi qatorning ham yaqinlashish radiusi R ga teng bo'lishini ko'rsatamiz: Koshi-Adamar formulasiga ko'ra,

$$\lim_{n \rightarrow \infty} \sqrt[n]{n|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{n} \cdot \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \rho, \quad R = \frac{1}{\rho}.$$

11.15.5. Eslatma. Yuqoridagi darajali qatorning xossalariga asosan, ushbu

$$\sum_{n=0}^{\infty} a_n x^n, \quad \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}, \quad \sum_{n=1}^{\infty} n a_n x^{n-1}$$

darajali qatorlar bir xil yaqinlashish radiusiga ega ekanligi kelib chiqadi.

11.15.6. Misol. Ushbu

$$x - 4x^2 + 9x^3 - 16x^4 + \dots$$

darajali qatorning yig'indisini toping.

Yechilishi. (11.14.5) formula orqali berilgan darajali qatorning yaqinlashish radiusini topamiz:

$$R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|,$$

bunda $a_n = (-1)^{n-1} n^2$. $R = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n-1} n^2}{(-1)^n (n+1)^2} \right| = 1$. Darajali qator yig'indisini

$S(x) = x - 4x^2 + 9x^3 - 16x^4 + \dots$ kabi belgilaymiz. Berilgan qator yig'indisi $S(x)$ ni x ($x \neq 0$) ga bo'lib, uni darajali qatorlarning 5^0 -xossasiga ko'ra, $|x| < 1$ intervalda hadma-had integrallaymiz:

$$\begin{aligned} \int \frac{S(x)}{x} dx &= x - 2x^2 + 3x^3 - 14x^4 + \dots + C = \\ &= (x^2 - x^3 + x^4 - \dots)' - x + x^2 - x^3 + \dots + C = \frac{x}{(1+x)^2} + C. \end{aligned}$$

Bu tenglikni ($|x| < 1, x \neq 0$) differensiallab, $S(x) = \frac{x(1-x)}{(1+x)^3}$ ni topamiz ($|x| < 1, x \neq 0$). Bu yerda $x \neq 0$ degan shartni olib tashlash mumkin, chunki $S(0) = 0$.

11.15.7. Misol. Ushbu

$$1 + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} x^{2n} = \frac{1}{\sqrt{1-x^2}} \quad (-1 < x < 1)$$

darajali qatorni hadma-had integrallashdan foydalanib,

$$1 + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \frac{1}{2n+1} = \frac{\pi}{2}$$

bo'lishini ko'rsating.

Yechilishi. Bu darajali qatorni 4^0 -xossaga asosan, $[0; x]$ ($[0; x] \subset (-1; 1), x > 0$) segmentda hadma-had integrallash mumkin. U holda arksinusning bizga ma'lum bo'lgan yoyilmasiga ega bo'lamiz:

$$\int_0^x \frac{dt}{\sqrt{1-t^2}} = \arcsin x = x + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \frac{x^{2n+1}}{2n+1}.$$

Bu hosil bo'lgan darajali qatorning yaqinlashish intervali $(-1; 1)$ bo'ladi. $x = \pm 1$ da

$$x + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \frac{x^{2n+1}}{2n+1}.$$

darajali qator

$$1 \pm \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \frac{1}{2n+1}$$

sonli qatorga aylanadi. Bu qator, Raabe alomatiga ko'ra, yaqinlashuvchi bo'ladi. Unda $x = 1$ da

$$1 + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \frac{1}{2n+1} = \arcsin 1 = \frac{\pi}{2}$$

bo'ladi.

53-§. Tyeylor qatori

11.16. Funktsiyalarni Teylor qatoriga yoyish.

$f(x)$ funksiya

x_0 ($x_0 \in R$) nuqtaning biror

$$U_{\delta}(x_0) = \{x \in R : x_0 - \delta < x < x_0 + \delta\} \quad (\delta > 0)$$

atrofida berilgan bo'lib, u shu atrofda istalgan tartibdagi hosilaga ega bo'lsa, ushbu

$$f'(x_0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n \quad (11.16.1)$$

darajali qator, u yaqinlashuvchi bo'ladimi, yaqinlashuvchi bo'lib, uning yig'indisi $f(x)$ funksiyaga teng bo'ladimi yoki yo'qmi, bundan qat'iy nazar, $f(x)$ funksiyaning $x = x_0$ nuqtadagi Teylor qatori deyiladi. Bu qator (11.13.1) darajali qatorga o'xshash bo'lib, bunda

$$f(x_0) = a_0, \quad \frac{f'(x_0)}{1!} = a_1, \quad \frac{f''(x_0)}{2!} = a_2, \quad \frac{f'''(x_0)}{3!} = a_3, \dots, \frac{f^{(n)}(x_0)}{n!} = a_n, \dots$$

lar *Teylor koeffitsiyentlari* deyiladi.

Xususiyl holda, ya'ni $x_0 = 0$ bo'lganda (11.15.1) Teylor qatori

$$f(0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

ko'rinishga keladi. Bu qator ko'p hollarda *Makloren qatori* deb yuritiladi.

11.16.2. Teorema. $f(x)$ funksiya biror $U_{\delta}(x_0)$ atrofida istalgan tartibdagi hosilaga ega bo'lib, (11.15.1) qator uning $x = x_0$ nuqtadagi Teylor qatori bo'lsin. Bu qator $U_{\delta}(x_0)$ da $f(x)$ ga yaqinlashishi uchun uning

$$f(x) = f(x_0) + \frac{f^{(1)}(x_0)}{1!} (x - x_0) + \frac{f^{(2)}(x_0)}{2!} (x - x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n + r_n(x).$$

Teylor formulasi qoldiq hadining $\forall x \in U_{\delta}(x_0)$ da nolga intilishi, ya'ni $\lim_{n \rightarrow \infty} r_n(x) = 0$ bo'lishi zarur va yetarli.

Isboti. Zarurligi. (11.15.1) darajali qator $(-R, R)$ da yaqinlashuvchi bo'lib, uning yig'indisi $S(x)$ bo'lsin. Darajali qator yaqinlashish ta'rifiga ko'ra,

$$\lim_{n \rightarrow \infty} S_n(x) = S(x) \quad (x \in (-R, R)),$$

bunda

$$S_n(x) = f(x_0) + \frac{f^{(1)}(x_0)}{1!} (x - x_0) + \frac{f^{(2)}(x_0)}{2!} (x - x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

$\forall x \in (-R, R)$ da $\lim_{n \rightarrow \infty} S_n(x) = S(x)$ bo'lgani uchun

$$\lim_{n \rightarrow \infty} [S_n(x) - S(x)] = \lim_{n \rightarrow \infty} r_n(x) = 0$$

bo'ladi.

Yetarliligi. $\forall x \in (-R, R)$ uchun $\lim_{n \rightarrow \infty} r_n(x) = 0$ bo'lsin. U holda

$$\lim_{n \rightarrow \infty} [S_n(x) - S(x)] = \lim_{n \rightarrow \infty} r_n(x) = 0.$$

Bundan, $\lim_{n \rightarrow \infty} S_n(x) = S(x)$.

Demak,

$$S(x) = f(x_0) + \frac{f^{(1)}(x_0)}{1!}(x-x_0) + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \dots$$

bo'ladi. U holda bu munosabat o'rinli bo'lsa, u xolda $f(x)$ funktsiya Teyler qatoriga yoyilgan deyiladi.

Ma'lumki, Teylor formulasi qoldiq hadi:

$$a) \text{ integral ko'rinishida} \quad r_n(x) = \frac{1}{n!} \int_{x_0}^x (x-t)^n f^{(n+1)}(t) dt;$$

$$b) \text{ Lagranj ko'rinishida} \quad r_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-x_0)^{n+1},$$

bunda $c = x_0 + \theta(x-x_0)$, $0 < \theta < 1$;

s) Koshi ko'rinishida

$$r_n(x) = \frac{f^{(n+1)}(c)}{n!} (1-\theta)^n (x-x_0)^{n+1}, \quad c = x_0 + \theta(x-x_0), \quad 0 < \theta < 1;$$

$$d) \text{ Peano ko'rinishida} \quad r_n(x) = o((x-x_0)^n)$$

bo'ladi.

11.16.3. Teorema. $f(x)$ funktsiya $(x_0 - \rho, x_0 + \rho)$ ($\rho > 0$) oraliqda darajali qatorga yoyilgan bo'lsa:

$$f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots + a_n(x-x_0)^n + \dots,$$

bu qator $f(x)$ funktsiyaning Teylor qatori bo'ladi, bunda

$$a_0 = f(x_0), \quad a_1 = \frac{f'(x_0)}{1!}, \quad a_2 = \frac{f''(x_0)}{2!}, \quad a_3 = \frac{f'''(x_0)}{3!}, \dots, a_n = \frac{f^{(n)}(x_0)}{n!}, \dots$$

Isboti. $R > 0$ uchun $f(x)$ funktsiya $(-R, R)$ da darajali qatorga yoyilgan bo'lsin, ya'ni

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, x \in (x_0 - R, x_0 + R). \quad (11.16.4)$$

(11.16.4) darajali qatomi n marta differensiallab, 5^0 -xossaga ko'ra,

$$f^{(n)}(x) = \sum_{k=n}^{\infty} a_k k(k-1) \dots (k-n+1)(x-x_0)^{k-n}, x(x_0 - R, x_0 + R)$$

tenglik o'rinli bo'ladi.

Agar $x = x_0$ bo'lsa, tenglikning o'ng tomonidagi yig'indida birinchi haddan tashqari barcha hadlar nolga teng bo'ladi.

Demak,

$$f^{(n)}(x_0)(x) = n(n-1) \dots 2 \cdot 1 a_n = n! a_n, \quad a_n = \frac{f^{(n)}(x_0)}{n!} \quad (n=1, 2, \dots). \blacksquare$$

11.16.5. Teorema. Agar $f(x)$ funksiya biror $(x_0 - R, x_0 + R)$ intervalda istalgan tartibdagi hosilaga ega bo'lib, shunday o'zgarmas $M > 0$ son topilsaki, barcha $x \in (x_0 - R, x_0 + R)$, hamda barcha $n \in \mathbb{N}$ la uchun

$$|f^{(n)}(x)| \leq M$$

bajarilsa, u holda $(x_0 - R, x_0 + R)$ intervalda $f(x)$ funsiya Teylar qatoriga yoyiladi, ya'ni

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n \quad (11.16.6)$$

bo'ladi.

Isboti. Ma'lumki, $f(x)$ funksiyaning Lagranj ko'rinishdagi qodliq hadli Teyler formulasi quyidagicha buladi:

$$f(x) = f(x_0) + \frac{f^{(1)}(x_0)}{1!} (x - x_0) + \frac{f^{(2)}(x_0)}{2!} (x - x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n + r_n(x),$$

$$r_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1}, \quad c = x_0 + \theta(x - x_0), \quad 0 < \theta < 1.$$

Shartga asosan,

$$|r_n(x)| = \left| \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1} \right| \leq M \frac{R^{n+1}}{(n+1)!}, \quad x \in (x_0 - R, x_0 + R).$$

Kavshanki, $\lim_{n \rightarrow \infty} \frac{R^{n+1}}{(n+1)!} = 0$. Demak, $\forall x \in (x_0 - R, x_0 + R)$ da $\lim_{n \rightarrow \infty} r_n(x) = 0$ buladi. Bundan, $f'(x)$ funksiyaning Teylor qatoriga yoyilishi kelib chiqadi. ■

11.16.7. Misol. Ushbu

$$f(x) = \frac{1}{1+x}$$

funksiyani $x = 2$ nuqta atrofida Teylor qatoriga yoying.

Yechilishi. Berilgan $f(x) = \frac{1}{1+x}$ funksiyaning istalgan tartibli hosilalarini topamiz, co'ngra uning va hosilalarining $x_0 = 2$ nuqtadagi qiymatlarini hisoblaymiz:

$$f(x) = \frac{1}{x+1} = (x+1)^{-1}, \quad f(2) = \frac{1}{3},$$

$$f^{(1)}(x) = \left(\frac{1}{1+x}\right)' = -\frac{1}{(x+1)^2}, \quad f^{(1)}(2) = -\frac{1!}{3^2},$$

$$f^{(2)}(x) = \frac{2!}{(x+1)^3}, \quad f^{(2)}(2) = \frac{2!}{3^3},$$

$$f^{(n)}(x) = \frac{(-1)^n n!}{(x+1)^{n+1}}, \quad f^{(n)}(2) = \frac{(-1)^n n!}{3^{n+1}},$$

Topilganlarni (11.16.1) formulaga qo'ysak, $f(x) = \frac{1}{1+x}$ funksiyaning

Teylor qatori

$$\begin{aligned} & \frac{1}{3} - \frac{1!}{3^2}(x-2) + \frac{2!}{3^3} \frac{(x-2)^2}{2!} + \dots + \frac{(-1)^n n! (x-2)^n}{3^{n+1} n!} + \dots = \\ & = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} (x-2)^n \end{aligned} \quad (11.16.8)$$

ko'rinishda bo'ladi. Bu qatorning yaqinlashish intervalini Koshi alomatidan foydalanib topamiz:

$$u_n(x) = \frac{(-1)^n}{3^{n+1}} (x-2)^n, \quad \lim_{n \rightarrow \infty} \sqrt[n]{|u_n(x)|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{(-1)^n}{3^{n+1}} (x-2)^n \right|} = \frac{|x-2|}{3} < 1.$$

Endi $|x-2| < 3$ tengsizlikni yechamiz: $-3 < x-2 < 3 \Rightarrow -1 < x < 5$.
 (11.16.8) qatorning yaqinlashish intervali $(-1; 5)$ bo'ladi. Intervalning chegaralarida, ya'ni $x = -1$ va $x = 5$ nuqtalarda mos ravishda

$$1+1+1+\dots \quad \text{va} \quad 1-1+1-1+\dots$$

uzoqlashuvchi qatorlar hosil bo'ladi.

Demak, (11.16.8) qatorning yaqinlashish sohasi $(-1; 5)$ intervaldan iborat bo'ladi. Bu (11.16.8) qator $(-1; 5)$ intervalda $f(x) = \frac{1}{1+x}$ funksiyaga yaqinlashishi uchun $f(x)$ funktsiya Teylor formulasining qoldiq hadi barcha $x \in (-1; 5)$ da nolga intiladi. Haqiqatan ham,

$$|r_n(x)| = \frac{|(-1)^{n+1}(x-2)^{n+1}|}{|(3+\theta(x-2))^{n+2}|} \leq \frac{1}{3\left(1+\theta\frac{(x-2)}{3}\right)^{n+2}} \xrightarrow{n \rightarrow \infty} 0,$$

bunda $0 < \theta < 1$.

Shunday qilib, **11.16.2-teorema**ga asosan, (11.16.8) qatorning yig'indisi $\frac{1}{1+x}$ ga teng bo'ladi, ya'ni $\frac{1}{1+x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} (x-2)^n$.

11.17. Elementar funksiyalarning Teylor qatorlari. (11.16.6) formulada $x_0 = 0$ deb, amaliyotda ko'p uchraydigan ba'zi elementar funksiyalarning darajali qatorlari yoyilmalarini keltiramiz:

1. Ko'rsatkichli funksiyalar:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (-\infty < x < +\infty, \quad R = \infty).$$

$$a^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \ln^n a, \quad a > 0, a \neq 1 \quad (-\infty < x < +\infty, \quad R = \infty).$$

2. Trigonometrik funksiyalar:

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad (-\infty < x < +\infty, \quad R = \infty). \quad (11.17.1)$$

$$\sin x = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} \quad (-\infty < x < +\infty, \quad R = \infty). \quad (11.17.2)$$

3. Darajali funksiyalar:

$$(x+1)^\alpha = 1 + \sum_{n=1}^{\infty} C_\alpha^n x^n \quad (11.17.3)$$

$$\text{bunda } C_{\alpha}^n = \frac{\alpha(\alpha-1) \cdot \dots \cdot (\alpha-(n-1))}{n!}.$$

Agar $\alpha \neq 0, \alpha \neq n, (n \in \mathbb{N})$ bo'lsa, (11.17.3) qatorning yaqinlashish radiusi 1 ga teng bo'ladi.

(11.17.3) formulaning muhim hususiy hollari:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad (|x| < 1, R=1);$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \quad (|x| < 1, R=1);$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad (|x| < 1, R=1).$$

4. Logarifmik funksiyalar:

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \quad (-1 < x \leq 1, R=1);$$

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} \quad (-1 \leq x < 1, R=1).$$

5. Giperbolik funksiyalar:

$$\operatorname{ch} x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}, \quad (-\infty < x < +\infty, R=\infty);$$

$$\operatorname{sh} x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}, \quad (-\infty < x < +\infty, R=\infty).$$

6. Teskari trigonometrik funksiyalar:

$$\operatorname{arctg} x = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1}, \quad (|x| \leq 1, R=1);$$

$$\arcsin x = \sum_{n=0}^{\infty} \frac{(2n+1)!! x^{2n+1}}{(2n)!!(2n+1)}, \quad (|x| < 1, R=1);$$

$$\arccos x = \frac{\pi}{2} - x - \sum_{n=1}^{\infty} \frac{(2n-1)!! x^{2n+1}}{(2n)!!(2n+1)}, \quad (|x| < 1, R=1);$$

$$\operatorname{arcctg} x = \frac{\pi}{2} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{2n+1}, \quad (|x| < 1, R=1).$$

11.17.4. Misol. Ushbu $f(x) = \cos^4 x$ funksiyani $x_0 = \frac{\pi}{8}$ nuqta

atrofida Teylor qatoriga yoying va yaqinlashish radiusini toping.

Yechilishi. Bizga ma'lumki, $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$, u holda

$$f(x) = \left[\frac{1}{2}(1 + \cos 2x) \right]^2 = \frac{3}{8} + \frac{1}{2} \cos 2x + \frac{1}{8} \cos 4x$$

bo'ladi. So'ngra $x = t + \frac{\pi}{8}$ almashtirishni bajaramiz:

$$f(x) = f\left(t + \frac{\pi}{8}\right) = g(t) = \frac{3}{8} + \frac{\sqrt{2}}{4}(\cos 2t - \sin 2t) - \frac{1}{8} \sin 4t. \quad (11.17.5)$$

Endi (11.17.1) va (11.17.1) yoyilmalardan foydalanib, $\cos 2t$, $\sin 2t$ va $\sin 4t$ uchun

$$\cos 2t = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} t^{2n}; \quad \sin 2t = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!} t^{2n+1},$$

$$\sin 4t = \sum_{n=0}^{\infty} (-1)^n \frac{4^{2n+1} t^{2n+1}}{(2n+1)!}$$

tengliklarga ega bo'lamiz. $\cos 2t$, $\sin 2t$ va $\sin 4t$ funksiyalarning yoyilmalarini (11.17.5) ga keltirib qo'ysak,

$$g(t) = \frac{3}{8} + \frac{\sqrt{2}}{4} \left(\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} t^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} t^{2n+1}}{(2n+1)!} \right) - \frac{1}{8} \sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n+1} t^{2n+1}}{(2n+1)!}$$

ifodaga ega bo'lamiz. Endi $t = x - \frac{\pi}{8}$ ekanligini hisobga olsak, natijada

$$f(x) = \frac{3}{8} + \sqrt{2} \left(\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n-2}}{(2n)!} \left(x - \frac{\pi}{8}\right)^{2n} - \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n-1}}{(2n+1)!} \left(x - \frac{\pi}{8}\right)^{2n+1} \right) - \sum_{n=0}^{\infty} \frac{(-1)^n 2^{4n-1}}{(2n+1)!} \left(x - \frac{\pi}{8}\right)^{2n+1}$$

bo'ladi. Hosil bo'lgan qatorning yaqinlashish radiusi $R = \infty$.

54-§. Xos parametrga bog'liq integrallar

12.1. Limit funksiya tushunchasi va unga tekis yaqinlashish.

a). *Parametrga bog'liq integralning ta'rif.* $f(x, y)$ funksiya $\{M\} = \{(x, y) \in E^2 : a \leq x \leq b, y \in Y \subset R\}$ to'plamda aniqlangan bo'lib, $y_0 \in R$ nuqta Y to'plamning limit nuqtasi bo'lsin. Har bir tayinlangan $y \in Y$ uchun $x \in [a, b]$ bo'yicha xos yoki xosmas ma'nosida integrallanuvchi, ya'ni $\int_a^b f(x, y) dx$ integral mavjud bo'lsin. U holda, qarayotgan integral tayinlangan y ning funksiyasini ifoda qiladi:

$$J(y) = \int_a^b f(x, y) dx. \quad (12.1.1)$$

Odatda (12.1.1) integralni *parametrga bog'liq integral*, y ni esa, *parametr* deb ataladi

Biz, 12-bobda asosan, $f(x, y)$ funksiyaning funksional xossalriga ko'ra, $J(y)$ funksiyaning funksional xossalari (limiti, uzluksizligi, differentsiallanuvchiligi, integrallanuvchiligi) ni o'rganamiz.

$J(y)$ funksiyaning funksional xossalarini o'rganashda limit funksiya tushunchasi va unga tekis yaqinlashish shartlari muhim ahamiyatga ega.

b) $f(x, y)$ funksiya $\{M\}$ to'plamda aniqlangan bo'lib, y_0 nuqta $Y = \{y\}$ to'plamning chekli limit nuqtasi bo'lsin.

12.1.2. Ta'rif. Agar $\lim_{y \rightarrow y_0} f(x, y)$ -mavjud bo'lsin, har bir $x \in [a, b]$ lar uchun $f(x, y)$ funksiyaning $y \rightarrow y_0$ dagi limitini mos qo'yish natijasida, ya'ni

$$\varphi : x \rightarrow \lim_{y \rightarrow y_0} f(x, y)$$

funksiya hosil bo'ladi. Shu funksiyaga $f(x, y)$ funksiyaning $y \rightarrow y_0$ dagi *limit funksiyasi* deyiladi va u

$$\lim_{y \rightarrow y_0} f(x, y) = \varphi(x), x \in [a, b] \quad (12.1.3)$$

kabi yoziladi.

12.1.4. Ta'rif. Agar: 1) $y \rightarrow y_0$ da $f(x, y)$ funksiyaning chekli limit funksiyasi mavjud, ya'ni $\exists \lim_{y \rightarrow y_0} f(x, y) = \varphi(x), x \in [a, b]$ bo'lsa;

2) $\forall \varepsilon > 0$ son olinganda ham shunday x ga bog'liq bo'lmagan $\delta = \delta(\varepsilon) > 0$ son topilib, $|y - y_0| < \delta$ bo'lganda

$$|f(x, y) - \varphi(x)| < \varepsilon$$

tengsizlik hamma $x \in [a, b]$ lar uchun birdaniga bajarilsa, u holda $f(x, y)$ funksiya $\varphi(x)$ limit funksiyaga $x \in [a, b]$ ga nisbatan *tekis yaqinlashadi* deyiladi.

Agar $y_0 = \pm\infty$ bo'lganda, masalan $+\infty$ bo'lsa, bu ta'rifdagi $|y - y_0| < \delta$ tengsizlik $y_0 > \Delta$ tengsizlikka almashtiriladi.

Biz yuqorida 11.3-bandda tekis yaqinlashishning hususiy holi $f_n(x)$ ketma-ketlikning limit funksiyasiga tekis yaqinlashish tushunchasi (bu yerda n -parametr) va tekis yaqinlashish shartlarini ko'rgan edik. Endi biz, bu bandda, umumiy holda limit funksiyaga tekis yaqinlashish shartini keltiramiz.

12.1.5. Teorema. $f(x, y)$ funksiya $y \rightarrow y_0$ da limit funksiya $\varphi(x)$ ga ega bo'lishi va unga tekis yaqinlashish uchun $\forall \varepsilon > 0$ olinganda ham x ga bog'liq bo'lmagan shunday $\delta = \delta(\varepsilon) > 0$ topilib, $|y - y_0| < \delta$, $|y' - y_0| < \delta$ tengsizliklarni qanoatlantiruvchi $\forall y, y' \in Y$ hamda $\forall x \in [a, b]$ lar uchun

$$|f(x, y) - f(x, y')| < \varepsilon \quad (12.1.6)$$

tengsizlikning bajarilishi zarur va yetarli.

Isbot. Zarurligi. $y \rightarrow y_0$ da $f(x, y)$ funksiya $\varphi(x)$ ga tekis intilsin, ya'ni 12.1.4-ta'rifga asosan, $\forall \varepsilon > 0$ son olinganda ham shunday x ga bog'liq bo'lmagan $\exists \delta > 0$ topilib $|y - y_0| < \delta$, $|y' - y_0| < \delta$ tengsizliklarni qanoatlantiruvchi $\forall y, y' \in Y$ hamda $\forall x \in [a, b]$ uchun

$$|f(x, y) - \varphi(x)| < \frac{\varepsilon}{2}, \quad |\varphi(x) - f(x, y')| < \frac{\varepsilon}{2} \quad (12.1.7)$$

tengsizliklar bajariladi. U holda (12.1.7) dan (12.1.6) kelib chiqadi, ya'ni

$$|f(x, y) - f(x, y')| \leq |f(x, y) - \varphi(x)| + |\varphi(x) - f(x, y')| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Yetarliligi. (12.1.6) shart bajarilgan bo'lsin. U holda, $f(x, y)$ funksiya limitning mavjudligi haqidagi Koshi teoremasiga asosan, $\varphi(x)$ funksiya ega bo'ladi, ya'ni $\lim_{y \rightarrow y_0} f(x, y) = \varphi(x)$. Ikkinchidan y o'zgaruvchining $|y - y_0| < \delta$ tengsizlikni qanoatlantiradigan tayin qiymatida (12.1.6) da $y' \rightarrow y_0$ da limitga o'tsak

$$|\varphi(x) - f(x, y)| < \varepsilon$$

tengsizlikga ega bo'lamiz. Bu esa, $y \rightarrow y_0$ da $f(x, y)$ funksiyaning $\varphi(x)$ limit funksiya $[a, b]$ da x ga nisbatan tekis yaqinlashishini anglatadi. ■

12.1.8. Teorema. Agar $y \rightarrow y_0$ da $f(x, y)$ funksiya $\varphi(x)$ ga $[a, b]$ da tekis yaqinlashuvchi bo'lsa, u holda y_0 ga intiluvchi Y to'plamdan olingan har bir $\{y_n\}_{y_n \in Y}$ ga mos kelgan

$$f_n(x) = f(x, y_n)$$

funksional ketma-ketlik ham $[a, b]$ da tekis yaqinlashuvchi bo'ladi.

Isboti. $y \rightarrow y_0$ da $f(x, y)$ funksiya $\varphi(x)$ ga da tekis yaqinlashsin. U holda, 12.1.4-ta'rifga asosan, $\forall \varepsilon > 0$ son olinganda ham shunday $\delta = \delta(\varepsilon) > 0$ topilib, $|y - y_0| < \delta$ tengsizlikni qanoatlantiruvchi $\forall y \in Y, \forall x \in [a, b]$ lar uchun

$$|f(x, y) - \varphi(x)| < \varepsilon$$

tengsizlik bajariladi. $\{y_n\}$ ketma-ketlik y_0 ga intilganligi uchun $\forall \delta > 0$ son olinganda ham, $\exists n_0 \in N, \forall n > n_0$ uchun $|y_n - y_0| < \delta$ tengsizlik bajariladi.

Demak, $\forall \varepsilon > 0, \forall \delta > 0, |y_n - y_0| < \delta, \forall y_n \in Y, \forall x \in [a, b]$ lar uchun $|f_n(x) - \varphi(x)| < \varepsilon$ tengsizlik o'rinli bo'ladi. Bu tengsizlik $\{f_n(x)\}$ ketma-ketlikning $[a, b]$ da $\varphi(x)$ ga tekis yaqinlashishini anglatadi. ■

12.1.9. Teorema. Agar $f(x, y)$ funksiya $\forall y \in Y$ lar uchun $x \in [a, b]$ bo'yicha uzluksiz bo'lib, $y \rightarrow y_0$ da limit funksiya $\varphi(x)$ ga tekis intilsa, u holda limit funksiya $\varphi(x)$ ham uzluksiz bo'ladi.

Isboti. Y to'plamdan y_0 ga intiluvchi $\forall \{y_n\}$ ketma-ketlikni olib, $(y_n \in Y, n=1, 2, \dots, y_n \rightarrow y_0)[a, b]$ kesmada aniqlangan $f_n(x) = f(x, y_n)$ funksional ketma-ketlikni hosil qilamiz. Teoremaning shartiga ko'ra: 1) $\{f_n(x)\}$ funksional ketma-ketlikning har bir hadi $[a, b]$ segmentda uzluksiz bo'ladi; 2) 12.1.5-teoremaga asosan, $y \rightarrow y_0$ da $\{f_n(x)\}$ funksional ketma-ketlik limit funksiya $\varphi(x)$ ga tekis yaqinlashuvchi bo'ladi. U holda 12.1.8-teoremaga asosan, $\varphi(x)$ funksiya $[a, b]$ segmentda uzluksiz bo'ladi. ■

12.2. Parametrga bog'liq integrallarda parametr bo'yicha integral ostida limitga o'tish. $f(x, y)$ funksiya $\{M\} = \{(x, y) \in R^2 : a \leq x \leq b, y \in Y \subset R\}$ to'plamda aniqlangan bo'lib, y_0 nuqta Y to'plamning limit nuqtasi bo'lsin.

12.2.1. Teorema. Agar $f(x, y)$ funksiya o'zgarmas $y (y \in Y)$ larda x bo'yicha $[a, b]$ da uzluksiz bo'lib, $y \rightarrow y_0$ da limit funksiya $\varphi(x)$ ga a ga nisbatan tekis intilsa, u holda

$$\lim_{y \rightarrow y_0} J(y) = \lim_{y \rightarrow y_0} \int_a^b f(x, y) dx = \int_a^b \varphi(x) dx \quad (12.2.2)$$

tenglik o'rinli bo'ladi.

Isboti. Yuqoridagi 12.1.9-teoremaga asosan, limit funksiya $\varphi(x)$ uzluksiz.

Demak, $\varphi(x)$ funksiya integrallanuvchi ya'ni $\int_a^b \varphi(x) dx$ – integral mavjud. Shartga ko'ra $y \rightarrow y_0$ da $f(x, y)$ funksiya $\forall \varepsilon > 0$ son olinganda ham $\exists \delta = \delta(\varepsilon) > 0, |y - y_0| < \delta$ ni qanoatlantiruvchi $\forall y \in Y$ hamda $\forall x \in [a, b]$ lar uchun

$$|f(x, y) - \varphi(x)| < \frac{\varepsilon}{2}$$

tengsizlik o'rinli bo'ladi. U holda $|y - y_0| < \delta$ bo'lganda

$$\left| \int_a^b f(x, y) dx - \int_a^b \varphi(x) dx \right| \leq \int_a^b |f(x, y) - \varphi(x)| dx < \frac{\varepsilon}{b-a} \int_a^b dx = \varepsilon$$

bo'ladi. Keyingi tengsizlikdan

$$\lim_{y \rightarrow y_0} \int_a^b f(x, y) dx = \int_a^b \varphi(x) dx$$

kelib chiqadi, ya'ni y parametr bo'yicha integral ostida limitga o'tish mumkin bo'ladi. ■

12.2.3. Teorema. Agar $f(x, y)$ funksiya $\{M\} = \{(x, y) \in E^2 : a \leq x \leq b, c \leq y \leq d\}$ to'rtburchakda ikki o'zgaruvchili funksiya sifatida uzluksiz bo'lsa, u holda (12.1.1) integral parametr y ning $[c, d]$ dagi uzluksiz funksiya bo'ladi.

Isboti. Shartga ko'ra $f(x, y) : [a, b; c, d]$ da uzluksiz bo'lgani uchun u Kantar teoremasiga asosan, tekis uzluksiz bo'ladi. Tekis uzluksiz ta'rifiga ko'ra, $\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0, |x'' - x'| < \delta, |y'' - y'| < \delta$ tengsizliklarni qanoatlantiruvchi x va y lar uchun

$$|f(x'', y'') - f(x', y')| < \varepsilon$$

tengsizlik o'rinli bo'ladi. Xususiyl holda, $x'' = x' = x, y'' = y_0, y' = y$ desak, $|f(x, y) - f(x, y_0)| < \varepsilon$ bajariladi. Bundan, $y \rightarrow y_0$ da $f(x, y)$ funksiya limit funksiya $f(x, y_0)$ ga x ga nisbatan tekis intiladi. U holda 12.2.1-teoremaga asosan,

$$\lim_{y \rightarrow y_0} \int_a^b f(x, y) dx = \int_a^b f(x, y_0) dx \quad \text{yoki} \quad \lim_{y \rightarrow y_0} J(y) = J(y_0)$$

bo'ladi. Bu bilan teorema isbot bo'ladi. ■

12.3. Integral ostida parametr bo'yicha differensiallash

12.3.1. Teorema. Agar $f(x, y) : [a, b; c, d]$ da aniqlangan bo'lib, har bir o'zgarmas $y \in [c, d]$ lar uchun x bo'yicha $[a, b]$ da uzluksiz, hamda to'rtburchakda $f'_y(x, y)$ xususiyl hosila mavjud va ikki argumentli funksiya sifatida uzluksiz bo'lsa, u holda $J(y)$ funksiya $[c, d]$ da hosilaga ega va

$$J'(y) = \int_a^b f'_y(x, y) dx \quad (12.3.2)$$

formula o'rinli bo'ladi.

(12.3.2) formulaga *Leybnits formulasi* deb ataladi.

Isboti. Teorema shartga ko'ra, $f(x, y)$ funksiya $[a, b]$ da y bo'yicha uzluksiz bo'lgani uchun (12.1.1) integral mavjud bo'ladi. (12.1.1) da $y = y_0$ belgilab, unga $\Delta y = k$ ortirma beramiz:

$$J(y_0) = \int_a^b f(x, y_0) dx, J(y_0 + k) = \int_a^b f(x, y_0 + k) dx$$

bulardan

$$\frac{J(y_0 + k) - J(y_0)}{k} = \int_a^b \frac{f(x, y_0 + k) - f(x, y_0)}{k} dx$$

ni tuzamiz. Tenglikning o'ng tomoni k parametrga bog'liq. Bundan $k \rightarrow 0$ da integral ostida limitga o'tish mumkinligini ko'rsatsak, u holda

$$J'(y) = \lim_{k \rightarrow 0} \frac{f(x, y_0 + k) - f(x, y_0)}{k}$$

hosilaning mavjudligi ham isbotlangan bo'ladi, ya'ni

$$\begin{aligned} J'(y_0) &= \lim_{y \rightarrow y_0} \int_a^b \frac{f(x, y_0 + k) - f(x, y_0)}{k} dx = \\ &= \int_a^b \lim_{k \rightarrow 0} \frac{f(x, y_0 + k) - f(x, y_0)}{k} dx = \int_a^b f'_y(x, y) dx \end{aligned}$$

tenglik o'rinli bo'ladi. Shu bilan teorema isbot bo'ladi. Buning uchun, avvalo, Logranj formulasiga asosan,

$$\frac{f(x, y_0 + k) - f(x, y_0)}{k} = f'_y(x, y_0 + \theta k) \quad (0 < \theta < 1) \quad (12.3.3)$$

tenglikni yozamiz.

Shartga ko'ra, $f'_y(x, y)$ funksiya $[a, b; c, d]$ da uzluksiz bo'lgani uchun y tekis uzluksiz ham bo'ladi. U holda tekis uzluksiz ta'rifiga asosan, $\forall \varepsilon$ son olinganda ham shunday $\delta = \delta(\varepsilon) > 0$ topilib

$$|x'' - x'| < \delta, |y'' - y'| < \delta$$

bo'lganda

$$|f'_y(x'', y'') - f'_y(x', y')| < \varepsilon$$

tengsizlik bajariladi. Bunda $x' = x'' = x$, $y' = y_0$, $y' = y_0 + \theta k$ hamda $|x| < \delta$ deb olsak, u holda, (12.3.3) ni e'tborga olsak, $\forall x$ lar uchun

$$\left| \frac{f(x, y_0 + k) - f(x, y_0)}{k} - f'_y(x, y_0) \right| < \varepsilon$$

tengsizlik bajariladi. Bundan, integral ostidagi (12.3.3) funksiyaning $f'_y(x, y_0)$ – limit funksiya ga tekis yaqinlashuvchi ekanligi kelib chiqadi. U holda 12.2.1-teorema ga asosan, integral ishorasi ostida parametr bo'yicha limitga o'tish mumkinligi isbotlanadi. ■

12.4. Integral ishorasi ostida parametr bo'yicha integrallash

12.4.1. Teorema. Agar $f(x, y)$ funksiya ikki argumentli funksiya sifatida $[a, b] \times [c, d]$ da uzluksiz bo'lsa, u holda

$$\int_c^d dy \int_a^b f(x, y) dx = \int_a^b dx \int_c^d f(x, y) dy \quad (12.4.2)$$

formula o'rinli bo'ladi.

Isboti. (12.4.2) tenglikka nisbatan umumiyroq bo'lgan ushbu

$$\int_c^\eta dy \int_a^b f(x, y) dx = \int_a^b dx \int_c^\eta f(x, y) dy \quad (12.4.3)$$

tenglikni isbot qilamiz. bunda $c \leq \eta \leq d$. Tenglikning chap tomonidagi ichki $\int_a^b f(x, y) dx = J(y)$ integral 12.2.1-teorema ga asosan, y bo'yicha uzluksiz bo'ladi va chap tomonning η bo'yicha hosilasi

$$\left(\int_c^\eta J(y) dy \right)' = J(\eta) = \int_a^b f(x, \eta) dx$$

teng bo'ladi. (12.4.3) ning o'ng tomonini

$$\int_a^b \varphi(x, \eta) dx, \quad \varphi(x, \eta) = \int_c^\eta f(x, y) dy$$

ko'rinishida yozib olamiz. $\varphi(x, \eta)$ funksiya 11.3.1-teoremaning hamma shartlarini qanoatlantiradi.

Haqiqatdan ham 12.2.1-teorema ga asosan, $\varphi(x, \eta)$ x bo'yicha uzluksiz (bu yerda x parametr). Uning hosilasi $\varphi'_\eta(x, \eta) = f(x, \eta)$ - ikki o'zgaruvchisi bo'yicha ham uzluksiz. Shuning uchun 11.3.1-teorema ga asosan, unga Leybnits formulasini qo'llash mumkin.

$$\left(\int_a^b \varphi(x, \eta) dx \right)'_{\eta} = \int_a^b \varphi'_{\eta}(x, \eta) dx = \int_a^b f(x, \eta) dx = J(\eta)$$

Shunday qilib (12.4.3) tenglikning ikala tomoni η bo'yicha bir hil hosilaga ega. Ular bir-biridan o'zgarmas songa farq qiladi. Lekin, $\eta = c$ bo'lganda (12.4.3) tenglikning ikala tomoni ham nolga aylanadi. Shuning uchun (12.4.3) tenglik η ning hamma qiymatida ular aynan bir-biriga teng.

Xususiyl holda, $\eta = d$ bo'lganda (12.4.3) dan (12.4.2) kelib chiqadi. ■

12.5. Integral chegarasi ham parametrga bog'liq bo'lgan hol.
Ushbu

$$J(y) = \int_{\alpha(y)}^{\beta(y)} f(x, y) dx \quad (12.5.1)$$

parametrga bog'liq integralni qaraymiz.

12.5.2. Teorema. $f(x, y)$ funksiya $[a, b; c, d]$ to'rtburchakda aniqlangan, $x = \alpha(y)$, $x = \beta(y)$ ($c \leq y \leq d$) chiziqlar $[c, d]$ da uzluksiz bo'lib, x ning o'zgarish sohasidan chiqmasin. U holda, (12.5.1) integral y bo'yicha $[c, d]$ ga uzluksiz funksiyaning ifoda qiladi.

Isboti. $y_0 \in [c, d]$ ni olib, (12.5.1) integralni quyidagicha yozib olamiz:

$$J(y) = \int_{\alpha(y_0)}^{\beta(y_0)} f(x, y) dx + \int_{\beta(y_0)}^{\beta(y)} f(x, y) dx - \int_{\alpha(y_0)}^{\alpha(y)} f(x, y) dx \quad (12.5.3)$$

Bunda, birinchi integralning chegarasi o'zgarmas bo'lgani uchun 12.2.1-teoremaga asosan, u $y \rightarrow y_0$ da

$$J(y_0) = \int_{\alpha(y_0)}^{\beta(y_0)} f(x, y_0) dx$$

ga intiladi, ya'ni $\lim_{y \rightarrow y_0} J(y) = J(y_0)$.

Qolgan integrallarni baholaymiz:

$$\left| \int_{\beta(y_0)}^{\beta(y)} f(x, y) dx \right| \leq M \cdot |\beta(y) - \beta(y_0)|, \quad \left| \int_{\alpha(y_0)}^{\alpha(y)} f(x, y) dx \right| \leq M \cdot |\alpha(y) - \alpha(y_0)|$$

$M = \max|f(x, y)|$. Shartga ko'ra $\alpha(y)$ va $\beta(y)$ lar uzluksiz $y \rightarrow y_0$ da (12.5.3) ning o'ng tomonidagi ikkinchi va uchinchi integrallar nolga intiladi.

Shunday qilib, (12.5.3) dan $\lim_{y \rightarrow y_0} J(y) = J(y_0)$ ekanligi kelib chiqadi. ■

12.5.4. Teorema. 12.5.2-teoremadagi $f(x, y)$, $\alpha(y)$, $\beta(y)$ funksiyalarga qo'yilgan shartlardan tashqari $f(x, y)$ funksiya $[a, b; c, d]$ da uzluksiz $f'_y(x, y)$, $[c, d]$ da $\alpha'(y)$ va $\beta'(y)$ uzluksiz hosilalar mavjud bo'lsa, u holda

$$J'(y) = \int_{\alpha(y)}^{\beta(y)} f'_y(x, y) dx + \beta'(y) \cdot f(\beta(y), y) - \alpha'(y) \cdot f(\alpha(y), y) \quad (12.5.5)$$

formula o'rinli bo'ladi.

Isboti. Bu yerda ham (12.5.3) tenglikdan foydalanamiz: birinchi integral, 12.3.1-teoremaga asosan, $y = y_0$ nuqtada

$$\int_{\alpha(y_0)}^{\beta(y_0)} f'_y(x, y) dx$$

hosilaga ega. Ikkinchi integralga o'rta qiymat haqidagi teoremani qo'llab

$$\frac{1}{y - y_0} \int_{\alpha(y_0)}^{\beta(y)} f'_y(x, y) dx = \frac{\beta(y) - \beta(y_0)}{y - y_0} f(\bar{x}, y), \bar{x} \in [\alpha(y_0), \beta(y)]$$

ni hosil qilamiz. Bundan, $y \rightarrow y_0$ da limitga o'tsak, natijada ikkinchi integralning $y = y_0$ nuqtadagi hosilasini, ya'ni

$$\beta'(y_0) \cdot f(\beta(y_0), y_0)$$

ni hosil qilamiz. Huddi shunday uchinchi integralning $y = y_0$ dagi hosilasi

$$- \alpha'(y_0) \cdot f(\alpha(y_0), y_0)$$

ga tengligi keltirib chiqariladi.

Shunday qilib, (12.5.1) integralning y bo'yicha hosilasining mavjudligi va y (12.5.5) formula orqali topilishini isbot qildik.

12.5.6. Misol. Ushbu

$$J = \int_0^1 \frac{x^b - x^a}{\ln x} dx \quad (0 < a \leq b)$$

integralni hisoblang.

Yechilishi. Integral ostidagi ifodani

$$\frac{x^b - x^a}{\ln x} = \int_a^b x^y dy$$

ko'rishga keltirib, berilgan integralni takroriy integralga keltiramiz.

12.4.1-teoreмага asosan, takroriy integralda integrallarning o'rinlarini almashtirib

$$J = \int_a^b dy \int_0^1 x^y dx = \int_a^b \frac{dy}{y+1} = \ln \frac{b+1}{a+1}$$

ega bo'lamiz.

Bu misolni boshqacha usul bilan, ya'ni a -ni belgilab, b ni ($b \geq a$) o'zgaruvchi deb qarab, unga 12.3.5-teoremani qo'llab

$$\frac{dJ}{db} = \int_0^1 x^b dx = \frac{1}{b+1}$$

topamiz. Bundan, $J = \ln(b+1) + C$, $b = a$ da $J = 0$ bo'ladi. Bundan, $C = -\ln(a+1)$ ni topamiz. Topilgan C ning qiymatini yuqoridagi tenglikga qo'yish natijasida, yuqoridagi

$$J = \ln \frac{b+1}{a+1}$$

natijaga ega bo'lamiz.

Yuqoridagi misolni yechish jarayonida 12.3.1-12.4.1-teoremlar qo'llaniladi, chunki bu teoremlarning hamma shartlari bajarilishini ko'rish qiyin emas. Shu shartlarning birontasi bajarilmasa, teoremaning tasdig'i o'rinli bo'lmasligi mumkin. Masalan, $f(x, y) = \frac{y^2 - x^2}{(x^2 + y^2)^2} : [0, 1; 0, 1]$

da berilgan bo'lsin. U holda

$$\int_0^1 f(x, y) dx = \frac{x}{x^2 + y^2} \Big|_0^1 = \frac{1}{1 + y^2} (y > 0),$$

$$\int_0^1 dy \int_0^1 f(x, y) dx = \arctg y \Big|_0^1 = \frac{\pi}{4},$$

lekin $\int_0^1 dx \int_0^1 f(x, y) dy = -\frac{\pi}{4}$, chunki, $f(x, y)$ funksiya $(0;0)$ nuqtada uzulishga ega.

12.5.7. Misol. Ushbu $\lim_{y \rightarrow 0} \int_{-\pi}^{\pi} (x^2 + \cos^2 xy) dx$ ni toping

Yechilishi. 1) Integral ostidagi $f(x, y) = x^2 + \cos^2 xy$ funksiya $M = \{(x, y) \in R^2 : -\pi \leq x \leq \pi; -1 \leq y \leq 1\}$ da uzluksiz.

2) $\lim_{y \rightarrow 0} (x^2 + \cos^2 xy) = x^2 + 1, \varphi(x) = 1 + x^2, \forall \varepsilon > 0, \delta = \frac{\sqrt{\varepsilon}}{\pi}$ olinganda $\forall x \in [-\pi, \pi]$ va $\forall y \in [-1, 1]$ uchun

$$|f(x, y) - \varphi(x)| = |x^2 + \cos^2 xy - x^2 - 1| = |\cos^2 xy - 1| = |\sin^2 xy| \leq |xy^2| < \varepsilon.$$

Demak, 12.2.3-teorema asosan, tenglik o'rinli bo'ladi:

$$\begin{aligned} \lim_{y \rightarrow 0} \int_{-\pi}^{\pi} (x^2 + \cos^2 xy) dx &= \int_{-\pi}^{\pi} \lim_{y \rightarrow 0} (x^2 + \cos^2 xy) dx = \int_{-\pi}^{\pi} (x^2 + 1) dx = \\ &= 2 \int_0^{\pi} (x^2 + 1) dx = \frac{2\pi}{3} (\pi^2 + 3) \end{aligned}$$

55-§. Parametrغا bog'liq xosmas integrallar

Biz yuqorida parametrغا bog'liq xos integrallarning funksional xossalarini o'rganganimizda integral ostidagi funksiyaning limit funksiya tushunchasi va unga tekis yaqinlashish tushunchalari muhim ahamiyatga ega ekanligini ko'rdik.

Parametrغا bog'liq xosmas integrallarni o'rganishda, xuddi yuqoridagi singari integrallarning tekis yaqinlashish tushunchasi va shartlari muhim ro'l o'ynaydi, shu sababli parametrغا bog'liq xosmas integrallarning funksional xossalarini o'rganganish uchun avval parametrغا bog'liq xosmas integrallarning tekis yaqinlashish tushunchasi va shartlarini keltiramiz

12.6. Parametrغا bog'liq xosmas integrallar. 1) $f(x, y)$ funksiya $\{M\} = \{(x, y) \in R^2 : a \leq x < \infty, y \in E \subset R\}$ to'plamda berilgan bo'lib, u

har bir belgilangan $y \in E$ da x o'zgaruvchining funksiyasi sifatida $[a, +\infty)$ da integrallanuvchi, ya'ni $\int_a^{\infty} f(x, y) dx$ integral mavjud bo'lsin. Ravshanki, bu integralning qiymati y o'zgaruvchiga bog'liq bo'ladi:

$$J(y) = \int_a^{\infty} f(x, y) dx \quad (12.6.1)$$

(12.6.1) integral parametrga bog'liq (chegarasi cheksiz) integral, y esa, parametr deyiladi.

$f(x, y)$ funksiya $\{M'\} = \{(x, y) \in R^2 : -\infty < x < a, y \in E \subset R\}$ ($\{M''\} = \{(x, y) \in R^2 : -\infty < x < +\infty, y \in E \subset R\}$) to'plamda berilgan va y o'zgaruvchining E dan olingan har bir belgilangan qiymatda x o'zgaruvchining funksiyasi sifatida $(-\infty, a]((-\infty, +\infty))$ da integrallanuvchi bo'lsin, ya'ni

$$\int_{-\infty}^a f(x, y) dx \quad \left(\int_{-\infty}^{+\infty} f(x, y) dx \right) \quad (12.6.2)$$

integral mavjud bo'lsin. (12.6.2) integral ham parametrga bog'liq integral deb ataladi.

2) $f(x, y)$ funksiya $\{M_1\} = \{(x, y) \in R^2 : a \leq x < b, y \in E \subset R\}$ to'plamda berilgan bo'lib, y ning E dan olingan har bir belgilangan qiymatda x ning funksiyasi sifatida ($x = b$ maxsus nuqta) $[a, b)$ da integrallanuvchi, ya'ni

$$\int_a^b f(x, y) dx (y \in E \subset R)$$

integral mavjud bo'lsin. Ravshanki, bu integral ham y ning qiymatiga bog'liq bo'ladi:

$$J_1(y) = \int_a^b f(x, y) dx. \quad (12.6.3)$$

Bu integralga chegaralanmagan funksiyaning parametrga bog'liq hosmas integral deyiladi.

Huddi shunday, $x = a$ maxsus nuqta ($x = a, x = b$ maxsus nuqtalar) bo'lganda ushbu

$$\int_a^b f(x, y) dx \left(\int_a^b f(x, y) dx \right)$$

parametrga bog'liq hosmas integrallarning ham ta'riflari yuqoridagi kabi beriladi

3) $f(x, y)$ funksiya $\{M_2\} = \{(x, y) \in R^2 : c < x < +\infty, y \in E \subset R\}$ to'plamda berilgan bo'lib, y ning E dan olingan har bir tayin qiymatda x ning funksiyasi sifatida ($x = c$ maxsus nuqta) $(c, +\infty)$ da integrallanuvchi, ya'ni

$$\int_c^{+\infty} f(x, y) dx$$

chegaranmagan funksiyaning chegarasi cheksiz hosmas integrali mavjud bo'lsin. Bu integral ham y ga bog'liq bo'ladi:

$$J_2(y) = \int_c^{+\infty} f(x, y) dx \quad (12.6.4)$$

(12.6.4) integral chegaralanmagan funksiyaning parametrga bog'liq hosmas integrali deb ataladi. Masalan, ushbu

$$J_1(y) = \int_2^{+\infty} \frac{dx}{x^y} (y > 0), \quad J_2(y) = \int_a^b \frac{dx}{(x-a)^y} (y > 0), \quad J_3(y) = \int_0^{+\infty} x^{y-1} e^{-x} dx (y > 0)$$

integral parametrga bog'liq hosmas integrallardir.

12.7. Parametrga bog'liq hosmas integralning tekis yaqinlashishi. 1) $f(x, y)$ funksiya $\{M\}$ to'plamda aniqlangan bo'lib (12.6.1) integral mavjud bo'lsin. U holda cheksiz oraliq bo'yicha olingan xosmas integralning ta'rifiga ko'ra,

$$J(y) = \int_a^{+\infty} f(x, y) dx = \lim_{t \rightarrow +\infty} \int_a^t f(x, y) dx = \lim_{t \rightarrow +\infty} F(t, y)$$

bo'ladi, bunda $F(t, y) = \int_a^t f(x, y) dx$.

Shunday qilib, $J(y)$ funksiya $F(t, y)$ funksiyaning ($y = \text{const}$) $t \rightarrow +\infty$ dagi limit funksiyasi bo'ladi.

12.7.1. Ta'rif. Agar $t \rightarrow +\infty$ da $F(t, y)$ funksiya o'z limit funksiyasi $J(y)$ ga E to'plamda tekis yaqinlashsa, ya'ni $\forall \varepsilon > 0$ olinganda ham, shunday $\delta = \delta(\varepsilon) > 0$ topilib, $\forall t > \delta$ va $\forall y \in E$ uchun

$$\left| \int_t^{+\infty} f(x, y) dx \right| < \varepsilon \quad (12.7.2)$$

tengsizlik bajarilsa, u holda $\int_a^{+\infty} f(x, y) dx$ integral E to'plamda tekis yaqinlashuvchi deyiladi.

12.7.3. Ta'rif. Agar $t \rightarrow +\infty$ da $F(t, y)$ funksiya o'z limit funksiyasi $J(y)$ ga E to'plamda notekis yaqinlashsa, ya'ni $\forall \delta > 0$ olinganda ham, shunday $\varepsilon_0 > 0, y_0 \in E$ va $t_1 > \delta$ tengsizlikni qanoatlantiruvchi $t_1 \in [a, +\infty)$ topilib,

$$\left| \int_{t_1}^{+\infty} f(x, y) dx \right| \geq \varepsilon_0 \quad (12.7.4)$$

tengsizlik bajarilsa, u holda $\int_a^b f(x, y) dx$ integral E to'plamda tekis yaqinlashuvchi deyiladi.

Birinchi tur parametrga bog'liq xosmas integralning ta'rifini quyidagicha ham berish mumkin.

12.7.5. Ta'rif. Agar

$$\lim_{A \rightarrow +\infty} \sup_{y \in E} \left| J(y) - \int_a^A f(x, y) dx \right| = \lim_{A \rightarrow +\infty} \sup_{y \in E} \left| \int_A^{+\infty} f(x, y) dx \right| = 0$$

($a < A < +\infty$) bo'lsa, u holda $J(y) = \int_a^{+\infty} f(x, y) dx$ xosmas integral E

to'plamda tekis yaqinlashuvchi deyiladi.

2) $f(x, y)$ funksiya $\{M_1\}$ to'plamda berilgan bo'lib, (12.6.3) integral mavjud bo'lsin. U holda, chegaralanmagan funksiya olingan xosmas integralning ta'rifiga ko'ra,

$$J_1(y) = \int_a^b f(x, y) dx = \lim_{t \rightarrow b-0} \int_a^t f(x, y) dx = \lim_{t \rightarrow b-0} F_1(t, y)$$

bo'ladi.

Demak $J_1(y)$ funksiya $F_1(t, y)$ funksiyaning $t \rightarrow b-0$ dagi limit funksiyasi bo'lar ekan.

12.7.6. Ta'rif. Agar $t \rightarrow b-0$ da $F_1(t, y)$ funksiya o'z limit funksiyasi $J_1(y)$ ga E to'plamda tekis yaqinlashsa, ya'ni $\forall \varepsilon > 0$ son olinganda ham, shunday y ga bog'liq bo'lmagan $\delta = \delta(\varepsilon) > 0$ topilib, $\eta \leq \delta$ bo'lganda

$$\left| \int_a^b f(x, y) dx - \int_a^{b-\eta} f(x, y) dx \right| = \left| \int_{b-\eta}^b f(x, y) dx \right| < \varepsilon$$

tengsizlik $\forall y \in E$ uchun bajarilsa, u holda $J_1(y) = \int_a^b f(x, y) dx$ integral E to'plamda *tekis yaqinlashuvchi* deyiladi.

14.1. Misol. Ushbu

$$\int_0^{+\infty} e^{-x} \arctg xy dx$$

integralning $(-\infty; +\infty)$ da y parametr bo'yicha tekis yaqinlashish xarakterini tekshiring.

Yechilishi. $\forall \varepsilon > 0$ son uchun $\exists \delta = \ln \frac{\pi}{\varepsilon} > 0$ topiladiki, $\forall t > \delta$ va

$\forall y \in R$ uchun

$$\left| \int_t^{+\infty} e^{-x} \arctg xy dx \right| \leq \frac{\pi}{2} \int_t^{+\infty} e^{-x} dx = \frac{\pi}{2} e^{-t} \leq \frac{\pi}{2} e^{-\delta} = \frac{\varepsilon}{2} < \varepsilon.$$

Demak, **12.7.1-ta'rifga** asosan, berilgan integral $(-\infty; +\infty)$ da y parametr bo'yicha tekis yaqinlashuvchi bo'ladi.

14.2. Misol. Ushbu

$$I(y) = \int_0^{\infty} \frac{dx}{(x-y)^2 + 4}$$

integralning a) $E = [0; +\infty)$; b) $E_1 = (-\infty; 0]$ to'plamlarda y parametr bo'yicha tekis yaqinlashish xarakterini tekshiring.

Yechilishi. a) Dastlab, berilgan integralni yaqinlashishga tekshiramiz. Chegarasi cheksiz xosmas integralning ta'rifiga ko'ra,

$$F(t, y) = \int_0^t \frac{dx}{(x-y)^2 + 4} = \frac{1}{2} \operatorname{arctg} \frac{x-y}{2} \Big|_0^t = \frac{1}{2} \left(\operatorname{arctg} \frac{t-y}{2} + \operatorname{arctg} \frac{y}{2} \right).$$

Bundan, har bir tayinlangan y uchun $t \rightarrow +\infty$ da $F(t, y)$ funksiyaning limitini topamiz:

$$I(y) = \lim_{t \rightarrow +\infty} F(t, y) = \lim_{t \rightarrow +\infty} \frac{1}{2} \left(\operatorname{arctg} \frac{t-y}{2} + \operatorname{arctg} \frac{y}{2} \right) = \frac{\pi}{4} + \frac{1}{2} \operatorname{arctg} y.$$

Demak, berilgan integral yaqinlashuvchi. Endi qaralayotgan integralni tekis yaqinlashishga tekshiramiz. $\varepsilon_0 = \frac{\pi}{6}$ desak, u holda $\forall \delta \in (0, +\infty)$ olinganda ham, $t_1 > \delta$ ni qanoatlantiruvchi barcha t_1 va $\exists y_0 = t_1$ uchun

$$F(t_1, y_0) = \int_{t_1}^{+\infty} \frac{dx}{(x-y_0)^2 + 4} = \frac{\pi}{4} - \operatorname{arctg} \frac{t_1 - y_0}{2} = \frac{\pi}{4} > \frac{\pi}{6} = \varepsilon_0.$$

Shunday qilib, $\forall \varepsilon_0 \in (0, \frac{\pi}{4})$ uchun (14.6) tengsizlik o'rinli.

Demak, **12.7.3**-ta'rifga ko'ra, berilgan integral $E = [0; +\infty)$ to'plamda y parametr bo'yicha notekis yaqinlashadi.

b) $\forall \varepsilon > 0$ son olinganda ham, $\delta = \delta(\varepsilon) = 2 \operatorname{ctg} 2\varepsilon$ deyilsa, u vaqtda $\forall t > \delta$ va $\forall y \in E_1$ uchun

$$\sup_{y \in E_1} |F(t, y) - I(y)| = \sup_{y \in E_1} \int_t^{+\infty} \frac{dx}{(x-y)^2 + 4} = \sup_{y \in E_1} \left\{ \frac{1}{2} \left(\frac{\pi}{2} - \operatorname{arctg} \frac{t-y}{2} \right) \right\} = \frac{1}{2} \operatorname{arctg} \frac{t}{2} < \varepsilon$$

tengsizlik bajariladi.

Demak, berilgan integral, **14.1**-ta'rifga ko'ra, E_1 da y parametr bo'yicha tekis yaqinlashuvchi bo'ladi.

12.8. Hosmas parametrغا bog'liq integrallarning tekis yaqinlashish shartlari va alomatlari. Funksiyaning limit funksiyaga ega bo'lishi va unga tekis yaqinlashishi haqidagi umumiy **12.1.5**-teoremadan foydalanib birinchi tur parametrغا bog'liq hosmas integralning tekis yaqinlashishi to'g'risidagi Koshi kriteriyasini keltiramiz.

$f(x, y)$ funksiya $\{M\} = \{(x, y) \in R^2 : a \leq x < \infty, y \in E \subset R\}$ to'plamda aniqlangan bo'lsin.

12.8.1. Teorema. $\int_a^x f(x, y) dx$ - birinchi tur parametrğa bog'liq

hosmas integralning E to'plamda y ga nisbatan tekis yaqinlashishi uchun $\forall \varepsilon > 0$ son olinganda ham, shunday y ga bog'liq bo'lmagan A_0 son mavjud bo'lib, $A' > A > A_0$ va $\forall y \in E$ uchun

$$\left| \int_a^{A'} f(x, y) dx - \int_a^A f(x, y) dx \right| = \left| \int_A^{A'} f(x, y) dx \right| < \varepsilon$$

tengsizlikning bajarilishi zarur va yetarli.

Bu teoremaning inkori uchun Koshi kriteriyasini keltiramiz:

12.8.2. Teorema. Agar $\exists \varepsilon_0 > 0$ topilib, $\forall \delta \in R$ uchun $\exists A_1, A_2, A_1 > \delta, A_2 > \delta$ va $\exists y \in E$ bo'lib

$$\left| \int_{A_1}^{A_2} f(x, y) dx \right| \geq \varepsilon_0$$

tengsizlik bajarilsa, u holda $\int_a^x f(x, y) dx$ - integral E to'plamda y ga nisbatan tekis yaqinlashmaydi.

$f(x, y)$ funksiya $\{M\} = \{(x, y) \in R^2 : a \leq x < b, y \in E \subset R\}$ to'plamda aniqlangan bo'lsin.

12.8.3. Teorema. $\int_a^b f(x, y) dx$ - parametrğa bog'liq bo'lgan ikkinchi

tur hosmas integral E to'plamda y ga nisbatan tekis yaqinlashuvchi bo'lishi uchun $\forall \varepsilon > 0$ son uchun $\exists b' \in [a, b)$ bo'lib, $\forall \xi, \xi' \in [b', b]$ va $\exists y \in E$ uchun

$$\left| \int_{\xi}^{\xi'} f(x, y) dx \right| < \varepsilon$$

tengsizlikning bajarilishi zarur va yetarli.

12.8.4. Teorema. Agar $\exists \varepsilon_0 > 0$, $\forall b' \in [a, b)$ $\exists \xi_0, \xi'_0 \in [b', b]$ va $\exists y_0 \in E$ uchun

$$\left| \int_{\xi_0}^{\xi'_0} f(x, y_0) dx \right| \geq \varepsilon_0$$

tengsizlik bajarilsa, u holda $\int_a^b f(x, y) dx$ - integral E to'plamda y ga nisbatan tekis yaqinlashmaydi.

14.5. Misol. Ushbu

$$I(y) = \int_0^{+\infty} \frac{\sin yx}{x} dx$$

integralning a) $E = [a; 1] (a > 0)$; b) $E_1 = [0; 1]$ to'plamlarda y parametr bo'yicha tekis yaqinlashish harakterini Koshi teoremasidan foydalanib tekshiring.

a) $\forall \varepsilon > 0$ son, olinganda ham, $\delta = \delta(\varepsilon) = \frac{3}{a\pi\varepsilon}$ deyilsa, u holda

$\forall t' = n\pi > \delta, t'' = 2n\pi > \delta (n \in \mathbb{N})$ va $\forall y \in E$ uchun

$$\begin{aligned} \left| \int_{t'}^{t''} \frac{\sin yx}{x} dx \right| &= \left| -\frac{1}{x} \frac{\cos xy}{y} \right|_{t'}^{t''} - \frac{1}{y} \int_{t'}^{t''} \frac{\cos xy}{x^2} dx = \\ &= \left| \frac{1}{t'} \frac{\cos yt'}{y} - \frac{1}{t''} \frac{\cos yt''}{y} - \frac{1}{y} \int_{t'}^{t''} \frac{\cos xy}{x^2} dx \right| \leq \\ &\leq \left| \frac{1}{t'y} \right| + \left| \frac{1}{t''y} \right| + \frac{1}{y} \left| \int_{t'}^{t''} \frac{|\cos xy|}{x^2} dx \right| \leq \frac{1}{at'} + \frac{1}{at''} + \frac{1}{a} \left| \int_{t'}^{t''} \frac{dx}{x^2} \right| < \frac{2}{at'} + \frac{2}{at''} < \varepsilon \end{aligned}$$

tengsizlik bajariladi.

Demak, berilgan parametrga bog'liq bo'lgan xosmas integral Koshi teoremasiga asosan, E to'plamda tekis yaqinlashuvchi bo'ladi.

b) $\forall \delta > 0$ son uchun $y_\delta = \delta, t'_\delta = \frac{\pi}{3\delta}, t''_\delta = \frac{\pi}{2\delta}$ deb tanlasak, u holda

$$\left| \int_{t'_\delta}^{t''_\delta} \frac{\sin y_\delta x}{x} dx \right| = \left| \int_{\frac{\pi}{3\delta}}^{\frac{\pi}{2\delta}} \frac{\sin \delta x}{x} dx \right| = \left| \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sin t}{t} dt \right| = \varepsilon_0,$$

bo'ladi, bunda $\varepsilon_0 > 0$.

Demak, berilgan integral, yuqoridagi natijaga asosan, $y \in E_1$ da notekis yaqinlashadi.

12.9. Tekis yaqinlashish alomatlar

12.9.1. Ta'rif. $\int_a^{+\infty} \varphi(x) dx$ integral yaqinlashuvchi bo'lsa va $x > a$ va $\forall y \in E$ lar uchun $|f(x, y)| \leq \varphi(x)$ tengsizlik o'rinli bo'lsa, u holda $\varphi(x)$ $\{M\}$ to'plamda $f(x, y)$ funksiya uchun *majorant funksiya* deyiladi.

12.9.2. Teorema (Veyershtrass alomati). Agar $f(x, y)$ funksiya $\{M\}$ to'plamda $\varphi(x)$ -majorant funksiyaga ega bo'lsa, u holda $J(y) = \int_a^{+\infty} f(x, y) dx$ integral E to'plamda y ga nisbatan tekis yaqinlashuvchi bo'ladi.

Isboti. Shartga ko'ra $f(x, y)$ funksiya $\{M\}$ to'plamda majorant $\varphi(x)$ funksiyaga ega bo'lgani uchun $\int_a^{+\infty} \varphi(x) dx$ integral yaqinlashuvchi. U holda, $\forall \varepsilon > 0$ son olinganda ham, shunday $A_0(\varepsilon) > a$ son topiladiki $A' > A > A_0$ bo'lganda

$$\int_A^{A'} \varphi(x) dx < \varepsilon$$

tengsizlik bajariladi. $\forall y \in E$ uchun

$$\left| \int_A^{A'} f(x, y) dx \right| \leq \int_A^{A'} \varphi(x) dx < \varepsilon$$

bo'ladi.

Demak, Koshi kriteriyasiga asosan, $\int_a^{+\infty} f(x, y) dx$ - integral E to'plamda y ga nisbatan tekis yaqinlashadi.

12.9.3. Misol. $\int_0^{+\infty} \frac{\cos xy}{1+x^2} dx$ integralni $E = [0, +\infty)$ da tekis yaqinlashishga tekshiring.

Yechilishi. Integral ostidagi funksiyani bohalaymiz:

$$f(x, y) = \frac{\cos xy}{1+x^2}, |f(x, y)| \leq \frac{1}{1+x^2} = \varphi(x)$$

Demak, $\frac{\cos xy}{1+x^2}$ funksiya uchun $\frac{1}{1+x^2}$ funksiya majorant bo'ladi.

Majorant funksiya dan olingan $\int_0^{+\infty} \frac{dx}{1+x^2}$ integral yaqinlashuvchi, ya'ni

$$\int_0^{+\infty} \frac{dx}{1+x^2} = \arctg x \Big|_0^{+\infty} = \frac{\pi}{2} - \text{yaqinlashuvchi.}$$

Shunday qilib, berilgan integral Veyershrass alomatiga ko'ra, $E = [0, +\infty)$ to'plamda tekis yaqinlashuvchi.

12.9.4. Teorema (Abel alomati). $f(x, y)$ va $g(x, y)$ funksiyalar $\{M_0\} = \{(x, y) \in R^2 : a \leq x < +\infty, y \in E = [c, d]\}$ to'plamda aniqlangan bo'lib, ular quyidagi shartlarni qanoatlantirsin:

1) Har bir berilgan $y \in E$ da $g(x, y)$ funksiya $[a, +\infty)$ da monoton bo'lsin;

2) $\forall (x, y) \in \{M\}$ uchun $|g(x, y)| \leq c (c = \text{const})$ bo'lsin;

3) $\int_a^{+\infty} f(x, y) dx$ - integral E to'plamda y ga nisbatan tekis yaqinlashuvchi bo'lsin. U holda $\int_a^{+\infty} f(x, y) g(x, y) dx$ integral E to'plamda y ga nisbatan tekis yaqinlashuvchi bo'ladi.

12.9.5. Teorema (Dirixle alomati). $f(x, y)$ va $g(x, y)$ funksiyalar $\{M_0\}$ to'plamda aniqlangan bo'lib, ular quyidagi shartlarni qanoatlantirsin:

1) $\forall A > a, \forall y \in E$ da

$$\left| \int_a^A f(x, y) dx \right| \leq c (c = \text{const})$$

tengsizlik o'rinli bo'lsin;

2) Har bir belgilangan $y \in E$ da $g(x, y)$ funksiya limit funksiya $\varphi(x) = 0$ ga tekis ya'ni $x \rightarrow +\infty$ da $g(x, y) \xrightarrow{E} 0$ yaqinlashsin. U holda

$$\int_a^{+\infty} f(x, y) g(x, y) dx$$

integral E to'plamda y ga nisbatan tekis yaqinlashuvchi bo'ladi.

Isboti. Teoremlarni isbot qilishda Koshi kriteriyasidan foydalanamiz: $\int_{A_1}^{A_2} f(x,y)g(x,y)dx$ -integralga ikkinchi o'rta qiymat haqidagi teoremani qo'llab

$$\int_{A_1}^{A_2} f(x,y)g(x,y)dx = g(A_1,y) \int_{A_1}^{A_2} f(x,y)dx + g(A_2,y) \int_{A_1}^{A_2} f(x,y)dx \quad (12.9.6)$$

tengsizlikni hosil qilamiz, bunda $A_3, [A_1, A_2]$ dagi ixtiyoriy nuqta. Abel

alomatida $\int_a^{+\infty} f(x,y)dx$ integral tekis yaqinlashuvchi bo'lgani uchun

$\forall \varepsilon > 0$ son olinganda ham, shunday y ga bog'liq bo'lmagan $A_0(\varepsilon) > a$ son mavjud bo'lib, istalgancha kata $A_2 > A_1 > A_0$ lar uchun

$$\left| \int_{A_1}^{A_2} g(x,y)dx \right| < \frac{\varepsilon}{2} \quad \text{va} \quad \left| \int_{A_1}^{A_2} f(x,y)dx \right| < \frac{\varepsilon}{2}$$

tengsizliklar bajariladi. Bundan

$$\begin{aligned} \left| \int_{A_1}^{A_2} f(x,y)g(x,y)dx \right| &\leq \left| g(A_1,y) \int_{A_1}^{A_2} f(x,y)dx \right| + \left| g(A_2,y) \int_{A_1}^{A_2} f(x,y)dx \right| \leq \\ &\leq C \cdot \varepsilon + C \cdot \varepsilon = 2C \cdot \varepsilon, \end{aligned}$$

bunda $|g(x,y)| \leq c$.

Demak, Koshi kriteriyasiga asosan $\int_a^{+\infty} f(x,y)g(x,y)dx$ integral E to'plamda y ga nisbatan tekis yaqinlashuvchi bo'ladi. Bu bilan Abel alomati isbot bo'ladi.

$$\text{Dirixle alomatida } \left| \int_a^A f(x,y)dx \right| \leq C, \text{ hamda } x \rightarrow +\infty \text{ da } g(x,y) \xrightarrow{\varepsilon} 0$$

bo'lgani sababli $\forall \varepsilon > 0$ son olinganda ham, shunday y ga bog'liq bo'lmagan $A_0(\varepsilon)$ son topilib, istalgancha kata $A_2 > A_1 > A_0$ lar uchun

$|g(x,y)| < \frac{\varepsilon}{2}$ tengsizlik bajariladi. Bu tengsizliklarni e'tiborga olib (12.9.6) tenglikdan

$$\left| \int_{A_1}^{A_2} f(x, y) g(x, y) dx \right| \leq C \cdot \varepsilon + C \cdot \varepsilon = 2C \cdot \varepsilon$$

bundan Dirixle alomatining tasdig'i kelib chiqadi. ■

12.9.6. Misol.

$$\int_1^{\infty} \frac{\arctg yx}{x^{\frac{3}{2}}} e^{-xy} dx \quad (y \in E = [a; +\infty)) \quad (a > 0)$$

integralni tekis yaqinlashishga tekshiring.

Yechilishi. $f(x, y) = \frac{\arctg yx}{x^{\frac{3}{2}}}$, $g(x, y) = e^{-xy}$ deb olib, Abel alomati

shartining bajarilishini tekshiramiz. Ushbu $\int_1^x \frac{\arctg yx}{x^{\frac{3}{2}}} dx$ integral

Veyershtass alomatiga ko'ra, tekis yaqinlashuvchi, $g(x, y) = e^{-xy}$ esa, y ning $E = [a; +\infty)$ dan olingan har bir tayin qiymatida x ning kamayuvchi funksiyasi va $\forall x \in [1; +\infty), \forall y \in E = [a; +\infty)$ uchun $|g(x, y)| = e^{-xy} \leq 1$ bo'ladi.

Demak, berilgan integral Abel alomatiga asosan, $[a; +\infty)$ da tekis yaqinlashuvchi.

12.9.7. Misol. Ushbu

$$\int_2^{+\infty} \frac{\cos xy \cdot \ln x}{\sqrt{x}} dx \quad (y \in E = [a; +\infty), a > 0)$$

integralni tekis yaqinlashishga tekshiring.

Yechilishi. Agar $f(x, y) = \cos xy$, $g(x, y) = \frac{\ln x}{\sqrt{x}}$ deyilsa, u holda

$\forall t > 0, \forall y \in E$ uchun

$$\left| \int_2^t f(x, y) dx \right| = \left| \int_2^t \cos xy dx \right| = \left| \frac{\sin ty - \sin 2y}{y} \right| \leq 2$$

bo'ladi. $x \rightarrow +\infty$ da $g(x, y) = \frac{\ln x}{\sqrt{x}}$ funksiya E da nolga tekis yaqinlashadi,

ya'ni $g(x, y) = \frac{\ln x}{\sqrt{x}} \xrightarrow{E} 0$.

Demak, berilgan integral Dirixle alomatiga asosan, $E = [a; +\infty)$ da tekis yaqinlashuvchi.

12.10. Parametrga bog'liq xosmas integrallarining funksional xossalari. Biz bu bandda ushbu

$$J(y) = \int_a^{+\infty} f(x, y) dx \quad (12.10.1)$$

parametrga bog'liq integralning funksional xossalarini, ya'ni limiti, uzluksizligi, differensiallanishi va integrallanishi haqidagi masalalarni qaraymiz.

$f(x, y)$ funksiya $\{M\} = \{(x, y) \in R^2 : a \leq x < +\infty, y \in E \subset R\}$ to'plamda aniqlangan bo'lib, $y_0 \in R$ esa, E to'plamning limit nuqtasi bo'lsin.

12.10.2. Teorema (integral ostida limitga o'tish). $f(x, y)$ funksiya $\{M\}$ to'plamda aniqlangan bo'lib, u quyidagi shartlarni qanoatlantirsin:

- 1) x bo'yicha $[a, +\infty)$ da uzluksiz;
- 2) $y \rightarrow y_0$ da ixtiyoriy chekli $[a, A] (a < A < \infty)$ oraliqda limit funksiya $\varphi(x)$ ga tekis yaqinlashsin;
- 3) (12.10.1) integral E to'plamda y ga nisbatan tekis yaqinlashuvchi bo'lsin. U holda

$$\lim_{y \rightarrow y_0} J(y) = \lim_{y \rightarrow y_0} \int_a^{+\infty} f(x, y) dx = \int_a^{+\infty} \varphi(x) dx \quad (12.10.3)$$

bo'ladi.

Isboti. Teoremaning 1) va 2) shartlari bajarilishidan 12.1.9–teorema asosan, $\varphi(x)$ funksiya $[a, +\infty)$ da uzluksiz bo'ladi. U holda, u ixtiyoriy $[a, A] (a < A < \infty)$ oraliqda integrallanuvchi bo'ladi. Teoremaning 3) shartiga asosan, (12.10.1) integral E to'plamda tekis yaqinlashuvchi bo'lgani uchun parametrga bog'liq xosmas integralning yaqinlashish (12.8.1–teorema) shartiga asosan $\forall \varepsilon > 0$ son uchun shunday y ga bog'liq bo'lmagan $A_0 > a$ son mavjud bo'lib, $A' > A > A_0$ va $\forall y \in E$ uchun

$$\left| \int_{A'}^{A'} f(x, y) dx \right| < \varepsilon$$

tengsizlik o'rinli bo'ladi. Keyingi tengsizlikda $y \rightarrow y_0$ da (12.2.1-teoremaga q.) limitga o'tib,

$$\left| \int_A^{A'} \varphi(x) dx \right| \leq \varepsilon$$

tengsizlikga ega bo'lamiz. Bundan $\varphi(x)$ funksiyaning $[a, +\infty)$ oraliqda (birinchi tur xos integral yaqinlashish sharti) integrallanuvchi ekanligi kelib chiqadi.

$A > a$ uchun ushbu

$$\left| \int_a^{+\infty} f(x, y) dx - \int_a^{+\infty} \varphi(x) dx \right|$$

ayirmani qaraymiz:

$$\begin{aligned} \left| \int_a^{+\infty} f(x, y) dx - \int_a^{+\infty} \varphi(x) dx \right| &\leq \int_a^A |f(x, y) - \varphi(x)| dx + \left| \int_A^{+\infty} f(x, y) dx \right| + \\ &+ \left| \int_A^{+\infty} \varphi(x) dx \right|, \quad (a < A < \infty) \end{aligned} \quad (12.10.4)$$

(12.10.4) tengsizlikning o'ng tomonidagi qo'shiluvchilarni ayrim-ayrim baholaymiz. (12.10.1) integral E to'plamda tekis yaqinlashuvchi bo'lgani uchun $\forall \varepsilon > 0$ son olinganda ham, $\exists \delta_1 = \delta_1(\varepsilon) > 0, \forall A > \delta_1, \forall y \in E$:

$$\left| \int_A^{+\infty} f(x, y) dx \right| < \frac{\varepsilon}{3} \quad (12.10.5)$$

bo'ladi. $\int_a^{+\infty} \varphi(x) dx$ integral yaqinlashuvchi bo'lgani

uchun $\forall \varepsilon > 0, \exists \delta_2 = \delta_2(\varepsilon) > 0, \forall A > \delta_2$:

$$\left| \int_a^{+\infty} \varphi(x) dx \right| < \frac{\varepsilon}{3} \quad (12.10.6)$$

tengsizlik o'rinli bo'ladi. Ravshanki, $\forall A > \delta_0 : (\delta_0 = \max(\delta_1, \delta_2))$ da (12.10.5) bilan (12.10.6) birdaniga bajariladi. $f(x, y)$ funksiya $y \rightarrow y_0$ da $[a, A]$ kesmada ($A > \delta_0$) limit funksiya $\varphi(x)$ ga tekis yaqinlashuvchi bo'lgani uchun $\forall \varepsilon > 0, \exists \delta_3 = \delta_3(\varepsilon) > 0, |y - y_0| < \delta_3, \forall y \in E, \forall x \in [a, A] (a < A < \infty)$

$$|f(x, y) - \varphi(x)| < \frac{\varepsilon}{3(A-a)} \quad (12.10.7)$$

tengsizlik bajariladi. (12.10.4)-(10.10.7) tengsizliklardan

$$\left| \int_a^{+\infty} f(x, y) dx - \int_a^{+\infty} \varphi(x) dx \right| < \varepsilon$$

tengsizlik kelib chiqadi. Bundan esa,

$$\lim_{y \rightarrow y_0} J(y) = \lim_{y \rightarrow y_0} \int_a^{+\infty} f(x, y) dx = \int_a^{+\infty} \varphi(x) dx.$$

Bu tenglikni ushbu

$$\lim_{y \rightarrow y_0} \int_a^{+\infty} f(x, y) dx = \int_a^{+\infty} \left[\lim_{y \rightarrow y_0} f(x, y) \right] dx$$

ko'rinishda ham yozish mumkin. ■

12.10.9. Teorema ($J(y) = \int_a^{+\infty} f(x, y) dx$ *funksiyaning uzluksizligi*).

Agar $f(x, y)$ funksiya $\{M_0\}$ da uzluksiz bo'lib, (12.10.1) integral $[c, d]$ kesmada tekis yaqinlashuvchi bo'lsa, u holda, $J(y)$ funksiya $[c, d]$ kesmada uzluksiz bo'ladi.

Isboti. Shartga ko'ra, $f(x, y)$ funksiya $\{M_0\}$ to'plamda uzluksiz bo'lgani uchun u $\{M_A\} = \{(x, y) \in R^2 : a \leq x \leq A, c \leq y \leq d\} (a < A < \infty)$ to'plamda tekis uzluksiz bo'ladi. U holda, tekis uzluksiz ta'rifiga ko'ra, $\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0, |x'' - x'| < \delta, |y'' - y'| < \delta$ tengsizliklarni qanoatlantiruvchi x va y lar uchun $|f(x'', y'') - f(x', y')| < \varepsilon$ tenglik bajariladi. Bundan $x' = x'' = x, y' = y_0, y'' = y$ deb olsak,

$$|f(x, y) - f(x, y_0)| < \varepsilon$$

tengsizlik bajariladi. Bundan esa, $y \rightarrow y_0$ da $f(x, y) \xrightarrow{[a, A]} f(x, y_0)$ ekanligi, ya'ni $f(x, y)$ funksiya $y \rightarrow y_0$ da $[a, A] (a < A < \infty)$ oraliqda limit funksiya $f(x, y_0)$ ga tekis yaqinlashishi kelib chiqadi.

Bundan tashqari teoremaning shartiga ko'ra (12.10.1) integral $[c, d]$ da tekis yaqinlashuvchi. Shunday qilib berilgan $f(x, y)$ funksiya

12.10.2-teoremaning hamma shartini qanoatlantirar ekan. Shuning uchun 12.10.2-teorema asosan,

$$\lim_{y \rightarrow y_0} J(y) = \lim_{y \rightarrow y_0} \int_a^{+\infty} f(x, y) dx = \int_a^{+\infty} \left[\lim_{y \rightarrow y_0} f(x, y) \right] dx = \int_a^{+\infty} f(x, y_0) dx = J(y_0).$$

Demak, $J(y) = \int_a^{+\infty} f(x, y) dx$ funksiya $\forall y_0 \in [c, d]$ nuqtada

uzluksiz. ■

12.10.10. Teorema (parametr bo'yicha integrallash). 12.10.9-teoremaning hamma shartlari bajarilganda quyidagi formula o'rinli bo'ladi:

$$\int_c^d J(y) dy = \int_c^d dy \int_a^{+\infty} f(x, y) dx = \int_a^{+\infty} dx \int_c^d f(x, y) dy \quad (12.10.11)$$

Isboti. $A > 0$ uchun

$$\int_c^d J(y) dy = \int_c^d dy \int_c^A f(x, y) dx + \int_c^d dy \int_A^{+\infty} f(x, y) dx \quad (12.10.12)$$

12.4.1-teorema asosan,

$$\int_c^d dy \int_c^A f(x, y) dx = \int_c^A dx \int_c^d f(x, y) dy \quad (12.10.13)$$

tenglik o'rinli bo'ladi. U holda (12.10.13) bilan (12.10.12) dan

$$\int_c^d J(y) dy - \int_c^A dx \int_c^d f(x, y) dy = \int_c^d dy \int_A^{+\infty} f(x, y) dx$$

tenglikni hosil qilamiz. $\forall \varepsilon > 0$ son olinganda, A ni shunday katta qilib olish mumkinki, natijada hamma y lar uchun

$$\left| \int_A^{+\infty} f(x, y) dx \right| < \varepsilon$$

bo'ladi. Keyingi tenglikdan

$$\left| \int_c^d J(y) dy - \int_c^A dx \int_c^d f(x, y) dy \right| < \varepsilon(d - c).$$

Bundan,

$$\int_c^d J(y) dy = \lim_{A \rightarrow +\infty} \int_c^A dx \int_c^d f(x, y) dy.$$

Bu tenglik (12.10.11) tenglikga teng kuchli. ■

12.10.14. Teorema (integral ostida differensiallash). $f(x, y)$

funksiya quyidagi shartlarni qanoatlantirsin:

- 1) $\{M_0\}$ to'plamda uzluksiz;
- 2) $\{M_0\}$ to'plamda $f'_y(x, y)$ xususiy hosila mavjud va u ikki argumentli funksiya sifatida shu to'plamda uzluksiz;
- 3) har bir tayinlangan $y \in E = [c, d]$ da

$$J(y) = \int_a^{+x} f(x, y) dx$$

integral yaqinlashuvchi;

$$4) \int_a^{+x} f'_y(x, y) dx \text{ integral } [c, d] \text{ da tekis yaqinlashuvchi bo'lsin. U}$$

holda $J(y)$ funksiya $[c, d]$ da $J'(y)$ hosilaga ega va

$$J'(y) = \int_a^{+x} f'_y(x, y) dx$$

bo'ladi.

Isboti. $f'_y(x, y)$ funksiya **12.10.10**-teoremaning hamma shartlarini qanoatlantiradi. Shuning uchun **12.10.10**-teoremda d ni $\forall y (c \leq y \leq d)$ ga almashtirib topamiz:

$$\int_c^y dy \int_a^{+x} f'_y(x, y) dx = \int_a^{+x} dx \int_c^y f'_y(x, y) dy = \int_a^{+x} f(x, y) dx - \int_a^{+x} f(x, c) dx.$$

Tenglikning chap tomonidagi integral ostidagi funksiya, ya'ni

$$\int_a^{+x} f'_y(x, y) dx \text{ integral } \mathbf{12.10.9}\text{-teoremaga ko'ra, } y \text{ ning uzluksiz}$$

funksiyasi bo'ladi. Shuning uchun tenglikning chap tomonining hosilasi

$$\int_a^{+x} f'_y(x, y) dx \text{ ga teng. U holda, o'ng tomonining hosilasi ham mavjud}$$

va y

$$\frac{d}{dy} J(y) = J'(y) = \frac{d}{dy} \int_a^{+x} f(x, y) dx.$$

$$\text{Demak, } J'(y) = \int_a^{+x} f'_y(x, y) dx \text{ bo'ladi. } \blacksquare$$

12.10.15. Eslatma. Biz yuqorida

$$J(y) = \int_a^{+\infty} f(x, y) dx$$

integralning funksional xossalarini o'rgandik (12.10.2, 12.10.9, 12.10.10, 12.10.14- teoremlar). Xuddi shunday teoremlar

$$J(y) = \int_a^b f(x, y) dx$$

xosmas parametrga bog'liq integral uchun ham o'rinli bo'ladi.

12.10.16. Misol. Ushbu

$$\lim_{y \rightarrow 1} \int_1^{+\infty} \arctg \frac{2y}{y^2 + x^3} dx = \int_1^{+\infty} \arctg \frac{2}{1+x^3} dx$$

tenglikning o'rinli ekanligini ko'rsating.

Yechilishi. 12.10.2-teoremadan foydalanib, yuqoridagi tenglikning to'g'ri ekanligini ko'rsatamiz:

1) integral ostidagi $f(x, y) = \arctg \frac{2y}{y^2 + x^3}$ funksiya $[1; +\infty)$ dan olingan har bir tayinlangan y uchun $[1; +\infty)$ da x ning funksiyasi sifatida uzluksiz:

$$2) y \rightarrow 1 \text{ da } \forall [1; t] \text{ (} 1 < t < \infty \text{) da } f(x, y) \xrightarrow{\rightarrow} \arctg \frac{2}{1+x^3}.$$

Haqiqatan ham, $\forall \varepsilon > 0$ son olinganda, $\delta = \varepsilon$ deb olinsa, $\forall x \in [1; +\infty)$ va $\forall y \in [1; +\infty)$ uchun

$$\left| \arctg \frac{2y}{y^2 + x^3} - \arctg \frac{2}{1+x^3} \right| \leq \left| \frac{2y}{y^2 + x^3} - \frac{2}{1+x^3} \right| \leq \frac{2|1-y|(x^3 + y^2)}{(y^2 + x^3)(1+x^3)} < \delta = \varepsilon$$

tengsizlik bajariladi;

$$3) \int_1^{+\infty} \arctg \frac{2y}{y^2 + x^3} dx \text{ integral } E = [1; +\infty) \text{ to'plamda } y \text{ parametr}$$

bo'yicha tekis yaqinlashuvchi.

Shunday qilib, 12.10.1-teoremaga ko'ra, yuqoridagi tenglik o'rinli.

12.10.17. Misol. Ushbu

$$F(\alpha) = \int_0^1 \frac{\sin \frac{\alpha}{x}}{x^\alpha} dx$$

funksiyaning $-\infty < \alpha < 2$ oraliqda uzlyksizligini ko'rsating.

Yechilishi. Integral ostida $x = \frac{1}{t}$ ($t > 0$) almashtirishni bajarib,

$$F(\alpha) = \int_1^{+\infty} \frac{\sin \alpha t}{t^{2-\alpha}} dt$$

ni hosil qilamiz. $-\infty < \alpha < \frac{1}{2}$ bo'lsin. U holda $\forall t \in [1, +\infty)$ uchun

$$\left| \frac{\sin \alpha t}{t^{2-\alpha}} \right| \leq \frac{1}{\sqrt{t^3}} \text{ tengsizlik o'rinli va } \int_1^{+\infty} \frac{dt}{\sqrt{t^3}} \text{ integral yaqinlashuvchi.}$$

Demak, Veyershtass alomatiga ko'ra,

$$\int_1^{+\infty} \frac{\sin \alpha t}{t^{2-\alpha}} dt$$

integral $-\infty < \alpha < \frac{1}{2}$ da tekis yaqinlashuvchi.

$$f(x, \alpha) = \frac{\sin \alpha x}{x^{2-\alpha}} \text{ funksiya } t \geq 1 \text{ bo'lganda uzluksiz. 12.10.9-}$$

teoremaga asosan, $F(\alpha)$ funksiya $(-\infty; \frac{1}{2}]$ da uzluksiz bo'ladi.

$$\text{Endi } \frac{1}{2} \leq \alpha \leq 2 - \varepsilon \text{ bo'lsin, } \varepsilon > 0. \text{ U holda } \left| \int_1^x \sin \alpha t dt \right| \leq \frac{2}{\alpha} \leq 4. \frac{1}{t^{2-\alpha}}$$

funksiya α ning belgilangan qiymatida $t \rightarrow +\infty$ da monoton ravishda nolga intiladi, chunki $\frac{1}{t^{2-\alpha}} \leq \frac{1}{t^\varepsilon}$ tengsizlik o'rinli.

Demak, Dirixle alomatiga asosan, berilgan integral tekis yaqinlashuvchi. Integral ostidagi funksiya $\forall t \geq 1$ va $\frac{1}{2} \leq \alpha \leq 2 - \varepsilon$ da uzluksiz. Bularni e'tiborga olsak, $F(\alpha)$ funksiyaning $\frac{1}{2} \leq \alpha \leq 2 - \varepsilon$ oraliqda uzluksiz ekanligi kelib chiqadi.

Shunday qilib, $F(\alpha)$ funksiya $-\infty < \alpha \leq 2 - \varepsilon$ oraliqda uzluksiz ekan. $\varepsilon > 0$ sonning ixtiyoriyligini hisobga olsak, bundan $F(\alpha)$ funksiyaning $-\infty < \alpha \leq 2$ oraliqda uzluksizligi kelib chiqadi.

12.10.18. Misol. Ushbu

$$I(\alpha, \beta) = \int_0^{+\infty} \frac{1 - \cos \alpha x}{x} e^{-\beta x} dx, \quad \beta > 0$$

integralni parametr bo'yicha differensiallash teoremasidan foydalanib, hisoblang.

Yechilishi. $\frac{1 - \cos \alpha x}{x}$ funksiyaning $x = 0$ nuqtadagi qiymatini 0 ga teng deb qabul qilsak, $f(x, \alpha) = \frac{1 - \cos \alpha x}{x} e^{-\beta x}$, $f'_\alpha(x, \alpha) = \sin \alpha x e^{-\beta x}$ funksiyalar $M = \{(x; \alpha) \in \mathbb{R}^2 : 0 \leq x < +\infty, \alpha \in \mathbb{R}\}$ to'plamda uzluksiz bo'ladi.

Berilgan integralni yaqinlashishga tekshiramiz:

$$\int_0^{+\infty} \frac{1 - \cos \alpha x}{x} e^{-\beta x} dx = 2 \int_0^1 \sin^2 \frac{\alpha x}{2} \frac{e^{-\beta x}}{x} dx + 2 \int_1^{+\infty} \sin^2 \frac{\alpha x}{2} \frac{e^{-\beta x}}{x} dx = I_1 + I_2$$

$$x \rightarrow +0 \text{ da } \frac{2 \sin^2 \frac{\alpha x}{2} \cdot e^{-\beta x}}{x} = O\left(\frac{\alpha^2 x}{4}\right).$$

$$x \rightarrow +\infty \text{ da esa, } 2 \sin^2 \frac{\alpha x}{2} \cdot \frac{e^{-\beta x}}{x} = o\left(\frac{1}{x^2}\right) \quad (\beta > 1)$$

bo'lganligi sababli, taqqoslash teoremasiga asosan, I_1 va I_2 integrallar yaqinlashuvchi bo'ladi.

Endi, $\int_0^{+\infty} \sin \alpha x e^{-\beta x} dx$ integralni $\alpha \in \mathbb{R}$ da tekis yaqinlashishga tekshiramiz: $\forall \beta > 0$ uchun

$$\left| e^{-\beta x} \sin \alpha x \right| \leq e^{-\beta x}, \quad \int_0^{+\infty} e^{-\beta x} dx$$

integral yaqinlashuvchi bo'lganligidan, Veyershtass teoremasiga ko'ra,

$$\int_0^{+\infty} e^{-\beta x} \sin \alpha x dx \quad \text{integral } \alpha \in \mathbb{R} \text{ to'plamda tekis yaqinlashuvchi.}$$

Demak, **12.10.4-teoremaga** asosan,

$$J'_\alpha(\alpha, \beta) = \int_0^\pi e^{-\beta x} \sin \alpha x dx$$

bo'ladi. Bu integralni ikki marta bo'laklab integralash natijasida, ushbu

$$J'_\alpha(\alpha, \beta) = \frac{\alpha}{\alpha^2 + \beta^2}$$

tenglikka ega bo'lamiz. Bu yerdan

$$J(\alpha, \beta) = \frac{1}{2} \ln |\alpha^2 + \beta^2| + C(\beta), \quad J(0, \beta) = 0$$

bo'lganligi uchun $C(\beta) = -\ln \beta$ bo'lib, $J(\alpha, \beta) = \frac{1}{2} \ln \left(1 + \frac{\alpha^2}{\beta^2} \right)$ bo'ladi.

56-§. Eyler integrallari

12.11. Beta funksiya. Ushbu

$$\int_0^1 x^{a-1} (1-x)^{b-1} dx \quad (12.11.1)$$

xosmas integralni qaraymiz.

Bunda, $a < 1$ bo'lganda, 0 nuqta maxsus nuqta, $b < 1$ bo'lganda esa, 1 -maxsus nuqta bo'ladi. Berilgan integralni ushbu

$$\int_0^1 x^{a-1} (1-x)^{b-1} dx = \int_0^{\frac{1}{2}} x^{a-1} (1-x)^{b-1} dx + \int_{\frac{1}{2}}^1 x^{a-1} (1-x)^{b-1} dx$$

ko'rinishda tasvirlab olamiz. U holda, birinchi integral $1-a < 1$, ya'ni $a > 0$ bo'lganda yaqinlashuvchi, ikkinchi integral esa, $1-b < 1$, ya'ni $b > 0$ da yaqinlashuvchi. Shunday qilib, (12.11.1) integral $a > 0$ va $b > 0$ bo'lganda yaqinlashuvchi. Odatda, (12.11.1) integralni «Beta» funksiya (yoki 1-tur Eyler integrali) deyiladi va u

$$B(a, b) = \int_0^1 x^{a-1} (1-x)^{b-1} dx \quad (12.11.2)$$

kabi belgilanadi. Beta funksiyaning quyidagi xossalarini keltirib chiqaramiz.

1° . $t = 1-x$ almashtirish natijasida $B(a, b) = B(b, a)$ ekanligiga, ya'ni $B(a, b)$ funksiyaning simmetirlikligiga ishonch hosil qilish qiyin emas.

2^o. (12.11.2) formulani ($b > 1$) bo'laklab integrallaymiz:

$$B(a, b) = \int_0^1 (1-x)^{b-1} d\left(\frac{x^a}{a}\right) =$$

$$\frac{x^a}{a} \cdot (1-x)^{b-1} \Big|_0^1 + \frac{b-1}{a} \int_0^1 x^a (1-x)^{b-2} dx = \frac{b-1}{a} \int_0^1 x^a (1-x)^{b-2} dx.$$

Bundan, ushbu

$$x^a (1-x)^{b-2} = x^{a-1} (1-x)^{b-2} - x^{a-1} (1-x)^{b-1}$$

ayniyatdan foydalanib

$$B(a, b) = \frac{b-1}{a} \int_0^1 x^{a-1} (1-x)^{b-2} dx - \frac{b-1}{a} \int_0^1 x^{a-1} (1-x)^{b-1} dx =$$

$$= \frac{b-1}{a} B(a, b-1) - \frac{b-1}{a} B(a, b),$$

$$B(a, b) = \frac{b-1}{a+b-1} B(a, b-1) \quad (12.11.3)$$

formulaga ega bo'lamiz.

Bitta funksiyaning simmetirlik xossasiga asosan (12.12.3) ni

$$B(a, b) = \frac{a-1}{a+b-1} B(a-1, b) \quad (a > 1) \quad (12.11.4)$$

ko'rinishda ham yozish mumkin.

Agar xususiyl holda $b = n (n \in \mathbb{N})$ bo'lsa, u holda (12.11.4) formulani ketma-ket qo'llash natijasida

$$B(a, n) = \frac{n-1}{a+n-1} \cdot \frac{n-2}{a+n-2} \cdots \frac{1}{a+1} B(a, 1)$$

topamiz. Bunda $B(a, 1) = \int_0^1 x^{a-1} dx = \frac{1}{a}$. $B(a, b)$ funksiyaning

simmetirlikligidan

$$B(n, a) = B(a, n) = \frac{1 \cdot 2 \cdot 3 \cdots (n-1)}{a(a+1)(a+2) \cdots (a+n-1)}. \quad (12.11.5)$$

12.11.6. Natija. $B(a, b)$ funksiya $b = m \in \mathbb{N}, a = n \in \mathbb{N}$ bo'lganda, (12.11.3) va (12.11.4) formulalarni ketma-ket qo'llash natijasida

$$B(m, n) = \frac{(m-1)!(n-1)!}{(m+n-1)!}. \quad (12.11.7)$$

Bu formulani $m=1$ yoki $n=1$ bo'lganda qo'llash mumkin ($0 \neq 1$ deb). Bundan xususiy holda, $m=1$, $n=1$, ($0 \neq 1$) bo'lganda $B(1,1)=1$ bo'ladi.

3^o. (12.11.2) formulada $x = \frac{y^a}{1+y}$ almashtirishni bajarish

natijasida $B(a,b)$ funksiyaning ushbu

$$B(a,b) = \int_0^{\infty} \frac{y^{a-1}}{(1+y)^{a+b-1}} dy \quad (12.11.8)$$

ko'rinishini topamiz. Bk yerda $b=1-a$ ($0 < a < 1$) deb,

$$B(a,1-a) = \int_0^{\infty} \frac{y^{a-1}}{1+y} dy \quad (12.11.9)$$

formulani topamiz. Ma'lumki, $\int_0^{\infty} \frac{y^{a-1}}{1+y} dy = \frac{\pi}{\sin a\pi}$ ($0 < a < 1$), xususiy

holda, $a=1-a=\frac{1}{2}$ bo'lganda

$$B\left(\frac{1}{2}, \frac{1}{2}\right) = \pi \quad (12.11.10)$$

ni topamiz.

12.11.11. Misol. Ushbu

$$I = \int_0^1 \frac{dx}{\sqrt{1-x^2}} \int_0^1 \frac{dy}{1+x^2 y^2}$$

integralni, integrallash tartibini almashtirish haqidagi teoremdan foydalanib, hisoblang.

Yechilishi. $f(x,y) = \frac{1}{\sqrt{1-x^2}(1+x^2 y^2)}$ funksiya

$M_2 = \{(x,y) \in R^2 : 0 \leq x < 1, 0 \leq y \leq 1\}$ to'plamda uzluksiz.

$$\int_0^1 \frac{1}{\sqrt{1-x^2}(1+x^2 y^2)} dx$$

integral y bo'yicha $[0;1]$ da tekis yaqinlashuvchi.

Haqiqatan ham, $\forall x \in [0;1]$ va tayinlangan $y \in [0;1]$ uchun

$$\frac{1}{\sqrt{1-x^2}(1+x^2 y^2)} \leq \frac{1}{\sqrt{1-x^2}} \quad \text{tengsizlik o'rinli,} \quad \int_0^1 \frac{dx}{\sqrt{1-x^2}} \quad \text{integral}$$

yaqinlashuvchi.

Demak, $\int_0^1 \frac{1}{\sqrt{1-x^2}(1+x^2y^2)} dx$ integral, Veyershtass teoremasiga asosan, tekis yaqinlashuvchi. **12.10.10-teoremaga asosan,**

$$I = \int_0^1 \frac{dx}{\sqrt{1-x^2}} \int_0^1 \frac{dy}{1+x^2y^2} = \int_0^1 dy \int_0^1 \frac{dx}{\sqrt{1-x^2}(1+x^2y^2)},$$

tenglik o'rinli bo'lgani uchun. $x = \cos \varphi$ almashtirishni bajarib, quyidagiga ega bo'lamiz: $I = \frac{\pi}{2} \int_0^1 \frac{dy}{\sqrt{1+y^2}} = \frac{\pi}{2} \ln(1+\sqrt{2})$.

12.12. Gamma funksiya va uning xossalari. Ushbu

$$\int_0^{\infty} e^{-x} x^{a-1} dx$$

integralni qaraymiz. Bu integralda ∞ va $0 (a < 1)$ nuqtalar maxsus nuqtalardir. Odatda, bu integral «Gamma» funksiya (yoki 2-tur Eyler integrali) deb yuritiladi va u

$$\Gamma(a) = \int_0^{\infty} x^{a-1} e^{-x} dx \quad (12.12.1)$$

kabi belgilanadi. Gamma funksiya $a > 0$ da yaqinlashuvchi. Haqiqatdan ham

$$\Gamma(a) = \int_0^1 x^{a-1} e^{-x} dx + \int_1^{\infty} x^{a-1} e^{-x} dx \quad (12.12.2)$$

Keyingi tenglikning o'ng tomonidagi ikkala integral ham ixtiyoriy chekli $[\alpha, \beta] (0, \infty)$ kesmada Veyershtass alomatiga asosan, a parametr ga nisbatan tekis yaqinlashuvchi. $0 < \beta < 1, \beta > 1$ bo'lsin. U holda, $0 \leq x \leq 1$ da

$$0 \leq x^{a-1} e^{-x} \leq x^{a-1}, \quad \int_0^1 x^{a-1} dx = \frac{1}{a}$$

Demak, $\int_0^1 x^{a-1} e^{-x} dx$ integral $[\alpha, \beta]$ da tekis yaqinlashuvchi. Xudi

shunday, $x \geq 1$ bo'lganda $0 \leq x^{a-1} e^{-x} \leq x^{\beta-1} e^{-x}, \int_1^{\infty} x^{\beta-1} e^{-x} dx$ integral

yaqinlashuvchi, chunki $\frac{x^{\beta-1}e^{-x}}{1} = \frac{x^{\beta+\lambda-1}}{e^x} \xrightarrow{x \rightarrow \infty} 0 (\lambda > 1)$ u holda $\int_1^{+\infty} x^{a-1} e^{-x} dx$

integral $[\alpha, \beta]$ da tekis yaqinlashuvchi.

(12.12.1) integralda $x = \ln \frac{1}{z}$ almashtirishni bajarib

$$\Gamma(a) = \int_0^1 \left(\ln \frac{1}{z} \right)^{a-1} dz$$

ni hosil qilamiz. Ma'lumki $\ln \frac{1}{z} = \lim_{n \rightarrow \infty} n \left(1 - z^{\frac{1}{n}} \right)$. Buni e'tiborga olgan holda.

$$\Gamma(a) = \lim_{n \rightarrow \infty} n^{a-1} \int_0^1 \left(1 - z^{\frac{1}{n}} \right)^{a-1} dz$$

ega bo'lamiz. Bunda $z = y^n$ almashtirishni bajarib,

$$\Gamma(a) = \lim_{n \rightarrow \infty} n^a \int_0^1 y^{n-1} (1 - y)^{a-1} dy$$

(12.11.5) ga asosan,

$$\int_0^1 y^{a-1} (1 - y)^{a-1} dy = B(n, a) = \frac{1.2.3 \dots (n-1)}{a(a+1)(a+2) \dots (a+n-1)}$$

Shunday qilib,

$$\Gamma(a) = \lim_{n \rightarrow \infty} n^a \frac{1.2.3 \dots (n-1)}{a(a+1)(a+2) \dots (a+n-1)}.$$

Odatda bu formula Eyler-Gauss formulasi deb yuritiladi.

1°. $\Gamma(a)$ funksiya $a > 0$ da uzluksiz va uzluksiz differensiallanuvchi.

Haqiqatdan ham, integral ostidagi $x^{a-1}e^{-x}$ funksiya $x > 0, a > 0$ da uzluksiz. $\int_0^{+\infty} x^{a-1}e^{-x} dx$ integral ixtiyoriy $[\alpha, \beta] \subset (0, +\infty)$ kesmada tekis yaqinlashuvchi bo'lgani uchun 12.10.2-teoremaga ko'ra, (12.12.2) formuladagi ikala integral ham $[\alpha, \beta] \subset (0, +\infty)$ da a parametring uzluksiz funksiyasi bo'ladi.

Demak, $a > 0$ da $\Gamma(a)$ funksiya uzluksiz.

(12.12.1) integralni formula ravishda integral ostida a parametr bo'yicha differensiallaymiz, ya'ni Leybnits qoidasini qo'llaymiz:

$$\Gamma'(a) = \int_0^1 x^{a-1} \ln x \cdot e^{-x} dx + \int_1^{\infty} x^{a-1} \ln x \cdot e^{-x} dx.$$

Endi, Leybnits qoidasini qo'llash qonuniy ekanligini ko'rsatamiz:

$$\int_0^1 x^{a-1} \ln x \cdot e^{-x} dx \text{ integral uchun } x=0 \text{ maxsus nuqta, } x=0 \text{ da}$$

$a \geq \alpha > 0$ uchun $|x^{a-1} \ln x \cdot e^{-x}| \leq x^{\alpha-1} |\ln x|$ bu integral uchun $x^{\alpha-1} |\ln x|$ -

majorant funksiya bo'ladi. $\int_1^{\infty} x^{a-1} \ln x \cdot e^{-x} dx$ integral uchun $x=\infty$ da

$a \leq A < \infty$ ga $x^A e^{-x}$ -funksiya majorant ($x > 0, \ln x < x$).

Demak, ikala integral ham a parametr bo'yicha ixtiyoriy $[\alpha, \beta] \subset (0, +\infty)$ da tekis yaqinlashuvchi. U holda, **12.10.10-teorema**ga asosan, Leybnits qoidasini qo'llash mumkin ekan.

Matematik induksiya usuli yordamida $a > 0$ da

$$\Gamma^{(n)}(a) = \int_0^{\infty} x^{a-1} (\ln x)^n \cdot e^{-x} dx, \quad n \in \mathbb{N}$$

ekanligini ko'rsatish mumkin.

2^o. $a > 0$ bo'lsin. (12.12.1) ni bo'laklab integrallash natijasida

$$a \int_0^{\infty} x^{a-1} e^{-x} dx = x^a e^{-x} \Big|_0^{\infty} + \int_0^{\infty} x^a e^{-x} dx$$

$$a \cdot \Gamma(a) = \Gamma(a+1), \quad \Gamma(a+1) = a \cdot \Gamma(a) \quad (12.12.3)$$

ni hosil qilamiz. Bu formulani takroriy qo'llash natijasida

$$\Gamma(a+n) = (a+n-1)(a+n-2) \dots (a+1)a\Gamma(a) \quad (12.12.4)$$

formula hosil bo'ladi. (12.12.4) formulada $a=1$ deb olsak, u vaqtda

$$\Gamma(n+1) = n \cdot (n-1) \cdot 2 \cdot 1 \cdot \Gamma(1),$$

$$\Gamma(1) = \int_0^{\infty} e^{-x} dx = 1 \quad (12.12.5)$$

ekanligini e'tiborga olsak, natijada

$$\Gamma(n+1) = n! \quad (12.12.6)$$

formulaga ega bo'lamiz.

3^o. (Gamma funksiyaning o'zgarishi) (12.12.5) bilan (12.12.6) dan Roll teoremasiga asosan, $\exists a_0 (1 < a_0 < 2)$ nuqta topiladiki, $\Gamma'(a_0) = 0$ bo'ladi. $\forall a \in (0, +\infty)$ da $\Gamma''(a) = \int_0^{+\infty} x^{a-1} e^{-x} \cdot \ln^2 x dx > 0$ bo'lganligi uchun $\Gamma'(a)$ funksiya $(0, +\infty)$ da qat'iy o'suvchi bo'ladi. $\Gamma'(a)$ funksiya a_0 nuqtadan boshqa nuqталarda nolga aylanmaydi.

Demak,

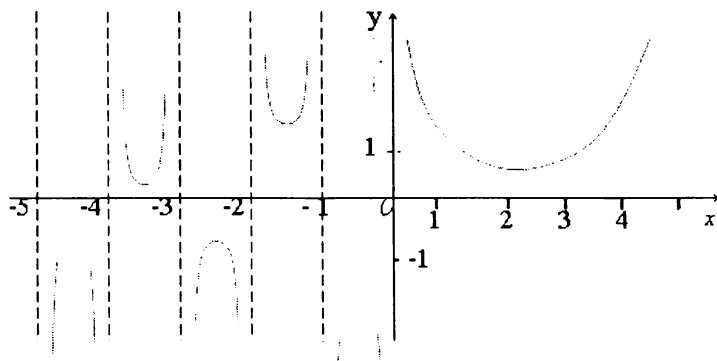
$$\Gamma'(a) = \int_0^{+\infty} x^{a-1} e^{-x} \cdot \ln x dx = 0$$

tenglama $(0, +\infty)$ oraliqda yagona yechimga ega. U holda

$$0 < a < a_0 \text{ da } \Gamma'(a) < 0, \quad a_0 < a < +\infty \text{ da } \Gamma'(a) > 0$$

bo'lib, $\Gamma(a)$ funksiya a_0 nuqtada minimumga ega bo'lib, $\Gamma(a)$ funksiya $a_0 < a$ bo'lganda o'suvchi bo'lgani uchun $a > n+1$ bo'lganda $\Gamma(a) > \Gamma(n+1) = n!$ bo'lib, undan $\lim_{a \rightarrow +\infty} \Gamma(a) = +\infty$ kelib chiqadi. (12.12.3)

dan $\Gamma(a) = \frac{\Gamma(a+1)}{a}$. $\Gamma(a)$ funksiyaning $a > 0$ da uzluksizligini (1^o-xossa) hamda $\Gamma(1) = 1$ e'tiborga olganda, keyingi tenglikdan $a \rightarrow +0$ da $\lim_{a \rightarrow +0} \Gamma(a) = +\infty$ ekanligini topamiz. $\Gamma(a)$ funksiyaning grafigi 12.1-chizmada ko'rsatilgan.



12.1-chizma.

4^o. Beta va gamma funksiyalar orasidagi bog'lanishni topish uchun (12.12.1) formulada ushbu $x = ty$ ($t > 0$) almashtirishni bajaramiz:

$$\frac{\Gamma(a)}{t^a} = \int_0^{+\infty} y^{a-1} e^{-ty} dy \quad (12.12.7)$$

(12.12.7) tenglikda birdaniga a ni $a+b$ ga, t ni $1+t$ ga almashtirib

$$\frac{\Gamma(a+b)}{(1+t)^{a+b}} = \int_0^{+\infty} y^{a+b-1} e^{-(1+t)y} dy$$

ega bo'lamiz. Bu tenglikning ikkala tomonini t^{a-1} ga ko'paytirib, t bo'yicha 0 dan $+\infty$ gacha integrallaymiz:

$$\Gamma(a+b) \int_0^{+\infty} \frac{t^{a-1}}{(1+t)^{a+b}} dt = \int_0^{+\infty} t^{a-1} dt \int_0^{+\infty} y^{a+b-1} e^{-(1+t)y} dy$$

Bu tenglikning chap tomoni ((12.11.8) ga q.) dagi integral $B(a,b)$ funksiyani ifoda qiladi, chap tomonidagi integrallarning o'rinlarini ([15], 18-bob, 3-§ q.) 5-teoremadan foydalanib almashtiramiz:

$$\begin{aligned} \Gamma(a+b)B(a,b) &= \int_0^{+\infty} y^{a+b-1} dy \int_0^{+\infty} t^{a-1} e^{-ty} dt = \int_0^{+\infty} y^{a+b-1} e^{-y} \cdot \frac{\Gamma(a)}{y^a} dy = \\ &= \Gamma(a) \int_0^{+\infty} y^{b-1} \cdot e^{-y} dy = \Gamma(a) \cdot \Gamma(b), \end{aligned}$$

bundan

$$B(a,b) = \frac{\Gamma(a) \cdot \Gamma(b)}{\Gamma(a+b)} \quad (12.12.8)$$

formulani hosil qilamiz. Bu formula Beta va gamma funksiyalar orasidagi bog'lanishni beruvchi formula bo'lib hisoblanadi.

5^o. To'ldirish formulasi. (12.12.8) formulada $0 < a < 1$ deb $b = 1 - a$ olsak, natijada to'ldirish formulasi deb ataluvchi

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin a\pi} \quad (12.12.9)$$

formulaga ega bo'lamiz. Bundan $a = \frac{1}{2}$ bo'lganda

$$(\Gamma(a) > 0) \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \quad (12.12.10)$$

bo'ladi. (12.12.1) formulada $a = \frac{1}{2}$ deb olsak, u holda

$$\int_0^{\infty} \frac{e^{-x}}{\sqrt{x}} dx = \sqrt{\pi}$$

bu integralda $x = z^2$ almashtirishni bajarsak, ehtimollar nazariyasida uchrovchi muhim $\int_0^{\infty} e^{-z^2} dz = \frac{\sqrt{\pi}}{2}$ formulaga ega bo'lamiz.

6^o. *Lejandr formulasi*. $B(a, b)$ funksiyani

$$B(a, a) = \int_0^1 x^{a-1} (1-x)^{a-1} dx = \int_0^1 \left[\frac{1}{4} - \left(\frac{1}{2} - x \right)^2 \right]^{a-1} dx = 2 \int_0^{\frac{1}{2}} \left[\frac{1}{4} - \left(\frac{1}{2} - x \right)^2 \right]^{a-1} dx$$

ko'rinishda yozib olib, $\frac{1}{2} - x = \frac{1}{2} \sqrt{t}$ almashtirishni bajaramiz:

$$B(a, a) = \frac{1}{2^{2a-1}} \int_0^1 t^{-\frac{1}{2}} (1-t)^{a-1} dt = \frac{1}{2^{2a-1}} B\left(\frac{1}{2}, a\right).$$

Bu tenglikning ikkala tomoniga (12.12.8) formulani qo'llab

$$\frac{\Gamma(a) \cdot \Gamma(a)}{\Gamma(2a)} = \frac{1}{2^{2a-1}} \cdot \frac{\Gamma\left(\frac{1}{2}\right) \cdot \Gamma(a)}{\Gamma\left(a + \frac{1}{2}\right)}$$

ni topamiz. Bundan, (12.12.10) ni e'tiborga olgan holda, Lejandr formulasi deb ataluvchi

$$\Gamma(a) \Gamma\left(a + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2a-1}} \Gamma(2a)$$

formulani hosil qilamiz.

12.12.11. Misol. Ushbu

$$\int_0^{\frac{\pi}{2}} \sin^{a-1} \varphi \cos^{b-1} \varphi d\varphi \quad (a, b > 0)$$

integralni hisoblang.

Yechilishi. Bu integralda, $x = \sin \varphi$ almashtirishni bajarib, uni

$$\int_0^1 x^{a-1} (1-x^2)^{b-1} dx$$

ko'rinishga keltiramiz. Bu integralda $x^2 = y$ almashtirishni olib, natijada

$0 < \alpha < 1$ shartda berilgan integral yaqinlashuvchi. So'ngra keyingi
 hosil bo'lgan integralga Beta funksiyaning 5^o-xossasini, Gamma
 funksiyaning 4^o- va 7^o — xossalari qo'llash natijasida

$$I = B(1 - \alpha, 1 - \alpha) = \frac{\Gamma(1 - \alpha) \Gamma(1 - \alpha)}{\Gamma(2 - 2\alpha)} = \frac{\pi \sin \alpha \pi}{\sin \alpha \pi}$$

bo'lishini topamiz.

$$\frac{1}{2} \int_0^1 y^{\frac{b}{2}-1} (1-y)^{\frac{b}{2}-1} dy = \frac{1}{2} B\left(\frac{a}{2}, \frac{b}{2}\right) = \frac{1}{2} \frac{\Gamma\left(\frac{a}{2}\right) \Gamma\left(\frac{b}{2}\right)}{\Gamma\left(\frac{a+b}{2}\right)}.$$

ega bo'lamiz.

$$\text{Demak, } \int_0^{\frac{\pi}{2}} \sin^{a-1} \varphi \cos^{b-1} \varphi d\varphi = \frac{1}{2} \frac{\Gamma\left(\frac{a}{2}\right) \Gamma\left(\frac{b}{2}\right)}{\Gamma\left(\frac{a+b}{2}\right)}.$$

12.12.12. Misol. Ushbu

$$I = \int_0^1 \sqrt{\frac{1-x}{x}} \frac{dx}{(x+2)^2}$$

integralni Eyler integrallari yordamida hisoblang.

Yechilishi. Berilgan integralda $\frac{3x}{2+x} = t$ almashtirish bajaramiz va

$$x = \frac{2t}{3-t}, \quad \frac{dx}{(x+2)^2} = \frac{dt}{6}$$

$$I = \frac{1}{2\sqrt{6}} \int_0^1 t^{-\frac{1}{2}} (1-t)^{\frac{1}{2}} dt$$

ifodani hosil kilamiz. Beta funksiyaning ta'rif va Gamma funksiyaning 4^0 -, 6^0 -, 7^0 - xossaligidan foydalanib, $\Gamma(2)=1$ ekanligini e'tiborga olgan holda,

$$I = \frac{1}{2\sqrt{6}} B\left(\frac{1}{2}, \frac{3}{2}\right) = \frac{\pi}{4\sqrt{6}}$$

bo'lishini topamiz.

12.12.13. Misol. Ushbu

$$I = \int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^{\alpha} x}{(\sin x + \cos x)^2} dx, \quad 0 < \alpha < 1$$

integralni hisoblang.

Yechilishi. Berilgan integralda $\operatorname{tg} x = t$ ($t > 0$) almashtirish bajarib,

$$I = \int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^{\alpha} x dx}{\cos^2 x (1 + \operatorname{tg} x)^2} = \int_0^{\infty} \frac{t^{\alpha} dt}{(1+t)^2}$$

munosabatga ega bo'lamiz.

$0 < \alpha < 1$ shartda berilgan integral yaqinlashuvchi. So'ngra keyingi hosil bo'lgan integralga Beta funksiyaning 5^o-xossasini, Gamma funksiyaning 4^o- va 7^o- xossalarini qo'llash natijasida

$$I = B(1 + \alpha, 1 - \alpha) = \alpha \Gamma(\alpha) \Gamma(1 - \alpha) = \frac{\pi \alpha}{\sin \alpha \pi}$$

bo'lishini topamiz.

$$\frac{1}{2} \int_0^1 y^{\frac{b}{2}-1} (1-y)^{\frac{b}{2}-1} dy = \frac{1}{2} B\left(\frac{a}{2}, \frac{b}{2}\right) = \frac{1}{2} \frac{\Gamma\left(\frac{a}{2}\right) \Gamma\left(\frac{b}{2}\right)}{\Gamma\left(\frac{a+b}{2}\right)}.$$

ega bo'lamiz.

$$\text{Demak, } \int_0^{\frac{\pi}{2}} \sin^{a-1} \varphi \cos^{b-1} \varphi d\varphi = \frac{1}{2} \frac{\Gamma\left(\frac{a}{2}\right) \Gamma\left(\frac{b}{2}\right)}{\Gamma\left(\frac{a+b}{2}\right)}.$$

12.12.12. Misol. Ushbu

$$I = \int_0^1 \sqrt{\frac{1-x}{x}} \frac{dx}{(x+2)^2}$$

integralni Eyler integrallari yordamida hisoblang.

Yechilishi. Berilgan integralda $\frac{3x}{2+x} = t$ almashtirish bajaramiz va

$$x = \frac{2t}{3-t}, \quad \frac{dx}{(x+2)^2} = \frac{dt}{6} \text{ ekanligini e'tiborga olib,}$$

$$I = \frac{1}{2\sqrt{6}} \int_0^1 t^{-\frac{1}{2}} (1-t)^{\frac{1}{2}} dt$$

ifodani hosil kilamiz. Beta funksiyaning ta'rif va Gamma funksiyaning 4^0 -, 6^0 -, 7^0 - xossaligidan foydalanib, $\Gamma(2)=1$ ekanligini e'tiborga olgan holda,

$$I = \frac{1}{2\sqrt{6}} B\left(\frac{1}{2}, \frac{3}{2}\right) = \frac{\pi}{4\sqrt{6}}$$

bo'lishini topamiz.

12.12.13. Misol. Ushbu

$$I = \int_0^{\frac{\pi}{2}} \frac{tg^{\alpha} x}{(\sin x + \cos x)^2} dx, \quad 0 < \alpha < 1$$

integralni hisoblang.

Yechilishi. Berilgan integralda $tgx = t$ ($t > 0$) almashtirish bajarib,

$$I = \int_0^{\frac{\pi}{2}} \frac{tg^{\alpha} x dx}{\cos^2 x (1 + tgx)^2} = \int_0^{\infty} \frac{t^{\alpha} dt}{(1+t)^2}$$

munosabatga ega bo'lamiz.

$0 < \alpha < 1$ shartda berilgan integral yaqinlashuvchi. So'ngra keyingi hosil bo'lgan integralga Beta funksiyaning 5^o-xossasini, Gamma funksiyaning 4^o- va 7^o- xossalari qo'llash natijasida

$$I = B(1 + \alpha, 1 - \alpha) = \frac{\Gamma(\alpha)\Gamma(1 - \alpha)}{\sin \alpha \pi} = \frac{\pi}{\sin \alpha \pi}$$

bo'lishini topamiz.

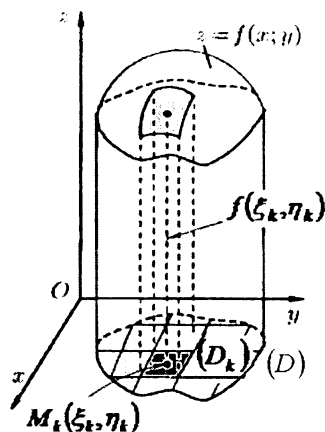
57-§. Ikki karrali integral va uni yig'indi bo'yicha hisoblash

13.1. Silindrik burusning hajmini topish haqidagi masala.

Ma'lumki ([15], 7-bob, 7.1-bandga q.) egri chiziqli trapetsiyaning yuzini topish haqidagi masala aniq integral tushinchasiga olib kelgan edi. Xuddi shunday silindrik brusning hajmini topish haqidagi masala ham ikki karrali integral tushinchasiga olib keladi. Biz bu bandda silindrik brusning hajmini topish haqidagi masalani qaraymiz.

Yuqoridan $z = f(x, y)$ sirt bilan, yon tomondan, yasovchilari Oz o'qiga parallel bo'lgan silindrik sirt, pastdan xOy tekisligidagi chegaralangan soha (D) bilan chegaralangan (V) jism berilgan bo'lib, shu jismning hajmi μV ni topish talab qilingan bo'lsin. Buning uchun (D) sohani unda butunlay yotuvchi nol yuzli egri chiziqlar yordamida ixtiyoriy ravishda n ta bo'lakka bo'lamiz: $(D_1), (D_2), \dots, (D_n)$.

Bo'luvchi chiziqlarni yo'naltiruvchi sifatida olib, Oz o'qiga parallel silindrik sirtlar o'tkazamiz. Natijada (V) jism, n ta $(V_k) (k = 1, 2, \dots, n)$ bo'laqlarga bo'linadi (13.1-chizma).



13.1-chizma.

(D_k) dan ixtiyoriy $M(\xi_k, \eta_k)$ nuqtani olamiz. Agar har bir silindrik sirtlarni balandligi $f(\xi_k, \eta_k)$ ga teng bo'lgan haqiqiy silindr deb faraz qilsak, u holda (V_k) silindrik brusning hajmi taxminan:

$f(\xi_k, \eta_k)D_k$ ga teng bo'ladi, bunda $D_k - (D_k)$ shaklning yuzi. U holda (V) jismning hajmi taxminan

$$V \approx \sum_{k=1}^n f(\xi_k, \eta_k)D_k, \quad (13.1.1)$$

chunki, $f(x, y) = f(\xi_k, \eta_k), (x, y) \in (D_k)$ bo'lsin deb hisobladik.

(V) jismning hajmini ifodalovchi (13.1.1) taqribiy formulaning aniqlik darajasini oshirish uchun (D) sohani bo'laklarga bo'lish sonini shunday o'rttirib boraylikki, natijada har bir $(D_k) (k=1, \dots, n)$ bo'lakning diametri nolga intilsin. U holda (D_i) sohalar diametrlarining

eng kattasi $\lambda \rightarrow 0$ da $\sum_{k=1}^n f(\xi_k, \eta_k)D_k$ yig'indining limiti izlanuvchi (V) jismning aniq hajmini ifodalaydi, ya'ni

$$V = \lim_{\lambda \rightarrow 0} \sum_{k=1}^n f(\xi_k, \eta_k)D_k.$$

13.2. Ikki karrali integralning ta'rif. Biror chegaralangan (D) $((D) \subset R^2)$ soha berilgan bo'lsin. (D) sohaning chegarasidagi ixtiyoriy ikki nuqtani birlashtiruvchi va to'laligicha shu sohada yotuvchi chiziqni (egri chiziqni) Γ chiziq deb belgilaymiz. Bunday Γ chiziqlar (D) sohani bo'laklarga ajratadi. $((D))$ sohada to'laligicha yotuvchi yopiq chiziqni ham Γ chiziq deb qaraymiz. Bunday chiziqlar ham (D) sohani bo'laklarga ajratadi. (D) sohani bo'laklarga ajratuvchi chekli sondagi Γ chiziqlar sistemasi $\{\Gamma: \Gamma \subset (D)\}$ (D) sohaning *bo'linishi* deb ataladi va u $P = \{\Gamma: \Gamma \subset (D)\}$ kabi belgilanadi. (D) sohani bo'laklarga ajratuvchi har bir Γ chiziq, P bo'linishning bo'luvchi chizig'i, (D) sohaning bo'lagi esa, P bo'linishning *bo'lagi* deyiladi.

P bo'linish bo'laklari diametrining eng kattasi uning diametri deb ataladi va u λ_p kabi belgilanadi. (D) sohaning bo'linishlari to'plamini $\Phi = \{P\}$ orqali belgilaymiz. $f(x, y)$ funksiya (D) $((D) \subset R^2)$ sohada berilgan bo'lsin (1.1-chizma). Bu erda va kelgusida hamma vaqt funksiyaning aniqlanish sohasi (D) ni yuzaga ega bo'lgan soha deb hisoblaymiz. Bu (D) sohaning $P \in \Phi$ bo'linishini va bu bo'linishning har

bir (D_k) ($k=1,2,\dots,n$) bo'lagidan ixtiyoriy $M_k(\xi_k, \eta_k)$ ($k=1,2,\dots,n$) nuqta olib, berilgan funksiyaning $M_k(\xi_k, \eta_k)$ ($k=1,2,\dots,n$) nuqtadagi $f(\xi_k, \eta_k)$ qiymatini (D_k) ($k=1,2,\dots,n$) sohaning D_k yuziga ko'paytirib, quyidagi

$$\sigma = \sum_{k=1}^n f(\xi_k, \eta_k) D_k$$

yig'indini tuzamiz va bu yig'indini $f(x,y)$ funksiyaning *integral yig'indisi* yoki *Riman yig'indisi* deb ataymiz.

Ravshanki, integral yig'indi berilgan $f(x,y)$ funksiya, (D) sohani bo'laklash usuliga, hamda tanlangan $M_k(\xi_k, \eta_k) \in (D_k)$ nuqtaga bog'liq bo'ladi $\sigma_p(f; \xi_k, \eta_k)$. Uni qisqacha σ_p deb belgilaymiz.

13.2.1. Ta'rif. Agar (D) sohaning har qanday $P_1, P_2, \dots, P_n, \dots$ bo'linishlar ketma-ketligi $\{P_n\}$ olinganda ham, unga mos kelgan $\sigma_1, \sigma_2, \dots, \sigma_n, \dots$ integral yig'indilar ketma-ketligi $\{\sigma_n\}$, $\lambda_p \rightarrow 0$ da $M_k(\xi_k, \eta_k)$ ($k=1,2,\dots,n$) nuqtalarni tanlab olishga bog'liq bo'lmasdan, hamma vaqt bitta I songa intilsa, bu I son σ integral yig'indining *limiti* deb ataladi va u

$$\lim_{\lambda_p \rightarrow 0} \sigma_p = \lim_{\lambda_p \rightarrow 0} \sum_{k=1}^n f(\xi_k, \eta_k) D_k = I$$

kabi belgilanadi.

Bu ta'rifni quyidagicha ham ta'riflash mumkin.

13.2.2. Ta'rif. Agar $\forall \varepsilon > 0$ olinganda ham, shunday $\delta > 0$ topilsaki, (D) sohaning diametri $\lambda_p < \delta$ bo'lganda, har qanday P bo'linish hamda har bir (D_k) bo'lakdagi ixtiyoriy $M_k(\xi_k, \eta_k)$ nuqtalari uchun,

$$|\sigma - I| < \varepsilon$$

tengsizlik bajarilsa, u holda I songa σ_p integral yig'indining *limiti* deb ataladi va u $\lim_{\lambda_p \rightarrow 0} \sigma_p = I$ kabi belgilanadi.

13.2.3. Ta'rif. Agar $\lambda_p \rightarrow 0$ da $f(x,y)$ funksiyaning σ_p integral yig'indisi chekli limitga ega bo'lsa, $f(x,y)$ funksiya (D) sohada integrallanuvchi (Riman ma'nosida) funksiya deyiladi. Bu σ_p integral

yig'indining chekli limiti I son esa, $f(x,y)$ funksiyaning (D) soha bo'yicha *ikki karrali integrali (Riman integrali)* deb ataladi va u

$$I = \lim_{\lambda_p \rightarrow 0} \sigma_p = \lim_{\lambda_p \rightarrow 0} \sum_{k=1}^n f(\xi_k, \eta_k) D_k = \iint_{(D)} f(x,y) dD$$

kabi belgilanadi.

Shunday qilib, 13.1 bandda keltirilgan (V) jismning hajmi $f(x,y)$ funksiyaning (D) soha bo'yicha *ikki karrali integralini* ifoda qilar ekan.

13.3. Ikki karrali integrallar uchun Darbu yig'indilari. $f(x,y)$ funksiya chegaralangan (D) $((D) \subset \mathbb{R}^2)$ sohada berilgan va chegaralangan bo'lsin. (D) sohaning biror P bo'linishini qaraylik. Bu bo'linishning har bir (D_k) $(k=1,2,\dots,n)$ bo'lagida $f(x,y)$ funksiya chegaralangan bo'lib, uning aniq quyi va aniq yuqori chegaralari

$$m_k = \inf_{(x,y) \in (D_k)} \{f(x,y)\}, \quad M_k = \sup_{(x,y) \in (D_k)} \{f(x,y)\}$$

mavjud bo'ladi. $f(x,y)$ funksiya (D) $((D) \subset \mathbb{R}^2)$ sohada chegaralangan bo'lgani uchun u har bir (D_k) $(k=1,2,\dots,n)$ da ham chegaralangan bo'ladi: $\forall (x,y) \in (D_k)$ $(k=1,2,\dots,n)$ uchun

$$m_k \leq f(x,y) \leq M_k \quad (13.3.1)$$

tengsizlik o'rinli bo'ladi.

13.3.2. Ta'rif. Ushbu

$$s_p = \sum_{k=1}^n m_k D_k, \quad S_p = \sum_{k=1}^n M_k D_k$$

yig'indilar, mos ravishda, *Darbuning quyi hamda yuqori yig'indilari* deb ataladi.

Darbu yig'indilari berilgan $f(x,y)$ funsiyaga va (D) to'plamning bo'laklashiga bo'g'liq: $s_p = s_p(f)$, $S_p = S_p(f)$. Har doim $s_p(f) \leq S_p(f)$ tengsizlik bajariladi. (13.3.1) ga asosan,

$$\sum_{k=1}^n m_k D_k \leq \sum_{k=1}^n f(\xi_k, \eta_k) D_k \leq \sum_{k=1}^n M_k D_k$$

tengsizlik hosil qilamiz. Bundan (D) sohaning berilgan bo'lishida (ξ_k, η_k) nuqtaning tanlanishiga bog'liq bo'lmagan holda

$$s_p(f) \leq \sigma_p \leq S_p(f)$$

tengsizlik o'rinli bo'ladi. (ξ_k, η_k) nuqtani tanlash hisobidan $f(\xi_k, \eta_k)$ qiymatini m_k va M_k larga yaqinlashtirish mumkin, shu bilan birga σ_p ni $s_p(f)$ va $S_p(f)$ yig'indilarga istalgancha yaqinlashtirish mumkin.

Shunday qilib, Darbuning quyi va yuqori yigindilari integral yig'indi uchun mos ravishda aniq quyi va aniq yuqori chegara bo'ladi.

(D) sohaning bo'linishlar to'plami $\Phi = \{P\}$ ning har bir $P \in \Phi$ bo'linishiga nisbatan $f(x, y)$ funksiyaning Darbu yig'indilari $s_p(f)$, $S_p(f)$ ni tuzib $\{s_p(f)\}$, $\{S_p(f)\}$ to'plamlarni qaraymiz. Ravshanki, bu to'plamlar chegaralangan bo'ladi.

13.3.3. Ta'rif. $\{s_p\}$ ($\{S_p\}$) to'plamlarning aniq yuqori (aniq quyi) chegarasi, ya'ni $\sup\{s_p\} = \underline{J}$ ($\inf\{S_p\} = \bar{J}$) miqdorlar, mos ravishda, $f(x, y)$ funksiyaning (D) sohadagi *quyi ikki karrali* (*yuqori ikki karrali*) *integrali* deb ataladi va u

$$\underline{J} = \iint_{(D)} f(x, y) dD = \sup\{s_p\} \quad \left(\bar{J} = \overline{\iint_{(D)} f(x, y) dD} = \inf\{S_p\} \right)$$

kabi belgilanadi.

13.3.4. Ta'rif. Agar $f(x, y)$ funksiyaning (D) soha bo'yicha olingan quyi hamda yuqori ikki karrali integrallari bir-biriga teng, ya'ni $\underline{J} = \bar{J}$ bo'lsa, $f(x, y)$ funksiya (D) sohada *integrallanuvchi* deb ataladi, ularning umumiy qiymati $\underline{J} = \bar{J} = J$ $f(x, y)$ funksiyaning (D) soha bo'yicha olingan ikki karrali deyiladi va u

$$J = \iint_{(D)} f(x, y) dD$$

kabi belgilanadi.

Agar $\underline{J} = \bar{J}$ bo'lsa, u holda $f(x, y)$ funksiya (D) sohada *integrallanmaydi* deyiladi.

Shunday qilib ikki karrali integralning ikki xil ta'rif berildi. Bu ta'riflar ekvivalent ta'riflardir.

Aniq integralda $f(x, y)$ funksiyaning Darbu yig'indilarinig xossalari keltirilgan bo'lib, ular isbotlangan edi. Xuddi shunday xossalari $f(x, y)$ funksiya Darbu yig'indilari uchun ham o'rinli bo'ladi.

Shuning uchun bu holda Darbu yig'indilarining xossalarini isbotsiz keltirish bilan kifoyalanamiz.

Darbu yig'indilari quyidagi xossalarga ega:

1⁰. (D) sohaning bo'lish chiziqlariga yangi bo'lish chizig'ini qo'shish bilan Darbuning quyi yig'indisi kamaymaydi, yuqori yig'indisi esa ortmaydi.

Bu xossa (D) sohaning bo'linishidagi bo'laklar soni orta borganda ularga mos Darbuning quyi yig'indisi kamaymasligi, yuqori yig'indining esa, oshmasligini anglatadi.

2⁰. (D) sohaning istalgan bo'linishiga mos kelgan Darbuning har bir quyi yig'indisi, har bir yuqori yig'indisidan ortmaydi.

Bu xossa (D) sohaning bo'linishlariga nisbatan tuzilgan quyi yig'indilar to'plami $\{s_p(f)\}$ ning har bir elementi yuqori yig'indilar to'plami $\{S_p(f)\}$ ning istalgan elementidan katta emasligini, yuqori yig'indilar to'plami $\{S_p(f)\}$ ning har bir elementi, quyi yig'indilar to'plami $\{s_p(f)\}$ ning har bir elementi kichik emasligini bildiradi.

3⁰. Agar $f(x, y)$ funksiya (D) sohada chegaralangan bo'lsa, u holda $\sup\{s_p(f)\} \leq \inf\{S_p(f)\}$ bo'ladi.

Bu xossa, $f(x, y)$ funksiyaning quyi ikki karrali integrali, uning yuqori ikki karrali integralidan katta emasligini bildiradi.

4⁰. Agar $f(x, y)$ funksiya (D) sohada chegaralangan bo'lsa, u holda $\forall \varepsilon > 0$ son olinganda ham, shunday $\delta > 0$ son topiladiki, uning (D) sohaning diametri $\lambda_p < \delta$ bo'lgan barcha bo'linishlari uchun

$$\begin{aligned} S_p(f) &< \bar{J} + \varepsilon \quad (0 \leq S_p(f) - \bar{J} < \varepsilon) \\ s_p(f) &> \underline{J} - \varepsilon \quad (0 \leq \underline{J} - s_p(f) < \varepsilon) \end{aligned} \quad (13.3.5)$$

bo'ladi.

Bu xossa $f(x, y)$ funksiyaning yuqori hamda quyi integrallari $\lambda_p \rightarrow 0$ da mos ravishda Darbuning yuqori hamda quyi yig'indilarning limiti ekanligini anglatadi:

$$\bar{J} = \lim_{\lambda_p \rightarrow 0} S_p(f), \quad \underline{J} = \lim_{\lambda_p \rightarrow 0} s_p(f) \quad .$$

Yuqoridagi mulohazalardan

$$s_p \leq \underline{J} \leq \bar{J} \leq S_p(f) \quad (13.3.6)$$

ekanligi kelib chiqadi.

13.4. Ikki karrali integralning mavjudlik sharti. $f(x, y)$ funksiya (D) sohada aniqlangan va chegaralangan bo'lsin.

13.4.1. Teorema (mavjudlik sharti). $f(x, y)$ funksiyaning (D) sohada integrallanuvchi bo'lishi uchun, $\forall \varepsilon > 0$ olinganda ham, shunday $\delta > 0$ topilib, (D) sohaning diametri $\lambda_p < \delta$ bo'lgan har qanday P bo'linishga nisbatan tuzilgan Darbu yig'indilari

$$S_p(f) - s_p(f) < \varepsilon \quad (13.4.2)$$

tengsizlikni qanoatlantirishi zarur va yetarli.

Isboti. Zarurligi. $f(x, y)$ funksiya (D) sohada integrallanuvchi bo'lsin, u holda $\underline{J} = \bar{J} = J$ bo'ladi, bunda $\underline{J} = \sup_p \{s_p(f)\}$, $\bar{J} = \inf_p \{S_p(f)\}$. $\forall \varepsilon > 0$ son olinganda ham, $\frac{\varepsilon}{2}$ ga ko'ra, shunday $\delta > 0$ topiladiki, (D) sohaning diametri $\lambda_p < \delta$ bo'lgan har qanday P bo'linishga nisbatan tuzilgan Darbu yig'indilari uchun (13.3.5) tengsizliklarga ko'ra,

$$S_p(f) - \bar{J} < \frac{\varepsilon}{2}, \quad \underline{J} - s_p(f) < \frac{\varepsilon}{2}, \quad S_p(f) - J < \frac{\varepsilon}{2}, \quad J - s_p(f) < \frac{\varepsilon}{2}$$

tengsizlikning o'rinli ekanligi kelib chiqadi, ya'ni (13.4.2) tengsizlik o'rinli bo'ladi.

Yetarliligi. (13.4.2) shart o'rinli bo'lsin. $f(x, y)$ funksiya (D) sohada chegaralangan bo'lganligi uchun

$$\underline{J} = \sup_p \{s_p(f)\}, \bar{J} = \inf_p \{S_p(f)\} \text{ lar mavjud bo'lib, Darbuning}$$

yig'indilarining 3^0 -xossasiga ko'ra, $\underline{J} \leq \bar{J}$ bo'ladi. (13.3.6)

munosabatdan $0 \leq \bar{J} - \underline{J} \leq S_p(f) - s_p(f)$ bo'ladi. (13.4.2) shart o'rinli

bo'lganligi uchun keyingi munosabatdan $0 \leq \bar{J} - \underline{J} < \varepsilon$ bo'ladi. Bu

tengsizlikdan $\underline{J} = \bar{J}$, bundan esa, **13.3.4**-ta'rifga asosan, $f(x, y)$ funksiyaning (D) sohada integrallanuvchi ekanligi kelib chiqadi.

Odatdagidek $f(x, y)$ funksiyaning $(D_i)(i=1, 2, \dots, n)$ sohadagi tebranishini ω_i deb belgilasak, u holda

$$S_p(f) - s_p(f) = \sum_{i=1}^n (M_i - m_i) D_i = \sum_{i=1}^n \omega_i D_i < \varepsilon \quad (13.4.3)$$

bo'lib, teoremaning (13.4.2) sharti (13.4.3) ko'rinishga keladi.

Demak, (13.4.2), (13.4.3) shartlar o'zaro ekvivalent shartlardir. ■

13.5. Integrallanuvchi funksiyalarning sinflari.

13.5.1. Teorema. Agar $f(x, y)$ funksiya chegaralangan yopiq (D) $((D) \subset R^2)$ sohada berilgan va uzluksiz bo'lsa, u shu sohada integrallanuvchi bo'ladi.

Isboti. Shartga ko'ra $f(x, y)$ funksiya (D) yopiq sohada uzluksiz bo'lgani uchun u shu sohada tekis uzluksiz bo'ladi. U holda Kantor teoremasidan kelib chiqqan natijaga ko'ra ([15], 4-bob, 4.10-bandga q.), $\forall \varepsilon > 0$ son olinganda ham, shunday $\delta > 0$ son topiladiki, (D) sohaning diametri $\lambda_p < \delta$ bo'lgan ixtiyoriy P bo'linishi olinganda ham, bu bo'linishlarning har bir bo'lagida funksiyaning tebranishi $\omega_i < \varepsilon$ bo'ladi. (D) sohaning diametri $\lambda_p < \delta$ bo'lgan ixtiyoriy P bo'linishiga mos kelgan Darbu yig'indilari ayirmasini qaraylik:

$$S_p(f) - s_p(f) = \sum_{i=1}^n \omega_i D_i < \varepsilon \sum_{i=1}^n D_i = \varepsilon \cdot D.$$

Bundan, 13.4.1-teoremaga asosan, $f(x, y)$ uzluksiz funksiyaning (D) sohada integrallanuvchi bo'ladi. ■

13.5.2. Lemma. (D) sohada nol yuzga ega bo'lgan (L) chiziq berilgan bo'lsin. U holda $\forall \varepsilon > 0$ son olinganda ham, shunday $\delta > 0$ son topiladiki, (D) sohaning diametri $\lambda_p < \delta$ bo'lgan P bo'linishi olinganda, bu bo'linishning (L) chiziq bilan umumiy nuqtaga ega bo'lgan bo'laklari yuzlarining yig'indisi ε dan kichik bo'ladi.

Isboti. Shartga ko'ra (L) chiziq nol yuzga bo'lgani uchun uni yuzi ε dan kichik bo'lgan (Q) ko'pburchak bilan o'rash mumkin. (L) chiziq bilan (Q) ko'pburchak chegarasi (k) umumiy nuqtaga ega emas deb, ular

orasidagi masofani qaraymiz. Bu chiziqlar orasidagi masofa o'zining eng kichik qiymati $\delta > 0$ ga erishadi. Haqiqatdan ham yuqoridagi ko'rsatilgan chiziqlar o'zlarining parametrik tenglamalari bilan berilgan bo'lsin:

$$(L): x = \varphi(t), \quad y = \psi(t), \quad (k): x = \varphi_1(u), \quad y = \psi_1(u).$$

$$t_0 \leq t \leq T \quad u_0 \leq u \leq U$$

Bunda $\varphi, \psi, \varphi_1, \psi_1$ – funksiyalarning har biri argumentlarining uzluksiz funksiyalari. U holda bu egri chiziqlar ixtiyoriy nuqtalari orasidagi masofa

$$\sqrt{[\varphi(t) - \varphi_1(u)]^2 + [\psi(t) - \psi_1(u)]^2}$$

$-[t_0, T; u_0, U]$ yopiq to'rt burchakda (t, u) ning uzluksiz funksiyasi bo'ladi. Shuning uchun u yuqoridagi ko'rsatilgan to'rtburchakda o'zining eng kichik qiymatiga erishadi. Biz uni $\delta > 0$ bilan belgilaymiz. (L) va (k) chiziqlar o'zaro kesishmaganligi uchun ular orasidagi eng kichik masofa $\delta > 0$ bo'ladi.

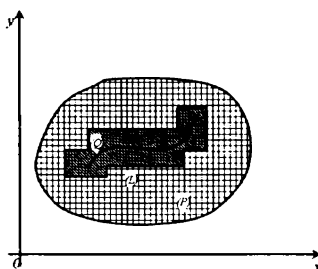
Agar (D) sohaning diametri $\lambda_p < \delta$ bo'lgan $\forall P$ bo'linishi olinganda, bu bo'linishning (L) chiziq bilan umumiy nuqtaga ega bo'lgan bo'laklari butunlay (Q) ko'pburchakning ichida joylashadi.

Demak, bunday bo'laklar yuzlarning yig'indisi berilgan ε dan kichik bo'ladi. ■

13.5.3. Teorema. Agar $f(x, y)$ funksiya (D) sohada chegaralangan va bu sohaning chekli sondagi nol yuzali chiziqlarida uzilishga ega bo'lib, (D) yopiq sohaning qolgan barcha nuqtalarida uzluksiz bo'lsa, bu funksiya (D) yopiq sohada integrallanuvchi bo'ladi.

Isboti. Soddalik (qulaylik) uchun $f(x, y)$ funksiya (D) sohada bitta nol yuzli (L) chiziqda uzilishga ega bo'lib, $((L) \subset (D))$ qolgan nuqtalarida uzluksiz bo'lsin.

$\forall \varepsilon > 0$ sonni olib, (L) chiziqni yuzi berilgan ε dan kichik bo'lgan nol yuzli (n)



13.2-chizma.

chiziqlarning ta'rifiga asosan ixtiyoriy (Q) ko'pburchak bilan o'raymiz (13.2-chizma).

Natijada (D) soha $(D)/(Q)$ va (Q) sohalarga ajraladi (13.2-chizma). (Q) ko'pburchakning chegarasi chekli sondagi sinq chiziqlardan iborat bo'lib, ularning har biri nol yuzli chiziqlardir. Shartga ko'ra, $f(x, y)$ funksiya $(D)/(Q)$ - sohada $((D)|(Q))$ - soha (D) sohadan (Q) ning ichki qismini olib tashlangan soha) uzluksiz bo'lgani uchun u bu sohada tekis uzluksiz ham bo'ladi. Shunday qilib berilgan $\forall \varepsilon > 0$ uchun shunday $\delta_1 > 0$ son topilib, (D) sohaning diametri $\lambda_{p_1} < \delta_1$ bo'lgan P_1 bo'linishning har bir bo'lagidagi $f(x, y)$ funksiyaning tebranishi $\omega_i < \varepsilon$ bo'ladi. Yuqoridagi 13.5.2-lemmaga asosan, shunday $\delta_2 > 0$ son topiladiki, (D) sohaning diametri $\lambda_{p_2} < \delta_2$ bo'lgan bo'linish olinganda, bu bo'linishning (Q) ko'pburchak bilan umumiy nuqtaga ega bo'lgan bo'laklari yuzlarining yig'indisi ε dan kichik bo'ladi. $\delta = \min\{\delta_1, \delta_2\}$ deb, (D) sohaning diametri $\lambda_p < \delta$ bo'lgan P bo'linishini qaraymiz. Bu bo'linishga nisbatan $f(x, y)$ funksiyaning Darbu yig'indilarini tuzib, ularning ayirmasini qaraymiz:

$$S_p(f) - s_p(f) = \sum_{i=1}^n \omega_i D_i. \quad (13.5.4)$$

(13.5.4) yig'indining (Q) ko'pburchakdan tashqarida joylashgan (D_i) bo'laklarga mos hadlaridan iborat yig'indisi $\sum_i' \omega_i D_i$ bo'lsin. (13.5.4) yig'indining qolgan barcha hadlaridan tashkil topgan yig'indisi esa,

$$\sum_{i''} \omega_i D_i$$

bo'lsin. Shunday qilib (13.5.4) yig'indisi ikki qismdan iborat bo'ladi:

$$\sum_{i=1}^n \omega_i D_i = \sum_{i'} \omega_i D_i + \sum_{i''} \omega_i D_i.$$

Ma'lumki, $(D)/(Q)$ sohadagi bo'laklarda $\omega_i < \varepsilon$ bo'lganligidan

$$\sum_{i'} \omega_i D_i < \varepsilon \sum_{i'} D_i < \varepsilon \cdot D.$$

Agar $f(x, y)$ funksiyaning (D) sohadagi tebranishini Ω deb belgilasak, u holda

$$\sum_{i^n} \omega_i D_i \leq \Omega \sum_{i^n} D_i \quad (\omega_i \leq \Omega)$$

bo'ladi. Ravshanki, (Q) ko'pburchak ichida butunlay yotgan P bo'linishning bo'laklari yuzlarining yig'indisi ε dan kichik hamda (Q) ko'pburchak chegarasi bilan umumiy nuqtaga ega bo'lgan bo'laklar yuzlarining yig'indisi ham ε dan kichik bo'ladi. U holda $\sum_{i^n} D_i < 2\varepsilon$.

Shunday qilib,

$$\sum_{i^n} \omega_i D_i < 2\Omega\varepsilon.$$

Demak, $\sum_{i=1}^n \omega_i D_i < \varepsilon D + 2\Omega\varepsilon = \varepsilon(D + 2\Omega)$. Bundan, $\lim_{\lambda \rightarrow 0} \sum_{i=1}^n \omega_i D_i = 0$

kelib chiqadi. (13.4.3) ga asosan, $f(x, y)$ funksiyaning (D) sohada integrallanuvchi ekanligi kelib chiqadi. ■

13.6. Ikki karrali integralning xossalari. Ikki karrali integral ham aniq integral singari qator xossalarga ega. Bu xossalar ham, aniq integral xossalari singari isbot qilinishini e'tborga olgan holda ularning ba'zilarini isbot qilish bilan chegaralanib, qolganlarini isbot qilishni o'quvchiga havola etamiz.

1°. $f(x, y)$ funksiya $(D)((D) \subset R^2)$ sohada integrallanuvchi bo'lsin. Bu funksiyaning (D) sohada to'lasincha yotuvchi nol yuzaga ega bo'lgan Γ chiziqdagi $(\Gamma \subset (D))$ qiymatlarinigina (chegaralan-ganligini saqlagan holda) o'zgartirishdan hosil bo'lgan $F(x, y)$ funksiya ham (D) sohada integrallanuvchi bo'ladi va

$$\iint_{(D)} f(x, y) dD = \iint_{(D)} F(x, y) dD$$

tenglik o'rinli.

Isboti. Haqiqatan ham, $\forall (x, y) \in (D) \setminus (L)$ uchun $f(x, y) = F(x, y)$. Shartga ko'ra, L -chiziq nol yuzli bo'lgani uchun 13.5.2-lemmaga asosan, $\forall \varepsilon > 0$ olinganda ham, shunday $\delta > 0$ son topiladiki, (D) sohaning diametri $\lambda_P < \delta$ bo'lgan har qanday P bo'linish olinganda ham, bu bo'linishning (L) chiziq bilan umumiy nuqtaga ega bo'lgan

bo'laklari yuzlarining yig'indisi berilgan $\varepsilon > 0$ dan kichik bo'ladi. P bo'linishga nisbatan $f(x, y)$ va $F(x, y)$ funksiyalarning integral yig'indilarini tuzamiz:

$$\sigma_P(f) = \sum_{k=1}^n f(\xi_k, \eta_k) D_k, \quad \sigma_P(F) = \sum_{k=1}^n F(\xi_k, \eta_k) D_k.$$

Ravshanki, $\sigma_P(f) = \sum_{k=1}^{n'} f(\xi_k, \eta_k) D_k + \sum_{k''=1}^n F(\xi_k, \eta_k) D_k$, bunda $\sum_{k'=1}^{n'}$ yig'indi (L) chiziq bilan umumiy nuqtaga ega bo'lgan (D_k) bo'laklar bo'yicha olingan, $\sum_{k''=1}^n$ yig'indi esa, qolgan barcha hadlardan tashkil topgan yig'indi. Xuddi shunday

$$\sigma_P(F) = \sum_{k'=1}^{n'} F(\xi_{k'}, \eta_{k'}) D_{k'} + \sum_{k''=1}^n F(\xi_{k''}, \eta_{k''}) D_{k''}$$

Bu tengliklardan, $\forall (x, y) \in (D) \setminus (L)$ uchun $f(x, y) = F(x, y)$ ekanligini e'tiborga olsak, unda

$$|\sigma_P(f) - \sigma_P(F)| \leq \sum_{k'=1}^{n'} |f(\xi_{k'}, \eta_{k'}) - F(\xi_{k'}, \eta_{k'})| D_{k'} \leq M \sum_{k'=1}^{n'} D_{k'} < M\varepsilon,$$

bunda $M = \sup |f(\xi_k, \eta_k) - F(\xi_k, \eta_k)|$, $\forall (x, y) \in (D) \setminus (L)$. Keyingi tengsizlikda $\lambda_P \rightarrow 0$ limitga o'tib $\iint_{(D)} f(x, y) dD = \iint_{(D)} F(x, y) dD$ ekanligini topamiz. ■

2°. $f(x, y)$ funksiya (D) sohada berilgan bo'lib, (D) soha nol yuzali Γ chiziq yordamida (D_1) va (D_2) sohalarga ajralgan bo'lsin. Agar $f(x, y)$ funksiya (D) sohada integrallanuvchi bo'lsa, u (D_1) va (D_2) sohalarda ham integrallanuvchi bo'ladi va aksinch (D_1) va (D_2) sohalarda integrallanuvchi bo'lsa, u holda (D) sohada ham integrallanuvchi bo'ladi hamda

$$\iint_{(D)} f(x, y) dD = \iint_{(D_1)} f(x, y) dD + \iint_{(D_2)} f(x, y) dD$$

munosabat o'rinli.

Isboti. Haqiqatan ham, (D_1) va (D_2) sohalarni ixtiyoriy ravishda bo'laklarga bo'lganda, (D) soha ham (D'_1) , $(D'_2), \dots, (D'_n)$ sohalarga bo'linadi. Agar (D') sohaga qarashli yig'indini k' bilan, (D'') sohaga qarashlisini k'' bilan belgilasak, u holda

$$\sum_{k=1}^n \omega_k D_k = \sum_{k'=1}^n \omega_{k'} D_{k'} + \sum_{k''=1}^n \omega_{k''} D_{k''}$$

bo'ladi. $f(x, y)$ funksiya (D) sohada integrallanuvchi bo'lsa, u holda $\lambda_p \rightarrow 0$ da chap tomondagi yig'indi nolga intiladi, natijada o'ng tomondagi yig'indilar ham nolga intiladi. Bundan $f(x, y)$ funksiyaning (D_1) va (D_2) sohalarda integrallanuvchi ekanligi kelib chiqadi.

Aksincha, yuqoridagi tenglikning o'ng tomonidagi yig'indilar $\lambda_p \rightarrow 0$ da nolga intilsa, u holda chap tomon ham nolga intiladi, lekin u ixtiyoriy bo'linish uchun tuzilmasidan (D_1) va (D_2) sohalarning bo'linish bo'yicha tuzilgan. (D) sohaning ixtiyoriy bo'linishidan yuqorigagi xususiy bo'linishga kelish uchun (D) sohaning bo'linish chiziqlariga Γ chiziqni qo'shish etarli. Bu bo'linishga mos kelgan yig'indi, oldingi bo'linishga mos kelgan yig'indidan farqi faqat Γ chiziq bilan umumiy nuqtaga ega bo'lgan qo'shiluvchilarga farq qiladi. Bu qo'shiluvchilar yuzlarining yig'indisi **13.5.2-lemmaga** asosan, $\lambda_p \rightarrow 0$ da nolga intiladi, ya'ni ikkala yig'indilarning bir-biridan farqi istalgancha kichik bo'ladi.

Shunday qilib, (13.4.3) shart umumiy holda bajariladi. Demak, $f(x, y)$ funksiya (D) sohada integrallanuvchi. Isbot qilishi kerak bo'lgan formula

$$\sigma_p(f) = \sum_{k=1}^n f(\xi_k, \eta_k) D_k = \sum_{k'=1}^n f(\xi_{k'}, \eta_{k'}) D_{k'} + \sum_{k''=1}^n f(\xi_{k''}, \eta_{k''}) D_{k''}$$

tenglikdan $\lambda_p \rightarrow 0$ da limitga o'tish natijasida kelib chiqadi, ya'ni

$$\iint_{(D)} f(x, y) dD = \iint_{(D_1)} f(x, y) dD_1 + \iint_{(D_2)} f(x, y) dD_2. \blacksquare$$

Xuddi shunday quyidagi xossalar ham integral yig'indi tuzib, so'ngra limitga o'tish bilan isbotlanadi.

3°. Agar $f(x, y)$ funksiya (D) sohada integrallanuvchi bo'lsa, u holda $C \cdot f(x, y)$ ($C = \text{const}$) funksiya ham shu sohada integrallanuvchi bo'ladi va

$$\iint_{(D)} C \cdot f(x, y) dD = C \iint_{(D)} f(x, y) dD$$

formula o'rinli.

4°. Agar $f(x, y)$ va $g(x, y)$ funksiyalar (D) sohada integrallanuvchi bo'lsa, u holda $f(x, y) \pm g(x, y)$ funksiya ham shu sohada integrallanuvchi bo'ladi va

$$\iint_{(D)} [f(x, y) \pm g(x, y)] dD = \iint_{(D)} f(x, y) dD \pm \iint_{(D)} g(x, y) dD$$

formula o'rinli.

13.6.1. Natija. Agar $f_1(x, y), f_2(x, y), \dots, f_n(x, y)$ funksiyalarning har biri (D) sohada integrallanuvchi bo'lsa, u holda

$$\sum_{k=1}^n C_k \cdot f_k(x, y) (C_k = \text{const}, k = 1, 2, \dots, n)$$

funksiya ham shu sohada integrallanuvchi bo'ladi va

$$\iint_{(D)} \sum_{k=1}^n C_k \cdot f_k(x, y) dD = \sum_{k=1}^n C_k \cdot \iint_{(D)} f_k(x, y) dD$$

tenglik o'rinli.

5°. Agar $f(x, y)$ funksiya (D) sohada integrallanuvchi bo'lib, $\forall (x, y) \in (D)$ uchun $f(x, y) \geq 0$ bo'lsa, u holda

$$\iint_{(D)} f(x, y) dD \geq 0$$

tengsizlik bajariladi.

13.6.2. Natija. Agar $f(x, y)$ va $g(x, y)$ funksiyalar (D) sohada integrallanuvchi bo'lib, $\forall (x, y) \in (D)$ uchun $f(x, y) \leq g(x, y)$ bo'lsa, u holda

$$\iint_{(D)} f(x, y) dD \leq \iint_{(D)} g(x, y) dD$$

tengsizlik o'rinli.

6°. Agar $f(x, y)$ funksiya (D) sohada integrallanuvchi bo'lsa, u holda $|f(x, y)|$ funksiya ham shu sohada integrallanuvchi bo'ladi va

$$\left| \iint_{(D)} f(x, y) dD \right| \leq \iint_{(D)} |f(x, y)| dD \quad (13.6.3)$$

tengsizlik o'rinli.

Isboti. $|f(x, y)|$ funksiyaning ixtiyoriy (D) sohadagi tebranish ω_k^* , $f(x, y)$ funksiyaning shu sohadagi tebranish ω_k oshmaydi, ya'ni $\omega_k^* \leq \omega_k$. U holda $\sum_{k=1}^n \omega_k^* D_k \leq \sum_{k=1}^n \omega_k D_k$ tengsizlik o'rinli bo'ladi. Bundan, $f(x, y)$ funksiya (D) sohada integrallanuvchi bo'lgani uchun tengsizlikning o'ng tomonidagi yig'indining $\lambda_p \rightarrow 0$ ning nolga intilishidan chap tomondagi yig'indining nolga intilishi kelib chiqadi, bundan esa, (13.4.3) shartga asosan, $|f(x, y)|$ funksiya (D) sohda integrallanuvchi bo'ladi. (13.6.3) tengsizlikning o'rinli bo'lishi esa ushbu

$$\left| \sum_{k=1}^n f(\xi_k, \eta_k) D_k \right| \leq \sum_{k=1}^n |f(\xi_k, \eta_k)| D_k$$

tengsizlikdan limitga o'tishdan natijasida kelib chiqadi.

7°. *O'rta qiymat haqidagi teorema.* $f(x, y)$ funksiya yuzaga ega bo'lgan (D) sohada aniqlangan va chegralangan bo'lsin, ya'ni $\exists M, m \in \mathbb{R}, \forall (x, y) \in (D)$ uchun: $m \leq f(x, y) \leq M$. Agar $f(x, y)$ funksiya (D) sohada integrallanuvchi bo'lsa, u holda shunday o'zgarmas μ son $m \leq \mu \leq M$ mavjudki,

$$\iint_{(D)} f(x, y) dD = D \mu \quad (13.6.4)$$

formula o'rinli, bu yerda $D = \text{yuzi}(D)$ sohaning yuzi.

Isboti. Ravshanki, $mD \leq \sum_{k=1}^n f(\xi_k, \eta_k) D_k \leq MD$, bundan $\lambda_p \rightarrow 0$ da limitga o'tish natijasida $mD \leq \iint_{(D)} f(x, y) dD \leq MD$ ni topamiz. Bu tengsizlikdan

$$m \leq \frac{\iint_{(D)} f(x, y) dD}{D} \leq M, \quad \mu = \frac{\iint_{(D)} f(x, y) dD}{D}$$

deb belgilasak, u holda (13.6.4) tenglikni hosil qilamiz $\iint_{(D)} f(x, y) dD = D \mu$. ■

13.6.5. Natija. Agar $f(x, y)$ funksiya yopiq (D) sohada uzluksiz bo'lsa, u holda shunday $(a, b) \in (D)$ nuqta topiladiki,

$$\iint_{(D)} f(x, y) dD = f(a, b) D$$

tenglik o'rinli bo'ladi.

8⁰. O'rta qiymat haqidagi umumlashgan teorema. Agar $g(x, y)$ funksiya (D) sohada integrallanuvchi bo'lib, u shu sohada o'z ishorasini o'zgartirmasa hamda $f(x, y)$ funksiya (D) sohada uzluksiz bo'lsa, u holda shunday $(a, b) \in (D)$ nuqta topiladiki,

$$\iint_{(D)} f(x, y) g(x, y) dD = f(a, b) \iint_{(D)} g(x, y) dD$$

tenglik o'rinli bo'ladi.

Biz kelgusida kerak bo'ladigan ba'zi tengliklarni keltiramiz:

$$\sum_{i=1}^n i = \frac{1}{2} n(n+1), \quad \sum_{i=1}^n (i+a) = \frac{1}{2} n(n+1) + an, \quad (13.6.6)$$

$$\sum_{i=1}^n (2i-1) = n^2,$$

$$\sum_{i=1}^n i^2 = \frac{1}{6} n(n+1)(2n+1), \quad (13.6.7)$$

$$\sum_{i=1}^n i^3 = \frac{1}{4} n^2 (n+1)^2,$$

$$\sum_{k=1}^n \sin k\alpha = \frac{\sin \frac{n+1}{2} \alpha \sin \frac{n}{2} \alpha}{\sin \frac{\alpha}{2}}, \quad \forall \alpha \neq 2m\pi, m \in Z, \quad (13.6.8)$$

$$\sum_{k=1}^n \cos k\alpha = \frac{\sin \frac{n}{2} \alpha \cos \frac{n+1}{2} \alpha}{\sin \frac{\alpha}{2}}, \quad \forall \alpha \neq 2m\pi, m \in Z,$$

$$\sum_{k=1}^n \sin(x+k\alpha) = \frac{\sin \frac{n}{2} \alpha \sin(x+\frac{n+1}{2} \alpha)}{\sin \frac{\alpha}{2}}, \quad \forall \alpha \neq 2m\pi, m \in Z$$

$$\sum_{k=1}^n \cos(x+k\alpha) = \frac{\sin \frac{n}{2} \alpha \cos(x+\frac{n+1}{2} \alpha)}{\sin \frac{\alpha}{2}}, \quad \forall \alpha \neq 2m\pi, m \in Z;$$

13.6.9. Misol. Ushbu $\iint_{(D)} (x+y) dx dy$ integralni integral yig'indi yordamida hisoblang, bunda $(D) = \{(x, y) \in R^2 : 0 \leq x \leq 1, 0 \leq y \leq 1\}$.

Yechilishi. Berilgan $f(x, y) = x + y$ funksiya (D) sohada uzluksiz, demak, 13.5.1- teoreмага ko'ra, u (D) sohada integrallanuvchi bo'ladi.

(D) sohani $x = \frac{i}{n}, y = \frac{j}{n} \ (i, j = \overline{1, n})$ to'g'ri chiziqlar yordamida bo'laklarga ajratamiz va har bir (D_{ij}) sohada $(x_i, y_j) = (\frac{i}{n}, \frac{j}{n})$ deb qaraymiz. U holda

$$\Delta x_i = x_i - x_{i-1} = \frac{1}{n}, \Delta y_j = y_j - y_{j-1} = \frac{1}{n}, D_{ij} = \Delta x_i \Delta y_j = \frac{1}{n^2}, f(x_i, y_j) = \frac{i}{n} + \frac{j}{n},$$

(13.6.6) formulani e'tiborga olgan holda, integral yig'indini tuzib, uni hisoblaymiz:

$$\begin{aligned} \sigma &= \sum_{i=1}^n \sum_{j=1}^n f(x_i, y_j) D_{ij} = \frac{1}{n^3} \sum_{i=1}^n \sum_{j=1}^n (i+j) = \frac{1}{n^3} \sum_{i=1}^n [in + \frac{n(n+1)}{2}] = \\ &= \frac{1}{n^3} [\frac{n^2(n+1)}{2} + \frac{n^2(n+1)}{2}] = \frac{n+1}{n}. \end{aligned}$$

Endi limitni hisoblaymiz: $\lim_{n \rightarrow \infty} \sigma = \lim_{n \rightarrow \infty} \frac{n+1}{n} = 1$.

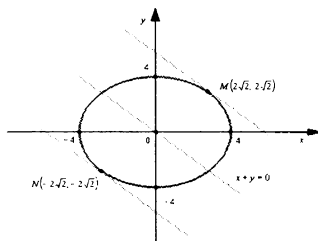
$$\text{Demak, } \iint_{(D)} (x+y) dx dy = \lim_{n \rightarrow \infty} \sigma = 1.$$

13.6.10. Misol. Quyidagi $\iint_{(D)} (x+y-5) dx dy$ ikki karrali integralni

baholang, bunda integrallash sohasi $(D) - x^2 + y^2 \leq 16$ doiradan iborat.

Yechilishi. $x^2 + y^2 \leq 16$ doirada $f(x, y) = x + y - 5$ funksiyaning eng kichik va eng katta qiymatini topamiz va 7^o- xossadan foydalanib, berilgan integralni baholaymiz. $x + y = 0$ to'g'ri chiziqda $z = x + y$ funksiya 0 qiymat qabul qiladi. $x + y = 0$ to'g'ri chiziqqa parallel bo'lgan $x + y = c$ to'g'ri chiziqda z funksiya c qiymatni qabul qiladi.

Demak, $z = x + y$ funksiya doiradagi $M(2\sqrt{2}, 2\sqrt{2})$ nuqtada mak-simal qiymat, $N(-2\sqrt{2}, -2\sqrt{2})$ nuqtada esa, minimal qiymat qabul qiladi. Bu nuqталarda $f(x, y)$ funksiya $f(M) = 4\sqrt{2} - 5$, $f(N) = -4\sqrt{2} - 5$ qiymatlarga ega bo'ladi (13.3 - chizma).



Bizga ma'lumki, doiraning yuzi $\pi R^2 = 16\pi$ teng.

13.3 - chizma.

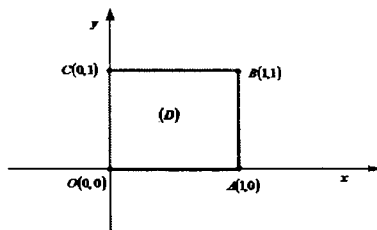
Ikki karrali integralning 7^o-xossasiga asosan, $(m = -4\sqrt{2} - 5, M = 4\sqrt{2} + 5)$ quyidagi

$$-16\pi(4\sqrt{2} + 5) \leq \iint_{(D)} (x + y - 5) dx dy \leq 16\pi(4\sqrt{2} + 5)$$

tengsizlik o'rinli bo'ladi.

13.6.11. Misol. $x = 0, x = 1, y = 0, y = 1$ to'g'ri chiziqlar bilan chegaralangan (D) sohada $f(x, y) = x + y$ funksiyaning integral o'rta qiymatini hisoblang.

Yechilishi. (D) soha — uchlari $O(0,0), A(1,0), B(1,1), C(0,1)$, nuqtalarda bo'lgan $OABC$ to'g'ri to'rtburchakdan iborat (13.4- chizma). O'rta qiymat haqidagi teorema asosan, $f(x, y)$ funksiya (D) yopiq sohada uzluksiz, u holda (D) sohada shunday $(\bar{x}, \bar{y}) \in (D)$ nuqta



13. 4-chizma.

topiladiki, $\frac{1}{D} \iint_{(D)} f(x, y) dD = f(\bar{x}, \bar{y})$ formula o'rinli bo'ladi, bunda

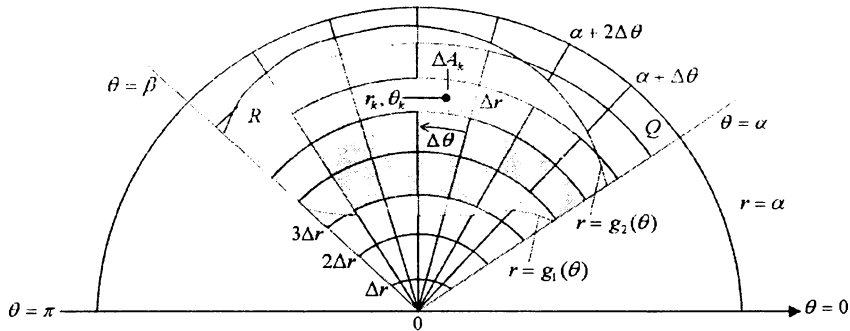
$D = (D)$ sohaning yuzasi.

Ravshanki, $OABC$ to'g'ri to'rtburchakning yuzi $D = 1$. U holda, 13.6.9-misolni e'tiborga olgan holda va 13.6.4-natijaga ko'ra, $\iint_{(D)} (x + y) dx dy = f(\bar{x}, \bar{y}) D$ bo'ladi.

Shunday qilib, integralning o'rta qiymati $f(\bar{x}, \bar{y})=1$.

13.7. Qutb koordinatalari sistemasida ikki karrali integral. Biz xOy tekislikdagi (D) soha bo'yicha $f(x, y)$ funksiyadan olingan ikki karrali integral aniqlaganda (D) soha tomonlari koordinatalar o'qlariga parallel bo'lgan to'g'ri to'rtburchaklardan iborat.

Faraz qilaylik, $f(r, \theta)$ funksiya (Δ) sohada aniqlangan bo'lib, bu soha $\theta = \alpha$ va $\theta = \beta$ nurlar hamda $r = g_1(\theta)$ va $r = g_2(\theta)$ uzluksiz egri chiziqlar bilan chegaralanganidir. Shuningdek, θ ning α dan β gacha bo'lgan har bir qiymati uchun, $0 \leq g_1(\theta) \leq g_2(\theta) \leq a$ munosabat o'rinli bo'lsin, deb faraz qilamiz. Bu holda (Δ) soha $0 \leq r \leq a$ va $\alpha \leq \theta \leq \beta$ tengsizliklar yordamida aniqlangan maxsus tuzilishga ega (Q) sohada yotadi (13.5-chizmaga q.).



13.5-chizma.

Berilgan $(\Delta): g_1(\theta) \leq r \leq g_2(\theta), \alpha \leq \theta \leq \beta$ soha maxsus ko'rinishdagi $(Q): 0 \leq r \leq a, \alpha \leq \theta \leq \beta$ sohada joylashgan. (Q) sohani doiraviy yoylar va nurlar vositasida qismlarga bo'lish, (Δ) ning bo'linishiga olib keladi.

Biz (Q) sohani aylana yoylari va nurlar yordamida bo'laklarga bo'lamiz (to'r bilan qoplaymiz). Yoylar - markazlari koordinatalar boshida bo'lib, radiuslari $\Delta r, 2\Delta r, \dots, m\Delta r$ bo'lgan konsentrik aylanalarga tegishli, bunda $\Delta r = \frac{a}{m}$. Nurlar esa, $\theta = \alpha, \theta = \alpha + \Delta\theta,$

$\theta = \alpha + 2\Delta\theta, \dots, \theta = \alpha + m'\Delta\theta = \beta$, munosabatlar vositasida berilgan, bunda $\Delta\theta = \frac{\beta - \alpha}{m'}$. Yoylar va nurlar (Q) ni «qutb to'g'ri to'rtburchaklari» deb ataladigan kichik qismlarga (bo'laklarga) bo'lishadi.

(Δ)sohaning ichida yotgan qutb to'g'ri to'rtburchaklarini ularning

$$\Delta A_1, \Delta A_2, \dots, \Delta A_n$$

yuzalari bilan birga raqamlab chiqamiz (tartiblaymiz).

(r_k, θ_k) nuqta orqali yuzasi ΔA_k bo'lgan qutb to'g'ri to'rtburchagining markazini belgilaymiz. Bunda «markaz» deganda, aylanalar yoylaridan hamda ularni kəsib o'tuvchi nurlardan baravar uzoqlikda joylashgan nuqta tushuniladi. Endi biz

$$S_n = \sum_{k=1}^n f(r_k, \theta_k) \Delta A_k \quad (13.7.1)$$

yig'indini tuzamiz.

Agar f funksiya (Δ) da uzluksiz bo'lsa, bu yig'indi to'rni biz Δr va $\Delta\theta$ lar nolga intiladigan qilib, bo'laklarga bo'lganimizda, chekli limitga intiladi. Ana shu limit f funksiyadan (Δ) soha bo'yicha olingan ikki karali integral deyiladi va

$$\lim_{n \rightarrow \infty} S_n = \iint_{(\Delta)} f(r, \theta) dA$$

kabi yoziladi. Bu limitni hisoblash uchun, avvalo biz S_n yig'indini, ΔA_k ni Δr va $\Delta\theta$ lar orqali ifodalashimiz kerak bo'ladi. Ma'lumki,

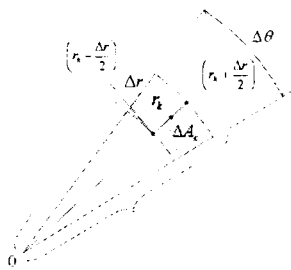
ΔA_k ni chegaralovchi ichki yoyning radiusi $r_k - \frac{\Delta r}{2}$ (13.6-chizma). Uni

chegaralovchi tashqi yoyning radiusi esa, $r_k + \frac{\Delta r}{2}$. U holda markazi

koordinatalar boshida bo'lib, bu yoylar yordamida hosil qilingan doiraviy sektorlarning yuzalari, mos ravishda, quyidagicha bo'ladi: Ichki

radiusli sektorning yuzi: $\frac{1}{2}(r_k - \frac{\Delta r}{2})^2 \Delta\theta$; tashqi radiusli

sektorning yuzi: $\frac{1}{2}(r_k + \frac{\Delta r}{2})^2 \Delta\theta$.



13.6- chizma. $\Delta A_k = (\text{katta sektor yuzi}) - (\text{kichik sektor yuzi}) = r_k \Delta r \Delta \theta$ ekanligining namoyishi.

Shunday qilib,

$$\Delta A_k = \text{katta sektorning yuzi} - \text{kichik sektorning yuzi} = \frac{\Delta \theta}{2} \left[\left(r_k + \frac{\Delta r}{2} \right)^2 - \left(r_k - \frac{\Delta r}{2} \right)^2 \right] = \frac{\Delta \theta}{2} (2r_k \Delta r) = r_k \Delta r \Delta \theta.$$

(13.7.2) U holda, (13.7.2) ni (13.7.1) ga keltirib qo'ysak,

$$S_n = \sum_{k=1}^n f(r_k, \theta_k) r_k \Delta r \Delta \theta$$

munosabatga ega bo'lamiz. Agar $(\Delta): g_1(\theta) \leq r \leq g_2(\theta), \alpha \leq \theta \leq \beta$ bo'lsa, u holda

$$\iint_{(\Delta)} f(r, \theta) dA = \int_{\alpha}^{\beta} d\theta \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r, \theta) r dr \quad (13.7.3)$$

takroriy formula o'rinli bo'ladi.

13.7.4 Misol. Ushbu $\iint_{(\Delta)} r \sin \theta dA$ ikki karrali integralni qutb koordinatalar sistemasidagi ta'rifi bo'yicha hisoblang, bunda

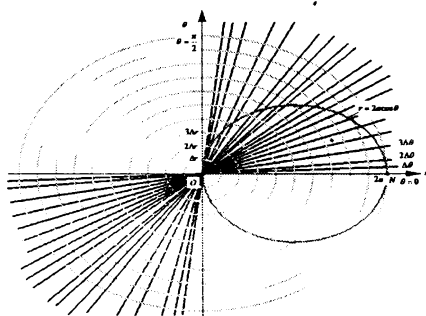
$$(\Delta) = \left\{ (r, \theta): 0 \leq \theta \leq \frac{\pi}{2}; 0 \leq r \leq 2a \cos \theta \right\}.$$

Yechilishi. (Δ) soha 13.7-chizma tasvirlangan. Qutb koordinatalar sistemasida (Δ) sohani konsentrik aylanalar va $O(0,0)$ nuqtadan chiqadigan nurlar yordamida n^2 ta $(A_{ik}) (i = \overline{1, n}, k = \overline{1, n})$ sohachalarga ajratamiz. Konsentrik aylanalarni quyi va yuqori chegara tenglamalari

mos ravishda $r = a = 0$, $r = b = 2a$ ga, $O(0,0)$ nuqtadan chiquvchi nurlarning quyi va yuqori tenglamalari mos ravishda $\theta = 0$, $\theta = \frac{\pi}{2}$ teng bo'ladi.

1) $ON = 2a - 0 = 2a$ kesmani n ta teng $\Delta r = \frac{2a}{n}$ bo'lakka ajratib, $(n-1)$ ta $r_i = \frac{1}{2} \Delta r i = \frac{2a}{2n} i = \frac{a}{n} i$ ($i = \overline{1, n-1}$) tenglamalar bilan tasvirlanadigan konsentrik aylanalar o'tkazamiz.

2) $\theta = \alpha = 0$ va $\theta = \frac{\pi}{2}$ nurlar orasidagi $\frac{\pi}{2}$ ga teng bo'lgan burchakni, har biri $\Delta\theta = \frac{\frac{\pi}{2} - 0}{n} = \frac{\pi}{2n}$ ga teng bo'lgan n ta burchaklarga ajratuvchi $(n-1)$ ta nur o'tkazamiz. Ularning tenglamalari $\theta_k = 0 + \Delta\theta \cdot k = \frac{\pi}{2n} k$ ($k = \overline{1, n-1}$) ko'rinishga ega bo'ladi.



13.7-chizma.

Endi (A_{ik}) sohaning yuzasi (13.7.2) formulaga asosan,

$$A_{ik} = r_i \Delta r \Delta \theta$$

ko'rinishda bo'ladi. (A_{ik}) sohaning o'rtasidan $N_{ik} = (r_i, \theta_k) = (\frac{a}{n} i, \frac{\pi}{2n} k)$ nuqtani olamiz. $f(r, \theta) = r \sin \theta$ funksiyaning N_{ik} nuqtadagi qiymatini hisoblaymiz

$$f(N_{ik}) = r_i \sin \theta_k = \frac{a}{n} i \sin \frac{\pi}{2n} k.$$

Endi (13.7.1) formulaga asosan, integral yig'indi tuzamiz:

$$\begin{aligned} \sigma(n, n) &= \sum_{i=1}^n \sum_{k=1}^n f(r_i, \theta_k) A_{ik} = \sum_{i=1}^n \sum_{k=1}^n r_i^2 \sin \frac{\pi}{2n} k \Delta r \Delta \theta = \\ &= \frac{2a}{n} \cdot \frac{\pi}{2n} \sum_{n=1}^n \left(\frac{a}{n} i \right)^2 \sum_{k=1}^n \sin \frac{\pi}{2n} k = \frac{a^3 \pi}{n^4} \sum_{i=1}^n i^2 \sum_{k=1}^n \sin \frac{\pi}{2n} k \end{aligned}$$

yig'indilarni hisoblashda (13.6.7), (13.6.8) formulalardan foydalanamiz:

$$\begin{aligned}\sigma(n, n) &= \frac{a^3 \pi}{n^4} \frac{1}{6} n(n+1)(2n+1) \cdot \frac{\sin \frac{(n+1)\pi}{4n} \sin \frac{\pi}{4}}{\sin \frac{\pi}{4n}} = \\ &= \frac{a^3}{6} \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right) \frac{\sqrt{2}}{2} \cdot 4 \cdot \sin \left(1 + \frac{1}{n}\right) \frac{\pi}{4} \cdot \frac{\frac{\pi}{4n}}{\sin \frac{\pi}{4n}}.\end{aligned}$$

oxirgi ko'paytuvchi uchun $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ birinchi ajoyib limitdan foydalanib, $\sigma(n, n)$ yig'indini $n \rightarrow \infty$ da limitini hisoblaymiz:

$$\begin{aligned}\lim_{n \rightarrow \infty} \sigma(n, n) &= \lim_{n \rightarrow \infty} \frac{a^3}{6} \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right) \frac{\sqrt{2}}{2} \cdot 4 \cdot \sin \left(1 + \frac{1}{n}\right) \frac{\pi}{4} \cdot \frac{\frac{\pi}{4n}}{\sin \frac{\pi}{4n}} = \\ &= \frac{a^3}{6} \cdot 2 \left(\frac{\sqrt{2}}{2}\right)^2 \cdot 4 = \frac{a^3}{6} \cdot 2 \cdot \frac{2}{4} \cdot 4 = \frac{2a^3}{3}.\end{aligned}$$

Endi (13.7.3) formuladan foydalanib, berilgan integralni hisoblaymiz:

$$\begin{aligned}\iint_{(\Delta)} r^2 \sin \theta dr d\theta &= \int_0^{\pi/2} \sin \theta d\theta \int_0^{2a \cos \theta} r^2 dr = \\ &= \int_0^{\pi/2} \sin \theta \left(\frac{r^3}{3}\right) \Big|_{r=0}^{r=2a \cos \theta} d\theta = \frac{8a^3}{3} \int_0^{\pi/2} \cos^3 \theta \sin \theta d\theta = \frac{8a^3}{3} \frac{\cos^4 \theta}{4} \Big|_0^{\pi/2} = \frac{2a^3}{3}.\end{aligned}$$

58-§. Ikki karrali integrallarni hisoblash va o'zgaruvchilarni almashtirish

Yuqorida $f(x, y)$ funksiyaning (D) sohada ikki karrali integrali integral yig'indining limiti sifatida ta'riflandi. Bu limit murakkab xarakterga ega bo'lib, uni shu ta'rif bo'yicha hisoblash sodda hollarda ham ancha qiyinchilik tug'diradi.

13.8. Ikki karrali integrallarni takroriy integrallarga keltirib hisoblash.

Soha to'g'ri to'rt to'rtburchak bo'lgan holda ikki karrali integralni hisoblash

13.8.1. Teorema. $f(x, y)$ funksiya $(D) = \{(x, y) \in R^2 : a \leq x \leq b, c \leq y \leq d\}$ sohada berilgan va integrallanuvchi bo'lsin. Agar $x(x \in [a, b])$ o'zgaruvchining har bir tayin qiymatida

$$J(x) = \int_c^d f(x, y) dy$$

integral mavjud bo'lsa, u holda

$$\int_a^b J(x) dx = \int_a^b \left[\int_c^d f(x, y) dy \right] dx$$

integral ham mavjud bo'ladi va

$$\iint_{(D)} f(x, y) dD = \int_a^b \left[\int_c^d f(x, y) dy \right] dx \quad (13.8.2)$$

formula o'rinni.

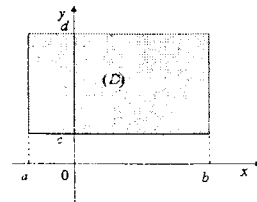
Isboti. (D) sohani $a = x_0 < x_1 < x_2 < \dots < x_n = b$ va $c = y_0 < y_1 < y_2 < \dots < y_m = d$ nuqtalar yordamida

$$(D_{ik}) = \{(x, y) \in R^2 : x_i \leq x \leq x_{i+1}, y_k \leq y \leq y_{k+1}\}$$

$(i = 0, 1, 2, \dots, n-1; k = 0, 1, 2, \dots, m-1)$ bo'laklarga bo'lamiz (13.8-chizma). Bu bo'linishni P_{nm} deb belgilaymiz. Uning diametri

$$\lambda_{P_{nm}} = \max \sqrt{\Delta x_i^2 + \Delta y_k^2}, \quad \Delta x_i = x_{i+1} - x_i, \quad \Delta y_k = y_{k+1} - y_k.$$

Shartga ko'ra $f(x, y)$ funksiya (D) sohada integrallanuvchi bo'lganligi uchun u shu sohada chegaralangan bo'ladi. Demak, $f(x, y)$ funksiya har bir (D_{ik}) sohada chegaralangan va shu sohada aniq quyi va aniq yuqori chegaraga ega bo'ladi:



13.8-chizma.

$m_{ik} = \inf \{f(x, y) : (x, y) \in (D_{ik})\}, M_{ik} = \sup \{f(x, y) : (x, y) \in (D_{ik})\},$
 $(i = 0, 1, 2, \dots, n-1; k = 0, 1, 2, \dots, m-1). \quad \forall (x, y) \in (D_{ik}) \text{ uchun } m_{ik} \leq f(x, y) \leq M_{ik}$
 Bu tengsizlikda x ni ixtiyoriy ravishda $x = \xi_i (\forall \xi_i \in [x_i, x_{i+1}])$ deb belgilab,
 bu tengsizlikni y bo'yicha y_k dan y_{k+1} gacha integrallaymiz:

$$m_{ik} \Delta y_k \leq \int_{y_k}^{y_{k+1}} f(\xi_i, y) dy \leq M_{ik} \Delta y_k,$$

bunda $\Delta y_k = y_{k+1} - y_k$. Keyingi tengsizliklarni k ning $k = 0, 1, 2, \dots, m-1$ qiymatlarida yozib, ularni hadlab qo'shib

$$\sum_{k=0}^{m-1} m_{ik} \Delta y_k \leq J(\xi_i) = \int_c^d f(\xi_i, y) dy \leq \sum_{k=0}^{m-1} M_{ik} \Delta y_k$$

tengsizlikni hosil qilamiz. Bu tengsizliklarni $\Delta x_i (\Delta x_i = x_{i+1} - x_i)$ ga ko'paytirib, so'ngra i bo'yicha 0 dan $n-1$ gacha yig'indi olamiz:

$$\begin{aligned}
 \sum_{i=0}^{n-1} \Delta x_i \sum_{k=0}^{m-1} m_{ik} \Delta y_k &\leq \sum_{i=0}^{n-1} J(\xi_i) \Delta x_i \leq \sum_{i=0}^{n-1} \Delta x_i \sum_{k=0}^{m-1} M_{ik} \Delta y_k, \\
 \sum_{i=0}^{n-1} \sum_{k=0}^{m-1} m_{ik} \Delta x_i \Delta y_k &\leq \sum_{i=0}^{n-1} J(\xi_i) \Delta x_i \leq \sum_{i=0}^{n-1} \sum_{k=0}^{m-1} M_{ik} \Delta x_i \Delta y_k. \quad (13.8.3)
 \end{aligned}$$

(D_{ik}) to'rtburchak diametrining eng kattasi nolga ($\lambda_p \rightarrow 0$) intilsin. U holda Δx larning kattasi ham nolga intiladi.

Ravshanki, (13.8.3) tengsizlikning chap va o'ng tomonlari mos ravishda Darbuning quyi va yuqori yig'indisini ifodalaydi. Shartga ko'ra $f(x, y)$ funksiya (D) sohada integrallanuvchi bo'lgani uchun

$$\lambda_p \rightarrow 0 \text{ da } s \rightarrow \iint_{(D)} f(x, y) dx dy, \quad S \rightarrow \iint_{(D)} f(x, y) dx dy.$$

U holda (13.8.3)ning o'rtasidagi ifoda ham $\lambda_p \rightarrow 0$ da $\iint_{(D)} f(x, y) dx dy$ ga intiladi, ya'ni

$$\int_a^b J(x) dx = \lim_{\lambda_p \rightarrow 0} \sum_{i=0}^{n-1} J(\xi_i) \Delta x_i = \iint_{(D)} f(x, y) dx dy.$$

$$\text{Bundan } \iint_{(D)} f(x, y) dx dy = \int_a^b J(x) dx = \int_a^b \left[\int_c^d f(x, y) dy \right] dx \text{ ekanligi kelib}$$

chiqadi. ■

13.8.4. Teorema. $f(x, y)$ funksiya (D) sohada berilgan va integrallanuvchi bo'lsin. Agar $y \in [c, d]$ o'zgaruvchining har bir tayin qiymatida $I(y) = \int_a^b f(x, y) dx$ integral mavjud bo'lsa, u holda

$\int_c^d \left[\int_a^b f(x, y) dx \right] dy$ integral ham mavjud bo'ladi va

$$\iint_{(D)} f(x, y) dD = \int_c^d \left[\int_a^b f(x, y) dx \right] dy$$

formula o'rinni.

13.8.4. Natija. Agar $f(x, y)$ funksiya chegaralangan yopiq $(D) \subset R^2$ sohada berilgan va uzluksiz bo'lsa,

$$\iint_{(D)} f(x, y) dD, \quad \int_a^b \left[\int_c^d f(x, y) dy \right] dx, \quad \int_c^d \left[\int_a^b f(x, y) dx \right] dy$$

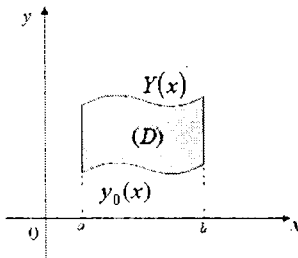
integrallarning har biri mavjud va ular o'zaro teng bo'ladi.

Soha egri chiziqli bo'lgan holda ikki karrali integralni hisoblash

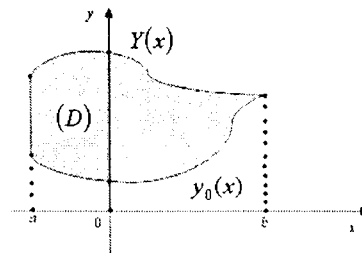
(D) soha, yuqoridan $y = Y(x)$, pastdan $y = y_0(x)$ egri chiziqlar bilan, yon tomonlardan esa $x = a$ va $x = b$ to'g'ri chiziqlar bilan chegaralangan soha, ya'ni

$$(D) = \{(x, y) \in R^2 : a \leq x \leq b, y_0(x) \leq y \leq Y(x)\}$$

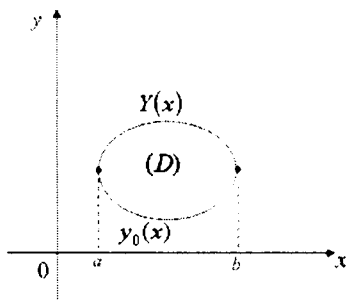
bo'lsin, bunda $y_0(x)$ va $Y(x)$ funksiyalar $[a, b]$ da uzluksiz deb faraz qilinadi (13.9-, 13.10-, 13.11-chizma).



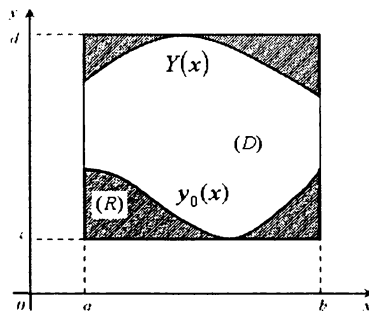
13.9-chizma.



13.10-chizma.



13.11-chizma.



13.12-chizma.

13.8.5. Teorema. $f(x, y)$ funksiya

$$(D) = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b, y_0(x) \leq y \leq Y(x)\}, y_0(x), Y(x) \in C([a, b])$$

sohada berilgan va integrallanuvchi bo'lsin (13.12-chizma). Agar

$$x \in [a, b] \text{ o'zgaruvchining har bir tayin qiymatida } J(x) = \int_{y_0(x)}^{Y(x)} f(x, y) dy$$

integral mavjud bo'lsa, u holda

$$\int_a^b dx \int_{y_0(x)}^{Y(x)} f(x, y) dy$$

integral ham mavjud bo'ladi va

$$\iint_{(D)} f(x, y) dx dy = \int_a^b dx \int_{y_0(x)}^{Y(x)} f(x, y) dy \quad (13.8.6)$$

tenglik o'rinli.

Isboti. Teoremaning isbotini 13.8.1-teoremaning isbotiga olib kelish yo'li bilan isbot qilamiz. Buning uchun (D) sohani $(R) = [a, b] \times [c, d]$ to'rtburchak bilan o'raymiz (13.12-chizma), bunda $c = \min_{a \leq x \leq b} y_0(x)$, $d = \max_{a \leq x \leq b} Y(x)$.

(R) da yordamchi $f^*(x, y)$ funksiyaning quyidagicha aniqlaymiz:

$$f^*(x, y) = \begin{cases} f(x, y), & (x, y) \in (D) \text{ b'lg anda,} \\ 0, & (x, y) \in (R) \setminus (D) \text{ b'lg anda.} \end{cases}$$

Endi $f^*(x, y)$ funksiyaning $(R)=[a, b, c, d]$ to'rtburchakda 13.8.1-teoremaning hamma shartlarini qanoatlantirishini ko'rsatamiz: $f^*(x, y)$ funksiya (D) sohada integrallanuvchi, chunki u (D) da $f(x, y)$ ga aynan teng, ya'ni

$$\iint_{(D)} f^*(x, y) dR = \iint_{(D)} f(x, y) dD \quad (13.8.7)$$

integral mavjud. $f^*(x, y)$ funksiyaning tuzilishiga ko'ra (Q) da $f^*(x, y)=0$ bo'lgani uchun

$$\iint_{(Q)} f^*(x, y) dQ = 0. \quad (13.8.8)$$

U holda ikki karrali integralning 2^o- xossasiga asosan, (13.8.7) va (13.8.8) larni e'tiborga olgan holda

$$\iint_{(R)} f^*(x, y) dR = \iint_{(Q)} f^*(x, y) dQ + \iint_{(D)} f^*(x, y) dD = \iint_{(D)} f(x, y) dD$$

ni hosil qilamiz. O'zgarmas $x(x \in [a, b])$ larda

$$\int_c^d f^*(x, y) dy = \int_c^{y_0(x)} f^*(x, y) dy + \int_{y_0(x)}^{Y(x)} f^*(x, y) dy + \int_{Y(x)}^d f^*(x, y) dy$$

integral mavjud, chunki o'ng tomondagi uchala integral ham mavjud. Haqiqatdan ham $y \in [c, y_0(x)]$, $y \in [Y(x), d]$ bo'lganda, $f^*(x, y)=0$ bo'lgani uchun birinchi va uchinchi integrallar mavjud bo'ladi, ikkinchi integral uchun $y \in [y_0(x), Y(x)]$ da $f^*(x, y) = f(x, y)$ bo'lgani uchun

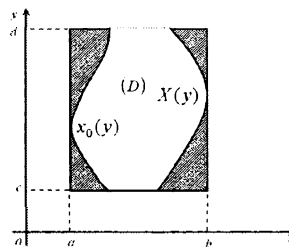
$$\int_c^d f^*(x, y) dy = \int_{y_0(x)}^{Y(x)} f(x, y) dy \quad (13.8.9)$$

tenglik o'rinli bo'ladi va u shartga ko'ra mavjud. Shunday qilib $f^*(x, y)$ 13.8.1-teoremaning hamma shartlarini qanoatlantiradi. U holda 13.8.1-teoreмага asosan, $\iint_{(R)} f^*(x, y) dR$ – integral mavjud bo'ladi va

$$\iint_{(R)} f^*(x, y) dR = \int_a^b dx \int_c^d f^*(x, y) dy$$

tenglik o'rinli bo'ladi. (13.8.7) va (13.8.9)ni e'tiborga olsak, u holda (13.8.6) formulaning o'rinli ekanligi klib chiqadi. ■

Agar (D) soha ushbu
 $(D) = \{(x, y) \in R^2 : c \leq y \leq d, x_0(y) \leq x \leq X(y)\}$
 (13.13-chizma) ko'rinishda bo'lsa, u holda quyidagi teorema o'rinli bo'ladi.



13.13-chizma.

13.8.10. Teorema. $f(x, y)$ funksiya

$$(D) = \{(x, y) \in R^2 : c \leq y \leq d, x_0(y) \leq x \leq X(y)\}, (x_0(y), X(y)) \in C[c; d]$$

sohada berilgan va integrallanuvchi bo'lsin. Agar $y \in [c, d]$

o'zgaruvchining har bir tayin qiymatida $J(y) = \int_{x_0(y)}^{X(y)} f(x, y) dx$ integral

mavjud bo'lsa, u holda ushbu

$\int_c^d dy \int_{x_0(y)}^{X(y)} f(x, y) dx$ integral ham mavjud bo'ladi va

$$\iint_{(D)} f(x, y) dx dy = \int_c^d dy \int_{x_0(y)}^{X(y)} f(x, y) dx \quad (13.8.11)$$

tenglik o'rinli.

13.8.12. Eslatma. Agar (D) sohaning chegarasi OY o'qiga hamda OX o'qiga parallel kesishsa (misol uchun 13.13-chizmadagidek) u holda 13.8.5 va 13.8.10- teoremlardagi shartlar bajarilganda (13.8.6) va (13.8.11) formulalar o'rinli bo'ladi. Ularni solishtirish natijasida

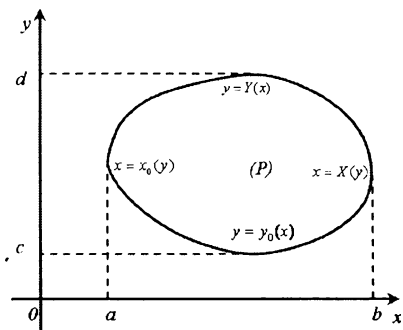
$$\int_a^b dx \int_{y_0(x)}^{Y(x)} f(x, y) dy = \int_c^d dy \int_{x_0(y)}^{X(y)} f(x, y) dx.$$

Agar (D) sohada $f(x, y)$ funksiya uzluksiz bo'lsa, u holda

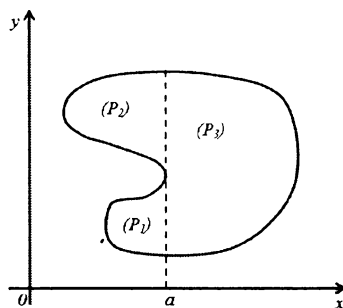
$\iint_{(D)} f(x, y) dD, \int_{y_0(x)}^{Y(x)} f(x, y) dy, \int_{x_0(y)}^{X(y)} f(x, y) dx$ integrallar mavjud bo'ladi va ikki

karrali integral (13.8.6) va (13.8.11) formulalar bilan hisoblanadi (13.14-chizma).

Agar (D) murakkab soha bo'lsa, u holda bu soha yuqorida



13.14-chizma.



13.15-chizma.

o'rganilgan sohalar ko'rinishiga keladigan qilib, chekli sondagi bo'laklarga ajratiladi, misol uchun 13.15-chizmadagidek. Natijada hisoblash talab qilingan ikki karrali integral, ajratilgan sohalar bo'yicha olingan ikki karrali integrallar yig'indisiga teng bo'ladi.

13.8.13. Eslatma. 1^0 . Agar $f(x,y)$ funksiya Oy o'qqa nisbatan sim-metrik bo'lgan (D) sohada aniqlangan va integrallanuvchi bo'lib, x o'zgaruvchiga nisbatan $f(x,y)$ juft funksiya, ya'ni $f(x,y)=f(-x,y)$ bo'lsa, u holda quyidagi

$$\iint_{(D)} f(x,y) dx dy = 2 \iint_{(D_1)} f(x,y) dx dy$$

formula o'rinli bo'ladi, bunda $(D_1) = (D) \cap \{(x,y): x \geq 0\}$.

2^0 . Agar $f(x,y)$ funksiya Oy o'qqa nisbatan simmetrik bo'lgan (D) sohada aniqlangan va integrallanuvchi bo'lib, x o'zgaruvchiga nisbatan $f(x,y)$ toq funksiya, ya'ni $f(x,y)=-f(-x,y)$ bo'lsa, u holda quyidagi

$$\iint_{(D)} f(x,y) dx dy = 0$$

formula o'rinli bo'ladi.

3⁰. Agar $f(x, y)$ funksiya Ox o'qqa nisbatan simmetrik bo'lgan (D) sohada aniqlangan va integrallanuvchi bo'lib, y o'zgaruvchiga nisbatan $f(x, y)$ juft funksiya, ya'ni $f(x, y) = f(x, -y)$ bo'lsa, u holda quyidagi

$$\iint_{(D)} f(x, y) dx dy = 2 \iint_{(D_1)} f(x, y) dx dy$$

formula o'rinli bo'ladi, bunda $(D_1) = (D) \cap \{(x, y) : y \geq 0\}$.

4⁰. Agar $f(x, y)$ funksiya Ox o'qqa nisbatan simmetrik bo'lgan (D) sohada aniqlangan va integrallanuvchi bo'lib, y o'zgaruvchiga nisbatan $f(x, y)$ toq funksiya, ya'ni $f(x, y) = -f(x, -y)$ bo'lsa, u holda quyidagi

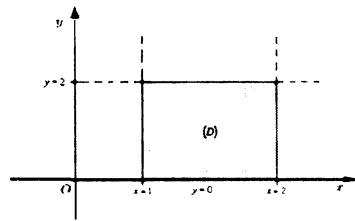
$$\iint_{(D)} f(x, y) dx dy = 0$$

formula o'rinli bo'ladi.

13.8.14. Misol. Ushbu $\iint_{(D)} (10 - x^2 - y^2) dx dy$ integralni hisoblang,

bunda $(D) - x=1, x=2, y=0, y=2$ to'g'ri chiziqlar bilan chegaralangan soha.

Yechilishi. Dastlab funksiyalar grafiklarini chizib olamiz (13.16-chizma). Berilgan ikki karrali integralga



13.16-chizma.

$\int_a^b \left[\int_c^d f(x, y) dy \right] dx$ formulani qo'llab uni

$$\iint_{(D)} (10 - x^2 - y^2) dx dy = \int_1^2 \left(\int_0^2 (10 - x^2 - y^2) dy \right) dx$$

takroriy integral ko'rinishda yozamiz.

Ichki integralni x o'zgaruvchini o'zgarmas miqdor deb, hisoblaymiz:

$$\int_0^2 (10 - x^2 - y^2) dy = \left(10y - x^2 y - \frac{y^3}{3} \right) \Big|_{y=0}^{y=2} = 20 - 2x^2 - \frac{8}{3} = \frac{52}{3} - 2x^2$$

Endi x o'zgaruvchi bo'yicha integralni hisoblaymiz:

$$\iint_{(D)} (10 - x^2 - y^2) dx dy = \int_1^2 \left(\frac{52}{3} - 2x^2 \right) dx = \left(\frac{52}{3}x - \frac{2x^3}{3} \right) \Big|_1^2 = \frac{38}{3}.$$

Misolni Maple tizimidan foydalanib, yechish:

$$> \text{int}\left(\text{int}\left(10 - x^2 - y^2, y = 0..2\right), x = 1..2\right); \quad \frac{38}{3}.$$

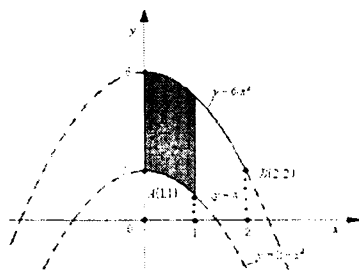
13.8.15.Misol. Ikki karrali integralni hisoblang:

$\iint_{(D)} (x + y) dx dy$, bunda $(D) - y = 6 - x^2, y = 2 - x^2$ parabolalar, $y = x$ va $x = 0$

to'g'ri chiziqlar bilan chegaralangan soha.

Yechilishi. Dastlab funksiyalar grafiklarini chizib olamiz (13.17-chizma), so'ngra

$$\begin{cases} y = 6 - x^2 \\ y = x \end{cases} \quad \text{va} \quad \begin{cases} y = 2 - x^2 \\ y = x \end{cases}$$



sistemalarni yechamiz:

$$x = 2, y = 2, x = 1, y = 1.$$

13.17-chizma.

Bu funksiyalar $A(1,1)$ va $B(2,2)$ nuqtalar kesishadi.

$$\begin{aligned} I &= \int_0^1 \left[\int_{2-x^2}^{6-x^2} (x+y) dy \right] dx + \int_1^2 \left[\int_x^{6-x^2} (x+y) dy \right] dx = \\ &= \int_0^1 \left(xy + \frac{y^2}{2} \right) \Big|_{2-x^2}^{6-x^2} dx + \int_1^2 \left(xy + \frac{y^2}{2} \right) \Big|_x^{6-x^2} dx = \\ &= \int_0^1 (-4x^2 + 4x + 16) dx + \int_1^2 \left(\frac{x^4}{2} - x^3 - \frac{15x^2}{2} + 6x + 18 \right) dx = \\ &= \left(-\frac{4}{3}x^3 + 2x^2 + 16x \right) \Big|_0^1 + \left(\frac{x^5}{10} - \frac{x^4}{4} - \frac{5}{2}x^3 + 3x^2 + 18x \right) \Big|_1^2 = \frac{1537}{60}. \end{aligned}$$

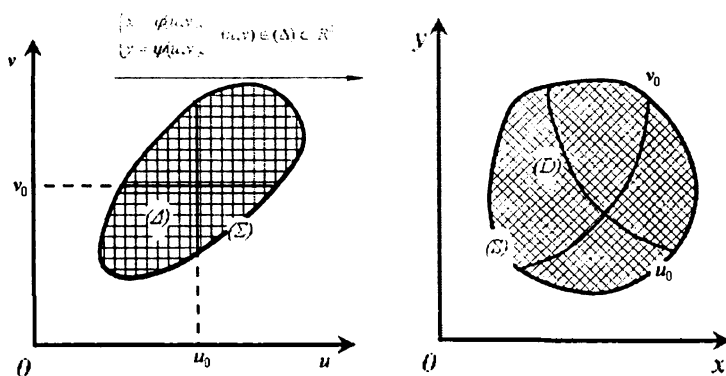
13.9. Ikki karrali integrallarda o'zgaruvchilarni almashtirish.

$f(x, y)$ funksiya (D) sohada berilgan va uning chekli

$$\iint_{(D)} f(x, y) dx dy \quad (13.9.1)$$

ikki karrali integrali mavjud va uni hisoblash talab qilingan bo'lsin. Agar $f(x, y)$ funksiya va (D) soha murakkab bo'lsa, (13.9.1) integralni hisoblash qiyinlashadi. Bunday hollarda x va y o'zgaruvchilarni ma'lum *qoidaga* ko'ra, boshqa o'zgaruvchilarga almashtirish natijasida integral ostidagi funksiya ham, integrallash sohasi ham soddalashib, ikki karrali integralni hisoblash osonlashadi.

xOy hamda uOv koordinatalar sistemasida (D) va (Δ) chegaralangan sohalar berilgan bo'lsin. Bu sohalarining chegaralari, mos ravishda, $(S) = \partial(D)$ va $(\Sigma) = \partial(\Delta)$ lar, sodda, bo'lakli silliq chiziqlardan iborat bo'lsin. (Δ) sohada



13.17- chizma.

$$\begin{cases} x = \varphi(u, v), \\ y = \psi(u, v), \end{cases} (u, v) \in (\Delta) \subset R^2 \quad (13.9.2)$$

uzluksiz funksiyalar sistemasi berilgan bo'lsin. Bu funksiyalar shunday funksiyalar bo'lsinki, ulardan tuzilgan (13.9.2) sistema (Δ) dagi (u, v) nuqtani (D) sohadagi (x, y) nuqtaga akslantirsin va bu akslantirishning akslaridan iborat $\{(x, y)\}$ to'plam (D) ga qarashli bo'lsin.

Demak, (13.9.2) sistema (Δ) sohani (D) sohaga akslantiradi (13.17-chizma).

(13.9.2) akslantirish quyidagi shartlarni qanoatlantirsin:

1°. (13.9.2) akslantirish o'zaro bir qiymatli bo'lsin, ya'ni (Δ) sohaning turli nuqtalarini (D) sohaning turli nuqtalariga akslantirsin. (D) sohaning har bir nuqtasi uchun (Δ) sohada unga bitta nuqta mos kelsin. Bu holda (13.9.2) sistema u va v larga nisbatan bir qiymatli yechiladi:

$$\begin{cases} u = \varphi_1(x, y), \\ v = \psi_1(x, y), \end{cases} \quad (x, y) \in (D) \subset R^2. \quad (13.9.3)$$

(13.9.3) akslantirish (13.9.2) akslantirishga nisbatan teskari akslantirish bo'lib, (D) sohani (Δ) sohaga akslantiradi, ya'ni

$$\begin{cases} x = \varphi(\varphi_1(x, y), \psi_1(x, y)), \\ y = \psi(\varphi_1(x, y), \psi_1(x, y)), \end{cases} \quad (x, y) \in (D) \subset R^2 \quad (13.9.4)$$

2°. $\varphi(u, v)$, $\psi(u, v)$ funksiyalar (Δ) sohada, $\varphi_1(x, y)$, $\psi_1(x, y)$ funksiyalar esa, (D) sohada uzluksiz va barcha xususiy hosilalarga ega bo'lib, bu xususiy hosilalar ham uzluksiz bo'lsin.

3°. $\forall (u, v) \in (\Delta)$ uchun (13.9.2) sistemadagi funksiyalarning xususiy hosilalaridan tuzilgan ikkinchi tartibli determinant

$$I(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \neq 0 \quad (13.9.5)$$

shartni qanoatlantirsin. Odatda, (13.9.5) ikkinchi tartibli determinant- (13.9.2) sistemaning *yakobiani* deyiladi va $I(u, v)$ yoki $\frac{D(x, y)}{D(u, v)}$ kabi belgilanadi.

13.9.6. Teorema. $f(x, y)$ funksiya (D) sohada berilgan va uzluksiz bo'lib, (13.9.2) akslantirish 1^o-, 2^o-, 3^o- shartlarni qanoatlantirsin. U holda

$$\iint_{(D)} f(x, y) dx dy = \iint_{(\Delta)} f(\varphi(u, v), \psi(u, v)) |I(u, v)| du dv \quad (13.9.7)$$

formula o'rinli. (13.9.7) formula ikki karrali integrallarda o'zgaruvchilarni almashtirish formulasi deyiladi.

(Δ) bog'lamli soha bo'lganda 2^o-va 3^o- shartlardan (13.9.5) yakobianning shu sohada o'z ishorasini saqlashi kelib chiqadi. Haqiqatdan ham, $I(u, v)$ funksiya (Δ) sohaning ikki turli nuqtalarida turli ishorali qiymatlarga ega bo'lsa, u holda ko'p o'zgaruvchili funksiyalar uchun Bolsano-Koshining birinchi teoremasiga asosan shunday (u_0, v_0) nuqta topiladiki, $I(u_0, v_0) = 0$ bo'ladi. Bu esa, $I(u, v) \neq 0$ bo'lishiga ziddir.

3) shartdan (13.9.3) sistemaning Yakobian, ya'ni

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \neq 0, \quad (x, y) \in (D)$$

funksional determinantning ham (D) sohada noldan farqli bo'lishi kelib chiqadi. Haqiqatdan ham (13.9.4)dan

$$\begin{aligned} \frac{\partial x}{\partial x} &= \frac{\partial x}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial x}{\partial v} \frac{\partial v}{\partial x} = 1, \\ \frac{\partial y}{\partial y} &= \frac{\partial y}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial y}{\partial v} \frac{\partial v}{\partial y} = 1, \\ \frac{\partial x}{\partial y} &= \frac{\partial x}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial x}{\partial v} \frac{\partial v}{\partial y} = 0, \\ \frac{\partial y}{\partial x} &= \frac{\partial y}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial y}{\partial v} \frac{\partial v}{\partial x} = 0, \end{aligned}$$

bo'lishini e'tiborga olsak, u holda $\frac{D(x, y)}{D(u, v)} \cdot \frac{D(u, v)}{D(x, y)} = 1$ bo'lib,

$\frac{D(u, v)}{D(x, y)} \neq 0$ bo'lishini topamiz.

Shunday qilib, (D) bog'lamli soha bo'lganda (13.9.5) yakobian ham shu sohada ishora saqlar ekan.

Yuqoridagi shartlardan yana quyidagilar kelib chiqadi. (13.9.2) akslantirish (Δ) sohaning ichki nuqtasini (D) sohaning ichki nuqtasiga akslantiradi. Haqiqatdan ham, oshkormas funksiya mavjudligi haqidagi teorema ko'ra (13.9.2) sistema (x_0, y_0) nuqtaning biror atrofida u va v larni x va y o'zgaruvchilarning funksiyasi sifatida aniqlaydi: $u = \varphi(x, y), v = \psi(x, y)$. Bunda $\varphi_1(x_0, y_0) = u_0, \psi_1(x_0, y_0) = v_0$ bo'ladi.

Demak, (x_0, y_0) nuqta (D) sohaning ichki nuqtasi. Bundan (13.9.2) akslantirish (Δ) sohaning chegarasi (Σ) ni (D) sohaning chegarasi (S) ga akslantirish kelib chiqadi.

Xuddi shunday (13.9.2) akslantirish (Δ) sohadagi silliq (bo'lakli silliq) egri chiziq

$$\left. \begin{aligned} u &= u(t) \\ v &= v(t) \end{aligned} \right\} (\alpha \leq t \leq \beta)$$

ni (D) sohadagi silliq (bo'lakli-silliq) egri chiziq

$$x = \varphi(u(t), v(t)) = x(t),$$

$$y = \psi(u(t), v(t)) = y(t)$$

ga akslantiradi. (Δ) sohada $u = u_0$ to'g'ri chiziqni olaylik. (13.9.2) akslantirish bu to'g'ri chiziqni (D) sohadagi

$$\left. \begin{aligned} x &= \varphi(u_0, v) \\ y &= \psi(u_0, v) \end{aligned} \right\} \quad (13.9.8)$$

egri chiziqqa akslantiradi. Xuddi shunday (Δ) sohadagi $v = v_0$ to'g'ri chiziqli (13.9.2) akslantirish (D) sohadagi

$$\left. \begin{aligned} x &= \varphi(u, v_0) \\ y &= \psi(u, v_0) \end{aligned} \right\} \quad (13.9.9)$$

egri chiziqqa akslantiradi. Odatda (13.9.8) va (13.9.9) egri chiziqlarni koordinat chiziqlari (13.9.8) ni v koordinat chizig'i, (13.9.9) chiziqni esa u koordinat chizig'i deb ataladi.

(13.9.2) akslantirish o'zaro bir qiymatli akslantirish bo'lgani uchun (D) sohaning har bir (x, y) nuqtasidan yagona v -koordinat chizig'i (v ning tayin o'zgarmas qiymatga mos kelgan chiziq), yagona u -koordinat chizig'i (u ning tayin o'zgarmas qiymatga mos bo'lgan chiziq) o'tadi.

Demak, (D) sohaning (x, y) nuqtasi yuqoridagi u va v lar bilan, ya'ni (Δ) sohaning (u, v) nuqtasi bilan to'la aniqlanadi. Shunga ko'ra, u va v larni (D) soha nuqtalarining koordinatalari deb qarash mumkin. (D) soha nuqtalarining bunday koordinatalarini egri chiziqli koordinatalar deb ataladi.

Shunday qilib, u va v lar (Δ) sohaning Dekart koordinatalari, ikkinchi tomondan u va v lar (D) soha nuqtalarining egri chiziqli koordinatalari bo'ladi.

Faraz qilaylik (13.9.2) sistema (Δ) sohani (D) sohaga akslantirib, u yuqoridagi 1) -3) shartlarni qanoatlantirsin. U holda, (D) sohaning yuzi

$$D = \iint_{(D)} |J(u, v)| du dv = \iint_{(\Delta)} \left| \frac{D(x, y)}{D(u, v)} \right| du dv \quad (13.9.10)$$

ga teng bo'ladi. $f(x, y)$ funksiya (D) sohada aniqlangan va uzluksiz bo'lsin.

(D) bo'lakli-silliqli chiziqli bilan chegaralangan soha bo'lsin. U holda $f(x, y)$ funksiya (D) da integrallanuvchi bo'ladi.

(Δ) sohani bo'lakli-silliqli chiziqli yordamida bo'laklarga bo'lamiz, ya'ni P_i bo'linishini qaraylik. (13.9.2) akslantirish natijasida (D) sohaning P_i bo'linishi hosil bo'ladi. Bu bo'linishga nisbatan $f(x, y)$ funksiyaning

$$\sigma_{P_i}(f) = \sum_{i=1}^n f(u_i, v_i) D_i$$

integral yig'indisini tuzamiz. Ma'lumki,

$$\lim_{\lambda_{P_i} \rightarrow 0} \sigma_{P_i}(f) = \lim_{\lambda_{P_i} \rightarrow 0} \sum_{i=1}^n f(u_i, v_i) D_i = \iint_{(D)} f(x, y) dx dy \quad (13.9.11)$$

Yuqoridagi (13.9.10) formulaga ko'ra,

$$D_i = \iint_{(D_i)} |J(u, v)| du dv$$

bo'ladi. O'rta qiymat haqidagi teorema asosan,

$$D_i = \left| J(u_i^*, v_i^*) \right| \Delta_i, (u_i^*, v_i^*) \in (\Delta_i).$$

Buni e'tiborga olgan holda (13.9.10) integral yig'indi

$$\sigma_{p_i}(f) = \sum_{i=1}^n f(u_i, v_i) \cdot |J(u_i^*, v_i^*)| \Delta_i$$

ko'rinishga keladi. (u_i, v_i) nuqta (D_i) sohadagi ixtiyoriy nuqta ekanligidan foydalanib, uni

$$\varphi(u_i^*, v_i^*) = u_i$$

$$\psi(u_i^*, v_i^*) = v_i$$

deb olish mumkin. U holda

$$\sigma_p(f) = \sum_{i=1}^n f(\varphi(u_i^*, v_i^*), \psi(u_i^*, v_i^*)) \cdot |J(u_i^*, v_i^*)| \Delta_i$$

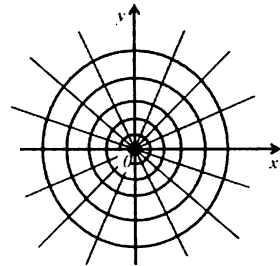
bo'ladi. Bundan $f(\varphi(u, v), \psi(u, v)) |J(u, v)|$ funksiya (Δ) da uzluksiz bo'lgani uchun

$$\begin{aligned} \lim_{\lambda_p \rightarrow 0} \sigma_p(f) &= \lim_{\lambda_p \rightarrow 0} \sum_{i=1}^n f(\varphi(u_i^*, v_i^*), \psi(u_i^*, v_i^*)) \cdot |J(u_i^*, v_i^*)| \Delta_i = \\ &= \iint_{(\Delta)} f(x(u, v), y(u, v)) |J(u, v)| du dv. \end{aligned} \quad (13.9.12)$$

$\lambda_{p_n} \rightarrow 0$ da $\lambda_{p_D} \rightarrow 0$ bo'lgani uchun (13.9.11) va (13.9.12) dan (13.9.7) o'rinli bo'lishi kelib chiqdi. ■

13.10. Dekart koordinatalar sistemasidan qutb koordinatalar sistemasiga o'tish. (r, θ) - qutb koordinatalar sistemasi berilgan bo'lsin. Bu sistemada $\theta = \text{const}$ munosabat 0 qutbdan chiqib qutb o'qining musbat yo'nalishi bilan (soat strelkasi yo'nalishiga qarama-qarshi) θ burchak tashkil etuvchi nurlar oilasini ifodalaydi ($0 \leq \theta \leq 2\pi$). $r = \text{const}$ munosabat esa, markazi 0 qutbda va radiusi r ga teng bo'lgan konsentrik aylanalar oilasini ifodalaydi, unda $r \geq 0$. Chiziqlarning yuqorida ko'rsatilgan ikki oilasi butun tekislikni yuzachalarga bo'ladi (13.18-chizma).

Har bir yuza ikki aylana ($r = r_1, r = r_2$) va ikki nur ($\theta = \theta_1, \theta = \theta_2$) bilan chegaralangan bo'lib, uning yuzi uchun $\Delta S = r dr d\theta$ formula (yuqori tartibli cheksiz kichik miqdorlar aniqligida) o'rinli. Agar ikki karrali integralda integrallash o'zgaruvchilari



13.18-chizma.

dekart koordinatalaridan iborat bo'lsa, unda (x, y) dekart koordinatalarini (r, θ) qutb koordinatalariga

$$x = r \cos \theta, \quad y = r \sin \theta \quad (0 \leq r < \infty, \quad 0 \leq \theta \leq 2\pi) \quad (13.10.1)$$

formulalar bo'yicha almashtiriladi. Bu holda funksional determinant

$$J(r, \theta) = \frac{D(x, y)}{D(r, \theta)} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r.$$

Natijada (13.9.7) formula ushbu

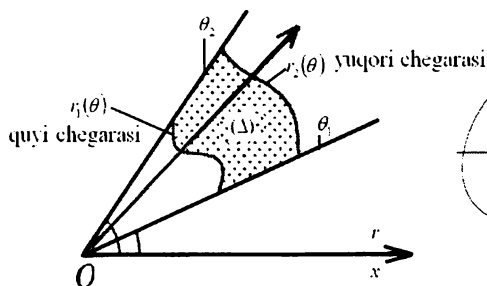
$$\iint_{(D)} f(x, y) dx dy = \iint_{(\Delta)} f(r \cos \theta, r \sin \theta) r dr d\theta \quad (13.10.2)$$

ko'rinishni oladi. Odatda, (13.10.2) munosabat, ikki karrali integralda *qutb koordinatalar sistemasiga o'tish formulasi* deyiladi.

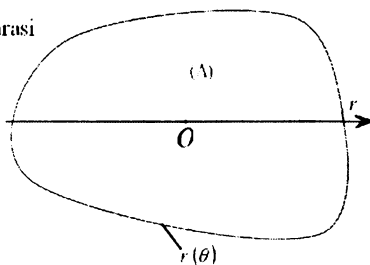
13.10.3. Eslatma. 1^o. Agar (Δ) soha $r_1(\theta)$ va $r_2(\theta)$ (bunda $\theta_1 \leq \theta \leq \theta_2, r_1 < r_2$) chiziqlar hamda qutb o'q bilan θ_1 va θ_2 burchaklar tashkil qiluvchi nurlar bilan chegaralangan bo'lsa (13.19-chizma), u holda

$$\iint_{(D)} f(x, y) dx dy = \iint_{(\Delta)} f(r \cos \theta, r \sin \theta) r dr d\theta = \int_{\theta_1}^{\theta_2} d\theta \int_{r_1(\theta)}^{r_2(\theta)} f(r \cos \theta, r \sin \theta) r dr d\theta$$

formula o'rinli bo'ladi.



13.19-chizma.



13.20-chizma.

2^o. Agar (Δ) soha $r(\theta)$ egri chiziq bilan chegaralangan va qutb boshini o'z ichida saqlasa (13.20-chizma), u holda

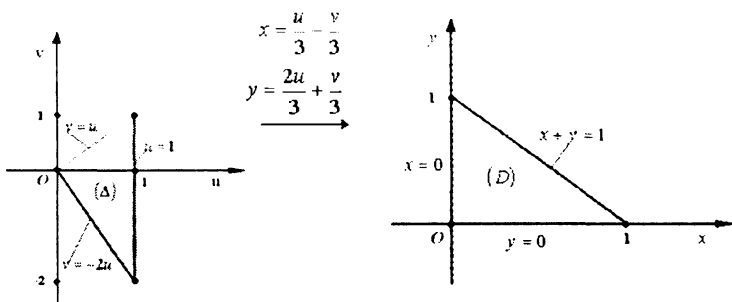
$$\iint_{(D)} f(x, y) dx dy = \iint_{(\Delta)} f(r \cos \theta, r \sin \theta) r dr d\theta = \int_0^{2\pi} d\theta \int_0^{r(\theta)} f(r \cos \theta, r \sin \theta) r dr d\theta$$

formula o'rinli bo'ladi.

13.10.4. Eslatma. Agar ikki karrali integral ostidagi funksiya yoki integrallanuvchi soha chegarasining tenglamalari $x^2 + y^2$ yig'indi orqali ifodalangan bo'lsa, ko'p hollarda (13.10.1) almashtirish yordamida qutb koordinatalar sistemasiga o'tib ikki karrali integralni hisoblash osonlashadi.

13.10.5. Misol. $\iint_{(D)} \sqrt{x+y}(y-2x)^2 dy dx$ ikki karrali integralni hisoblang, bunda (D) soha, $x=0$, $y=0$, $x+y=1$ to'g'ri chiziqlar bilan chegaralangan.

Yechilishi. xOy koordinatalar tekisligida (D) sohani chegaralaydigan sohani chizamiz:



13.21 - chizma.

$u = x + y$ va $v = y - 2x$ almashtirish yordamida integral ostidagi funksiya sodda funksiyaga keltiriladi. Berilgan almashtirishni x va y larga nisbatan yechamiz:

$$x = \frac{u}{3} - \frac{v}{3}, y = \frac{2u}{3} + \frac{v}{3} \quad (13.10.6)$$

(13.10.6) tenglamalar sistemasidan (u, v) koordinatalar tekisligida (Δ) sohani chizib olamiz (13.21- chizma).

xOy koordinatalar tekisligidagi (D) sohaning chegaralari	(Δ) sohaning chegaralari uchun (u, v) tekisligidagi mosliklar	uOv koordinatalar tekisligidagi (Δ) sohaning chegaralari
$x + y = 1$ $x = 0$ $y = 0$	$\left(\frac{u}{3} - \frac{v}{3}\right) + \left(\frac{2u}{3} + \frac{v}{3}\right) = 1$ $\frac{u}{3} - \frac{v}{3} = 0$ $\frac{2u}{3} + \frac{v}{3} = 0$	$u = 1$ $v = u$ $v = -2u$

(13.10.6) almashtirishning yakobianini hisoblaymiz:

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{3} \neq 0.$$

(13.9.2) formulaga asosan, berilgan ikki karrali integralni yozamiz va hisoblaymiz:

$$\begin{aligned} \iint_{(D)} \sqrt{x+y} (y-2x)^2 dx dy &= \int_{u=0}^{u=1} \int_{v=-2u}^{v=-4} u^{1/2} v^2 |J| du dv = \\ &= \int_0^1 du \int_{-2u}^{-4u} u^{1/2} v^2 \frac{1}{3} dv = \frac{1}{3} \int_0^1 u^{1/2} \left[\frac{1}{3} v^3 \right]_{v=-2u}^{v=-4u} du = \\ &= \frac{1}{9} \int_0^1 u^{1/2} (u^3 + 8u^3) du = \int_0^1 u^{7/2} du = \frac{2}{9} u^{9/2} \Big|_0^1 = \frac{2}{9}. \end{aligned}$$

Misolni Maple tizimidan foydalanib, yechish:

> int(int((1/3)*(sqrt(u)*v^2),v=-2*u..u),u=0..1);

$$\frac{2}{9}.$$

13.10.5. Misol. Ushbu $I = \iint_{(D)} e^{x^2+y^2} dx dy$ ikki karrali integralni qutb

koordinatalar sistemasidan foydalanib hisoblang, bunda

$$(D) = \{(x, y): x^2 + y^2 \leq 4, x \geq 0, y \geq 0\}.$$

Yechilishi. Integral ostidagi $f(x,y)=e^{x^2+y^2}$ funksiya va integrallanuvchi soha chegarasining tenglamasi $x^2 + y^2 = 4$ aylanadan iborat bo'lib, $x^2 + y^2$ bog'liq. **13.10.4**-eslatmaga asosan, (13.10.1) almashtirish yordamida (r,φ) qutb koordinatalar sistemasiga o'tamiz:

$$f(x,y)=e^{x^2+y^2}=e^{r^2\cos^2\varphi+r^2\sin^2\varphi}=e^{r^2}, \quad r^2\cos^2\varphi+r^2\sin^2\varphi=4 \text{ yoki}$$

$$r^2=4, r=2, \quad x=0 \Rightarrow r\cos\varphi=0 \Rightarrow \varphi=\frac{\pi}{2}, \quad y=0 \Rightarrow r\sin\varphi=0 \Rightarrow \varphi=0.$$

Demak, $(\Delta)=\left\{(r,\varphi): 0 \leq \varphi \leq \frac{\pi}{2}, 0 \leq r \leq 2\right\}$ (13.22-chizma). Endi

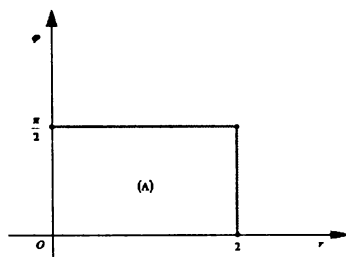
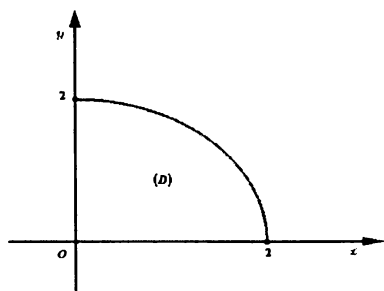
(13.10.2) formulaga asosan, berilgan ikki karrali integralni hisoblaymiz:

$$I = \iint_{(D)} e^{x^2+y^2} dx dy = \iint_{(\Delta)} e^{r^2} r dr d\varphi = \int_0^{\frac{\pi}{2}} d\varphi \int_0^2 r e^{r^2} dr = \frac{\pi}{4} (e^4 - 1).$$

Misolni Maple tizimidan foydalanib, yechish:

`>int(int(y*exp(y^2),y=0..2),x=0..Pi/2);`

$$\frac{1}{4} e^4 \pi - \frac{1}{4} \pi$$



13.22-chizma.

13.11. Ikki karrali integral yordamida jismning hajmini hisoblash. R^3 fazoda yuqoridan $z = f(x, y)$ sirt bilan, yon tomonlaridan yasovchilari Oz o'qqa parallel bo'lgan silindrik sirt hamda pastdan xOy tekislikdagi (D) soha bilan chegaralangan (V) jismning V hajmi ushbu

$$V = \iint_{(D)} f(x, y) dx dy \quad (13.11.1)$$

formula yordamida hisoblanadi.

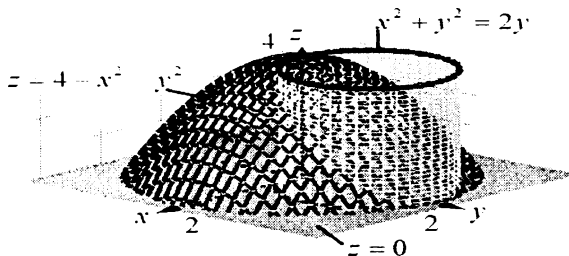
13.11.2. Misol. Ushbu $z = 4 - x^2 - y^2$, $x^2 + y^2 = 2y$, $z = 0$ sirtlar bilan chegaralangan jismning hajmini toping.

Yechilishi. Berilgan jism yuqoridan $z = 4 - x^2 - y^2$ paraboloid, quyidan esa, $x^2 + y^2 = 2y$ doiraviy silindrik sirt bilan chegaralangan (13.23-chizma). Jismning hajmini hisoblaymiz:

$$V = \iiint_{(D)} (4 - x^2 - y^2) dx dy.$$

Integrallash chegarasi (D) soha-doira hamda integral ostidagi funksiya $x^2 + y^2$ bog'liq bo'lganligi uchun, qutb koordinatalar sistemasiga o'tib hisoblaymiz. $x^2 + y^2 = 2y$ aylana tenglamasi qutb koordinatalar sistemasida $r^2 = 2r \sin \varphi$, ya'ni $r = 2 \sin \varphi$ ko'rinishda bo'lib, (D) sohada φ burchak 0 va π oraliqda o'zgaradi, integral ostidagi funksiya $4 - r^2$ bo'ladi. (13.11.1) formulaga asosan, jismning hajmini topamiz:

$$\begin{aligned} V &= \iint_{(D)} (4 - r^2) r dr d\varphi = \int_0^\pi d\varphi \int_0^{2\sin\varphi} (4r - r^3) dr = \\ &= \int_0^\pi \left(2r^2 - \frac{r^4}{4} \right) \Big|_0^{2\sin\varphi} d\varphi = \int_0^\pi (8\sin^2\varphi - 4\sin^4\varphi) d\varphi = \\ &= \int_0^\pi [4(1 - \cos 2\varphi) - (1 - \cos 2\varphi)^2] d\varphi = \\ &= \int_0^\pi \left(4 - 4\cos 2\varphi - 1 + 2\cos 2\varphi + \frac{1 + \cos 4\varphi}{2} \right) d\varphi = \frac{5\pi}{2} \text{ (kb. birl)}. \end{aligned}$$



13.23-chizma.

Misolni Maple tizimidan foydalanib, yechish:

`int(int(4*y-y^3,y=0..2*sin(x)),x=0..Pi);`

$$\frac{5}{2}\pi.$$

13.12. Ikki karrali integral yordamida silliq sirtning yuzini hisoblash. (S) –sirt tenglamasi $z = f(x, y)$ oshkor ko‘rinishda berilgan bo‘lsin. Bu sirtning xOy tekislikdagi proyeksiyasi kvadratlanuvchi (D) soha bo‘lib, (D) sohaning har bir nuqtasida $z = f(x, y)$ funksiya uzluksiz $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$ xususiy hosilalarga ega bo‘lsin. Bu holda (S) –silliq sirtning yuzi

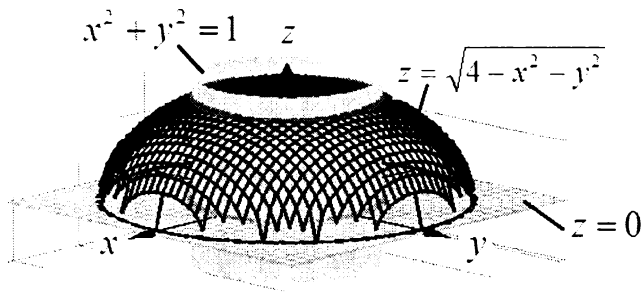
$$S = \iint_{(D)} \sqrt{1 + p^2 + q^2} dx dy \quad (13.12.1)$$

formula orqali hisoblanadi.

13.12.2. Misol. $x^2 + y^2 + z^2 = 4$ ($z \geq 0$) yarim sferaning $x^2 + y^2 = 1$ silindr bilan ajratilgan qismi yuzini toping.

Yechilishi. (S) sirtning tenglamasi $z = \sqrt{4 - x^2 - y^2}$ ko‘rinishda. $x^2 + y^2 = 1$ aylana bilan chegaralangan (D) soha-doiradan iborat (13.24-chizma).

$$z'_x = p = -\frac{x}{\sqrt{4 - x^2 - y^2}}, \quad z'_y = q = -\frac{y}{\sqrt{4 - x^2 - y^2}}$$



13.24-chizma.

(13.12.1) formuladan foydalanib, izlanayotgan sirt yuzini topamiz:

$$S = \iint_{(D)} \sqrt{1 + \frac{x^2 + y^2}{4 - x^2 - y^2}} dx dy = 2 \iint_{(D)} \frac{dx dy}{\sqrt{4 - x^2 - y^2}}.$$

Qutb koordinatalar sistemasidan foydalanib ikki karrali integralni hisoblaymiz. $x^2 + y^2 = 1$ aylana tenglamasi $r = 1$ ko'rinishda bo'lib, bunda $0 \leq \varphi \leq 2\pi$.

Demak,

$$\begin{aligned} S &= 2 \iint_{(D)} \frac{r dr d\varphi}{\sqrt{4 - r^2}} = 2 \int_0^{2\pi} d\varphi \int_0^1 \frac{r dr}{\sqrt{4 - r^2}} = \\ &= 2 \int_0^{2\pi} [-\sqrt{4 - r^2}]_0^1 d\varphi = 2 \cdot 2\pi [-\sqrt{3} + 2] = 4\pi(2 - \sqrt{3}) \text{ (kv. birlik).} \end{aligned}$$

Misolni Maple tizimidan foydalanib, yechish:

```
> int(int(2*y/sqrt(4-y^2),y=0..1),x=0..2*Pi);
```

$$4 \frac{\pi}{2 + \sqrt{3}}.$$

13.13. Tekis shaklning yuzi. Ikki karrali integrallar yordamida tekis shaklning yuzasini topish mumkin. Integralning ta'rifidan (D) shaklning yuzi uchun

$$D = \iint_{(D)} dx dy \quad (13.13.1)$$

formula bevosita kelib chiqadi.

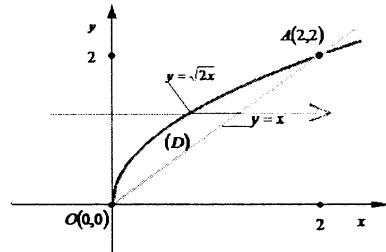
13.13.2. Misol. $y = \sqrt{2x}$ parabola va $y = x$ to'g'ri chiziqlar bilan chegaralangan tekis shaklning yuzini hisoblang.

Yechilishi. Tekis shaklning yuzini (13.13.1) formuladan foydalanib hisoblaymiz. Integrallash chegarasini topish uchun, dastlab $y = \sqrt{2x}$ parabola va $y = x$ to'g'ri chiziqlarning kesishish nuqtalarini topamiz:

$$\begin{cases} y^2 = 2x \\ y = x \end{cases} \Rightarrow x^2 = 2x \Rightarrow x_1 = 0, y_1 = 0 \text{ va}$$

$$x_2 = 2, y_2 = 2.$$

Demak, berilgan parabola va to'g'ri chiziqning kesishish nuqtalari $O(0,0)$ va



13.25-chizma.

$A(2,2)$ bo'la-di. (D) sohaning Oy o'qdagi proyeksiyasi $[0,2]$ kesmadan iborat. Shunday qilib, (13.13.1) formulaga asosan,

$(D) = \left\{ (x, y) \in R^2 : 0 \leq y \leq 2, \frac{y^2}{2} \leq x \leq y \right\}$ yuzani hisoblaymiz (13.25-chizma):

$$D = \int_0^2 dy \int_{y^2/2}^y dx = \int_0^2 \left(x \Big|_{x=y^2/2}^{x=y} \right) dy = \int_0^2 \left(y - \frac{y^2}{2} \right) dy = \left(\frac{y^2}{2} - \frac{y^3}{6} \right) \Big|_0^2 = \frac{2}{3}.$$

Misolni Maple tizimidan foydalanib, yechish:

> int(int(1,x=y^2/2..y),y=0..2);

$$\frac{2}{3}.$$

13.14. Ikki karrali integrallar yordamida mexanikaga oid masalalarni yechish. 1. Faraz qilaylik, bir jinsli (D) - xOy tekislikda berilgan zichligi $\rho(x, y)$ ga teng bo'lgan, plastinkaberilgan bo'lsin. U holda plastinkaning massasi ushbu

$$m = \iint_{(D)} \rho(x, y) dx dy$$

formula yordamida hisoblanadi.

2. Plastinkaning Ox va Oy o'qlarga nisbatan statik momentlari

$$M_x = \iint_{(D)} y \rho(x, y) dx dy, \quad M_y = \iint_{(D)} x \rho(x, y) dx dy$$

formulalar yordamida hisoblanadi.

3. Plastinka og'irlik markazining koordinatalari esa,

$$x_0 = \frac{M_y}{m}, \quad y_0 = \frac{M_x}{m}$$

formulalar yordamida hisoblanadi.

4. Plastinkaning Ox va Oy o'qlarga nisbatan inersiya momentlari

$$I_x = \iint_{(D)} y^2 \rho(x, y) dx dy, \quad I_y = \iint_{(D)} x^2 \rho(x, y) dx dy$$

formulalar yordamida hisoblanadi.

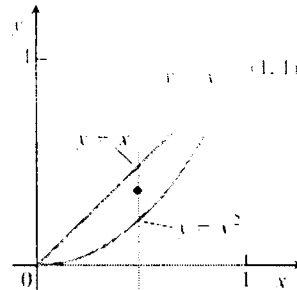
5. Plastinkaning koordinatalar boshiga nisbatan inersiya momenti

$$I_o = I_x + I_y = \iint_{(D)} (y^2 + x^2) \rho(x, y) dx dy$$

formula yordamida hisoblanadi.

13.14.1. Misol. $y = x^2$ parabola va $y = x$ to'g'ri chiziqlar bilan chegaralangan, zichligi $\rho(x, y) = 1$ bo'lgan bir jinsli plastinkaning og'irlik markazini toping.

Yechilishi. Dastlab bir jinsli plastinkaning massasini topamiz (13.26- chizma):



13.26- chizma.

$$m = \int_0^1 dx \int_{x^2}^x 1 \cdot dy = \int_0^1 [y]_{y=x^2}^{y=x} dx = \int_0^1 (x - x^2) dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{6}.$$

Ox va Oy o'qlariga nisbatan statistik momentlarini topamiz:

$$M_x = \int_0^1 dx \int_{x^2}^x y dy = \int_0^1 \left[\frac{y^2}{2} \right]_{y=x^2}^{y=x} dx = \int_0^1 \left[\frac{x^2}{2} - \frac{x^4}{2} \right] dx = \left[\frac{x^3}{6} - \frac{x^5}{10} \right]_0^1 = \frac{1}{15},$$

$$M_y = \int_0^1 dx \int_{x^2}^x x dy = \int_0^1 (xy) \Big|_{y=x^2}^{y=x} dx = \int_0^1 (x^2 - x^3) dx = \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{1}{12}.$$

m , M_x va M_y lardan foydalanib, plastinka og'irlik markazining koordinatalarini hisoblaymiz:

$$x_0 = \frac{M_y}{m} = \frac{12}{1} = \frac{1}{2}, \quad y_0 = \frac{M_x}{m} = \frac{15}{1} = \frac{2}{5}.$$

Demak, $C\left(\frac{1}{2}; \frac{2}{5}\right)$.

60 - §. Uch karrali integrallar

Biz yuqorida ikki o'zgaruvchili funksiyaning Riman integrali tushunchasi qanday kiritilishini ko'rib o'tdik va uni batafsil o'rgandik. Huddi shunday uch o'zgaruvchili hamda n o'zgaruvchili funksiyalarning Riman integrallari tushunchasi kiritiladi va ikki karrali integralda kiritilgan mulohaza va tasdiqlarning deyarli hammasi o'zgarishsiz qaytariladi. Shuning uchun biz uch karrali integral to'g'risidagi faktlar, tasdiqlarni isbotsiz keltirish bilan chegaralanamiz. Teorema va tasdiqlarning isbotini o'quvchiga havola hilamiz.

13.15. Uch karrali integralning ta'rif. R^3 fazoda chegaralangan (V) sohada $f(x, y, z)$ funksiuya berilgan bo'lsin. Biz bu erda va kelgusidsida hamma vaqt funksiyaning aniqlanish sohasi (V) ni hajimga ega bo'lgan soha deb qaraymiz. R^3 fazoda biror kublanuvchi (ochiq yoki yopiq) (V) soha berilgan bo'lib, bu sohada chegaralangan $f(x, y, z)$ funksiya berilgan bo'lsin. (V) sohaning ixtiyoriy P bo'linishini olib, bu bo'linishning har bir (V_i) , bo'lagidan $\forall M_i(\xi_i, \eta_i, \zeta_i)$ ($i=1, 2, \dots, n$), nuqtani olib, berilgan funksiyaning $M_i(\xi_i, \eta_i, \zeta_i)$ nuqtadagi $f(\xi_i, \eta_i, \zeta_i)$ qiymatini (V_i) sohaning V_i hajmiga ko'paytirib, quyidagi

$$\sigma_p(f) = \sum_{i=1}^n f(\xi_i, \eta_i, \zeta_i) V_i \quad (13.15.1)$$

integral yig'indini tuzamiz. Odatda (13.15.1) yig'indi $f(x, y, z)$ funksiyaning Riman *integral yig'indisi* deb ataladi.

(V) sohaning shunday

$$P_1, P_2, \dots, P_m, \dots \quad (13.15.2)$$

bo'linishlar ketma-ketligini qaraylikki, ularning diametrlaridan tashkil topgan

$$\lambda_{p_1}, \lambda_{p_2}, \dots, \lambda_{p_m}, \dots$$

ketma - ketlik nolga intilsin. (13.15.2) bo'linishlar ketma-ketligiga mos ushbu

$$\sigma_{p_1}, \sigma_{p_2}, \dots, \sigma_{p_m}, \dots \quad (13.15.3)$$

integral yig'indilar ketma-ketligini tuzamiz.

13.15.4. Ta'rif. Agar (V) sohaning har qanday (13.15.2) ko'rinishdagi bo'linishlar ketma-ketligi olinganda ham, unga mos (13.15.3) integral yig'indilar ketma-ketligi $M_i(\xi_i, \eta_i, \zeta_i)$ nuqtalarni tanlab olishga bog'liq bo'lmagan holda hamma vaqt bitta J songa intilsa, u holda bu J son $\sigma_p(f)$ – integral yig'indining limiti deyiladi va u

$$\lim_{\lambda_p \rightarrow 0} \sigma_p(f) = \lim_{\lambda_p \rightarrow 0} \sum_{i=1}^n f(\xi_i, \eta_i, \zeta_i) V_i = J$$

kabi belgilanadi.

13.15.5. Ta'rif. Agar $\lambda_p \rightarrow 0$ da $f(x, y, z)$ funksiyaning integral yig'indisi $\sigma_p(f)$ chekli limitga intilsa, u holda $f(x, y, z)$ funksiya (V) sohada (Riman ma'nosida) integrallanuvchi deyiladi. Bu $\sigma_p(f)$ yig'indining chekli limiti J esa, $f(x, y, z)$ funksiyaning (V) soha bo'yicha uch karrali integrali deb ataladi va u

$$J = \iiint_{(V)} f(x, y, z) dV = \iiint_{(V)} f(x, y, z) dx dy dz$$

kabi belgilanadi.

Demak,

$$J = \iiint_{(V)} f(x, y, z) dV = \lim_{\lambda_p \rightarrow 0} \sigma_p(f) = \lim_{\lambda_p \rightarrow 0} \sum_{i=1}^n f(\xi_i, \eta_i, \zeta_i) V_i.$$

Ravshanki, (13.15.1) integral yig'indining chekli limiti faqat chegaralangan funksiyalar uchun mavjud bo'lishi mumkin. Chegaralangan funksiyalarning integral yig'indisidan tashqari Darbu yig'indilari tushunchasini ham kiritish mumkin.

$f(x, y, z)$ funksiya $(V) \subset R^3$ sohada chegaralangan bo'lsin:

$$m \leq f(x, y, z) \leq M, \quad (\forall (x, y, z) \in (V))$$

(V) sohaning bo'linishlar to'plami $\{P\}$ ning P bo'linishiga nisbatan Darbu yig'indilari

$$s_p(f) = \sum_{i=1}^n m_i V_i, S_p(f) = \sum_{i=1}^n M_i \cdot V_i$$

ni $\left(m_i = \inf_{(V_i)} \{f\}, M_i = \sup_{(V_i)} \{f\} \right)$ tuzib $\{s_p(f)\}, \{S_p(f)\}$ to'plamlarni qaraymiz.

Ma'lumki, bu to'plamlar chegaralangan.

13.15.6. Ta'rif. $\{s_p(f)\}$ to'plamning aniq yuqori chegarasiga $f(x, y, z)$ funksiyaning quyi uch karrali integrali deyiladi va u
$$J = \iiint_{(V)} f(x, y, z) dV$$
 kabi belgilanadi.

$\{S_p(f)\}$ to'plamning aniq quyi chegarasiga $f(x, y, z)$ funksiyaning yuqori uch karrali integrali deyiladi va u
$$\bar{J} = \overline{\iiint_{(V)} f(x, y, z) dV}$$
 kabi belgilanadi.

13.15.7. Ta'rif. Agar $f(x, y, z)$ funksiyaning quyi va yuqori uch karrali integrallari o'zaro teng bo'lsa, u holda $f(x, y, z)$ funksiya (V) sohada *integrallanuvchi* deyiladi va ularning umumiy qiymati

$$J = \iiint_{(V)} f(x, y, z) dV = \overline{\iiint_{(V)} f(x, y, z) dV}$$

$f(x, y, z)$ funksiyaning *uch karrali integrali* deb ataladi.

Demak,

$$\iiint_{(V)} f(x, y, z) dV = \iiint_{(V)} f(x, y, z) dV = \overline{\iiint_{(V)} f(x, y, z) dV}.$$

13.16. Uch karrali integralning mavjudlik sharti. $f(x, y, z)$ funksiya (V) $((V) \subset R^3)$ sohada berilgan bo'lsin.

13.16.1. Teorema. $f(x, y, z)$ funksiyaning (V) sohada integrallanuvchi bo'lishi uchun $\forall \varepsilon > 0$ son olinganda ham shunday $\delta > 0$ son topilib, (V) sohaning diametri $\lambda_p < \delta$ bo'lgan har qanday P bo'linishga nisbatan Darbu yig'indilarining

$$S_p(f) - s_p(f) < \varepsilon \text{ yoki } \sum_{i=1}^n \omega_i V_i < \varepsilon, \omega_i = M_i - m_i$$

tengsizliklarni qanoatlantirishi zarur va yetarli.

13.17. Integrallanuvchi funksiyalarning sinflari.

13.17.1. Teorema. Agar $f(x, y, z)$ funksiya chegaralangan yopiq (V) sohada uzluksiz bo'lsa, u shu sohada integrallanuvchi bo'ladi.

13.17.2. Teorema. Agar $f(x, y, z)$ funksiya (V) sohada chegaralangan va bu sohaning chekli sondagi nol hajmli sirtlarida uzilishga ega bo'lib, qolgan barcha nuqtalarida uzluksiz bo'lsa, u shu sohada integrallanuvchi bo'ladi.

13.18. Uch karrali integralning xossalari.

1⁰. Uch karrali integralning mavjudligi va uning qiymati funksiyaning chekli sondagi nol hajmli sirtlardagi qiymatiga bog'liq bo'lmaydi.

2⁰. Agar (V) soha nol hajmli (S) sirt bilan (V_1) va (V_2) sohalarga ajratilgan bo'lib, $f(x, y, z)$ funksiya (V) sohada integrallanuvchi bo'lsa, funksiya (V_1) va (V_2) sohalarda ham integrallanuvchi bo'ladi va aksincha $f(x, y, z)$ funksiya (V_1) va (V_2) sohalarning har birida integrallanuvchi bo'lsa, funksiya (V) da ham integrallanuvchi bo'ladi va ushbu

$$\iiint_{(V)} f(x, y, z) dV = \iiint_{(V_1)} f(x, y, z) dV + \iiint_{(V_2)} f(x, y, z) dV$$

tenglik o'rinli bo'ladi.

3⁰. Agar $f(x, y, z)$ funksiya (V) da integrallanuvchi bo'lsa, u holda $C \cdot f(x, y, z)$ ($C = \text{const}$) funksiya ham shu sohada integrallanuvchi bo'ladi va ushbu

$$\iiint_{(V)} C f(x, y, z) dV = C \iiint_{(V)} f(x, y, z) dV$$

tenglik o'rinli bo'ladi.

4⁰. Agar $f(x, y, z)$ va $g(x, y, z)$ funksiyalar (V) sohada integrallanuvchi bo'lsa, u holda $f(x, y, z) \pm g(x, y, z)$ funksiya ham shu sohada integrallanuvchi va ushbu

$$\iiint_{(V)} (f(x, y, z) \pm g(x, y, z)) dV = \iiint_{(V)} f(x, y, z) dV \pm \iiint_{(V)} g(x, y, z) dV$$

tenglik o'rinli bo'ladi.

5⁰. Agar $f(x, y, z)$ funksiya (V) sohada integrallanuvchi bo'lib, $\forall (x, y, z) \in (V)$ uchun $f(x, y, z) \geq 0$ bo'lsa, u holda

$$\iiint_{(V)} f(x, y, z) dV \geq 0$$

bo'ladi.

6⁰. Agar $f(x, y, z)$ funksiya (V) sohada integrallanuvchi bo'lsa, u holda $|f(x, y, z)|$ funksiya ham shu sohada integrallanuvchi bo'ladi va ushbu

$$\left| \iiint_{(V)} f(x, y, z) dV \right| \leq \iiint_{(V)} |f(x, y, z)| dV$$

tengsizlik o'rinli bo'ladi.

7⁰. Agar $f(x, y, z)$ funksiya (V) sohada integrallanuvchi bo'lib,

$$m \leq f(x, y, z) \leq M$$

bo'lsa, u holda

$$\iiint_{(V)} f(x, y, z) dv = \mu \cdot V$$

formula o'rinli bo'ladi ($m \leq \mu \leq M$).

8⁰. Agar $f(x, y, z)$ funksiya yopiq (V) sohada uzluksiz bo'lsa, u holda bu sohada shunday $(\bar{x}, \bar{y}, \bar{z}) \in (V)$ nuqta topiladiki,

$$\iiint_{(V)} f(x, y, z) dV = f(\bar{x}, \bar{y}, \bar{z}) \cdot V \quad (13.18.1)$$

formula o'rinli bo'ladi.

Uch karrali integralning soha bo'yicha funksiya tushunchasini, xususiyl holda, sohaning additiv funksiyasi tushunchasini, soha funksiyasining berilgan nuqtadagi hosila tushunchalari ham ikki o'Ichovli holdagidek kiritiladi.

Sohaning additiv funksiyasiga misol sifatida ushbu o'zgaruvchan (V) soha bo'yicha olingan

$$\Phi((V)) = \iiint_{(V)} f(x, y, z) dV \quad (13.18.2)$$

integral misol bo'laoladi. Soha funksiyasi $\Phi(V)$ ning berilgan M nuqtadagi hosilasi deb, ushbu nisbatning

$$\lim_{(V) \rightarrow M} \frac{\Phi(V)}{V}$$

ga aytiladi.

9⁰. Agar integral ostidagi funksiya uzluksiz bo'lsa, u holda (13.18.1) integralning $M(x, y, z)$ nuqtadagi soha bo'yicha hosilasi integral ostidagi funksiyaning shu nuqtadagi qiymatiga teng bo'ladi, ya'ni $f(M) = f(x, y, z)$.

Shunday qilib, yuqoridagi shartlarda (13.18.2) integral qandaydir ma'noda f funksiya uchun boshlang'ich funksiyasi bo'lar ekan.

13.18.3. Misol. Ushbu $\iiint_{(V)} (x + yz) dx dy dz$ uch karali integralni

integral yig'indi yordamida hisoblang, bunda

$(V) = \{(x, y, z): 1 \leq x \leq 3, 1 \leq y \leq 2, 1 \leq z \leq 2\}$ –parallelepiped.

Yechilishi. (V) sohani chizamiz (13.27-chizma). (V) sohani P bo'lish yordamida n^3 ta sohalarga (shunday bo'lishni qaraylikki, $n \rightarrow \infty$ da $\lambda_p \rightarrow 0$ bo'lsin) bo'lamiz. Buning uchun (x, y) yoki $z = 0$, (y, z) yoki $x = 0$, (z, x) yoki $y = 0$ koordinata tekisliklariga parallel bo'lgan

$$z = z_k, z_k = 1 + \frac{1}{n}k, \left(k = \overline{1, n}, \Delta z = z_k - z_{k-1} = 1 + \frac{1}{n}k - \left(1 + \frac{1}{n}(k-1) \right) = \frac{1}{n} \right);$$

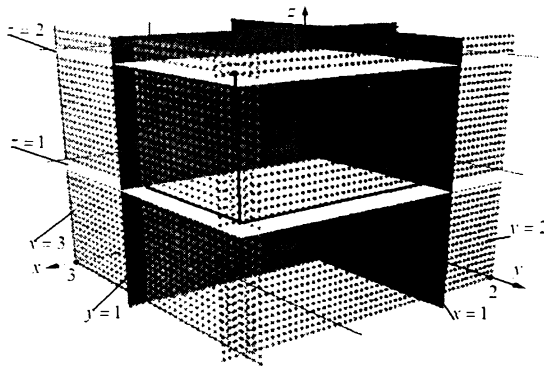
$$x = x_i, x_i = 1 + \frac{2}{n}i \left(i = \overline{1, n}, \Delta x_i = \frac{2}{n} \right);$$

$$y = y_j, y_j = 1 + \frac{1}{n}j \left(j = \overline{1, n}, \Delta y_j = \frac{1}{n} \right)$$

tekisliklar o'tkazamiz. Hosil bo'lgan har bir $(V_{i,j,k})$ soha uchun

$$x_{i-1} < x \leq x_i, y_{j-1} \leq y \leq y_j, z_{k-1} \leq z \leq z_k \left(i = \overline{1, n}, j = \overline{1, n}, k = \overline{1, n} \right)$$

shartlar bajariladi va uning hajmi $\Delta V_{i,j,k} = \Delta x_i \cdot \Delta y_j \cdot \Delta z_k = \frac{2}{n^3}$ teng bo'ladi.



13.27-chizma.

Integral ostidagi $f(x, y, z) = x + yz$ funksiya qiymatini ixtiyoriy $(x_i, y_j, z_k) \in (V_{i,j,k})$ nuqtada hisoblaymiz: $f(x_i, y_j, z_k) = x_i + y_j z_k$, so'ngra natijani $(V_{i,j,k})$ sohashaning $\Delta V_{i,j,k}$ hajmiga ko'paytirib, berilgan uch karrali integral uchun integral yig'indini tuzamiz:

$$\begin{aligned}
 \sigma_P(n) &= \sum_{j=1}^n \sum_{k=1}^n \sum_{i=1}^n (x_i + y_j z_k) \Delta x_i \Delta y_j \Delta z_k = \\
 &= \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \left[1 + \frac{2}{n} i + \left(1 + \frac{1}{n} j \right) \cdot \left(1 + \frac{1}{n} k \right) \right] \frac{2}{n^3} = \\
 &= \frac{2}{n^3} \sum_{i=1}^n \sum_{j=1}^n \left[n + \frac{2}{n} i n + \left(1 + \frac{1}{n} j \right) \cdot n + \frac{1}{n} \left(1 + \frac{1}{n} j \right) \frac{(n+1)n}{2} \right] = \\
 &= \frac{2}{n^3} \sum_{i=1}^n \sum_{j=1}^n \left(2n + 2i + j + \frac{n+1}{2} + \frac{1}{n} j \frac{n+1}{2} \right) = \\
 &= \frac{2}{n^3} \sum_{i=1}^n \left[2n^2 + 2in + \frac{n(n+1)}{2} + \frac{n(n+1)}{2} + \frac{(n+1)^2}{4} \right] = \\
 &= \frac{2}{n^3} \sum_{i=1}^n \left[2n^2 + 2ni + n(n+1) + \frac{(n+1)^2}{4} \right] = \\
 &= \frac{2}{n^3} \left[2n^3 + 2n \frac{n(n+1)}{2} + n^2(n+1) + \frac{n(n+1)^2}{4} \right] = \\
 &= 2 \left(2 + 1 + \frac{1}{n} + 1 + \frac{1}{n} + \frac{1}{4} \left(1 + \frac{1}{n} \right)^2 \right) = 2 \left[4 + \frac{1}{4} \left(1 + \frac{1}{n} \right)^2 + \frac{2}{n} \right].
 \end{aligned}$$

Endi $n \rightarrow \infty$ da $\sigma_P(n)$ ning limitini topamiz:

$$\lim_{n \rightarrow \infty} \sigma_{p'}(n) = 2[4 + \frac{1}{4}] = 8\frac{1}{2}.$$

Berilgan uch karrali integralni takroriy integralga keltirish usuli bilan hisoblaymiz:

$$\begin{aligned} \int_1^3 dx \int_1^2 dy \int_1^2 (x + yz) dz &= \int_1^3 dx \int_1^2 \left(xz + y \frac{z^2}{2} \right) \Big|_1^2 dy = \\ &= \int_1^3 dx \int_1^2 \left(x + \frac{y}{2}(4-1) \right) dy = \int_1^3 dx \int_1^2 \left(x + \frac{3}{2}y \right) dy = \\ &= \int_1^3 \left(xy + \frac{3}{2} \frac{y^2}{2} \right) \Big|_1^2 dx = \int_1^3 \left(x + \frac{3}{4}(4-1) \right) dx = \\ &= \int_1^3 \left(x + \frac{9}{4} \right) dx = \frac{x^2}{2} \Big|_1^3 + \frac{9}{4} x \Big|_1^3 = \frac{1}{2}(9-1) + \frac{68}{4} = 4 + 4\frac{2}{4} = 8\frac{1}{2}. \end{aligned}$$

Demak, ikki holda ham, uch karrali integralning natijasi bir xil bo'ldi.

Misolni Maple tizimidan foydalanib, yechish:

> with(student);

> Tripleint((x+y*z),z=1..2,y=1..2,x=1..3);

$$\int_1^3 \int_1^2 \int_1^2 (x + yz) dz dy dx$$

> value(Tripleint((x+y*z),z=1..2,y=1..2,x=1..3));

>

$$\frac{17}{2}.$$

13.19. Uch karrali integralni takroriy integralga keltirish yo'li bilan hisoblash. $f(x, y, z)$ funksiya

$$(V) = \{ (x, y, z) \in R^3 : (x, y) \in (D), z_1(x, y) \leq z \leq z_2(x, y) \}$$

(bunda $z_1(x, y), z_2(x, y)$ funksiyalar kvadratlanuvchi sohada uzluksiz) sohada aniqlangan bo'lsin.

13.19.1. Teorema. Agar: 1) ushbu

$$\iiint_{(V)} f(x, y, z) dv$$

integral mavjud bo'lsa;

2) $\forall (x, y) \in (D)$ uchun

$$J(x,y) = \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z) dz$$

aniq integral mavjud bo'lsa, u holda

$$\iint_{(D)} J(x,y) dx dy = \iint_{(D)} dx dy \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z) dz$$

takroriy integral ham mavjud va ushbu

$$\iiint_{(V)} f(x,y,z) dv = \iint_{(D)} dx dy \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z) dz \quad (13.19.2)$$

formula o'rinli bo'ladi.

(13.19.2) formulada takroriy integralning birinchisi ikki karra integral bo'lib, ichki integral esa, aniq integraldan iborat. Ba'zi hollarda takroriy integralning birinchisi aniq integral bo'lib, ichki integral ikki karra integral bo'lishi mumkin, ya'ni

$$(V) = \{(x,y,z) \in R^3 : a \leq x \leq b, (y,z) \in (G) - \text{kvadratlanuvchi soha}\}$$

bo'lsin.

13.19.3. Teorema. Agar: 1) $\iiint_{(V)} f(x,y,z) dv$ integral mavjud bo'lsa

$$2) \forall x \in [a,b] \text{ uchun } J(x) = \iint_{(G)} f(x,y,z) dy dz - \text{ikki karra integral}$$

mavjud bo'lsa, u holda

$$\int_a^b J(x) dx = \int_a^b dx \iint_{(G)} f(x,y,z) dy dz$$

takroriy integral ham mavjud bo'ladi va ushbu

$$\iiint_{(V)} f(x,y,z) dv = \int_a^b dx \iint_{(G)} f(x,y,z) dy dz$$

tenglik o'rinli bo'ladi.

Agar $(D) = \{(x,y) \in R^2 : a \leq x \leq b, y_1(x) \leq y \leq y_2(x)\}$ (bunda $y_1(x)$ va $y_2(x)$ funksiyalar $[a,b]$ da uzluksiz) soha egri chiziqli trapetsiyadan iborat bo'lsa, u holda

$$\iint_{(D)} J(x,y) dx dy - \text{ikki karra integral ushbu ko'rinishdagi}$$

$$\iint_{(D)} J(x, y) dx dy = \int_a^b dx \int_{y_1(x)}^{y_2(x)} J(x, y) dy = \int_a^b dx \int_{y_1(x)}^{y_2(x)} dz \int_{z_1(x, y)}^{z_2(x, y)} f(x, y, z) dz$$

takroriy integralga keltiriladi. Shunday qilib, bu holda uch karrali integralni hisoblash uchta ketma-ket bir karrali integrallarni hisoblashga olib kelinadi:

$$\iiint_{(V)} f(x, y, z) dv = \int_a^b dx \int_{y_1(x)}^{y_2(x)} dy \int_{z_1(x, y)}^{z_2(x, y)} f(x, y, z) dz. \quad (13.19.4)$$

Xususiylashtirilgan holda, $(V) = [a, b; c, d; k, l]$ to'g'ri burchakli parallelepipeddan iborat bo'lib, uning yz tekislikdagi proyeksiyasi $(R) = [c, d; k, l]$ to'g'ri to'rtburchakdan iborat bo'lsa, u holda (13.19.4) formulaning ko'rinishi

$$\iiint_{(V)} f(x, y, z) dv = \int_a^b dx \iint_{(R)} f(x, y, z) dR$$

shakilda bo'ladi. Bunda ichkaridagi ikki karrali integralni takroriy integralga keltirsak, natijada

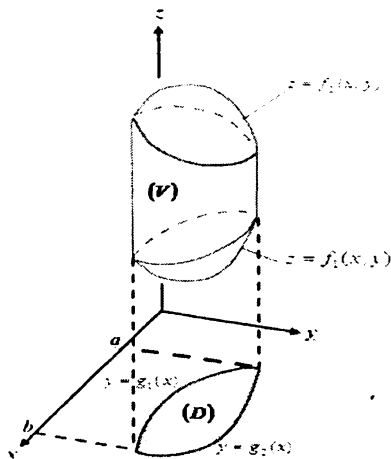
$$\iiint_{(V)} f(x, y, z) dv = \int_a^b dx \int_c^d dy \int_k^l f(x, y, z) dz$$

formulaga ega bo'lamiz, ya'ni bu holda uch karrali integralni hisoblash uchta takroriy aniq integrallarni hisoblashga keltiriladi.

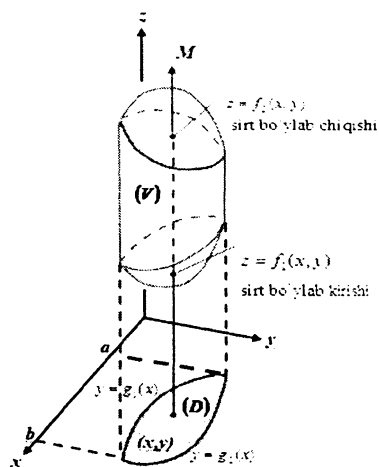
13.20. Uch karrali integralda integrallash chegaralarini topish algoritmi. (V) soha bo'yicha olingan $\iiint_{(V)} F(x, y, z) dv$ uch karrali

integralni, dastlab z ga nisbatan, so'ngra y ga nisbatan va nihoyat, x ga nisbatan hisoblash quyidagi bosqichlarini o'z ichiga oladi.

1-qadam. Chizish. (V) sohani uning xOy tekislikdagi (vertikal proyeksiyasi) (D) sohasi bilan birga chizish. (V) sohani pastdan va yuqoridan chegaralovchi sirtlarni hamda (D) ni chegaralovchi egri chiziqlarni aniqlash (13.28-chizma).



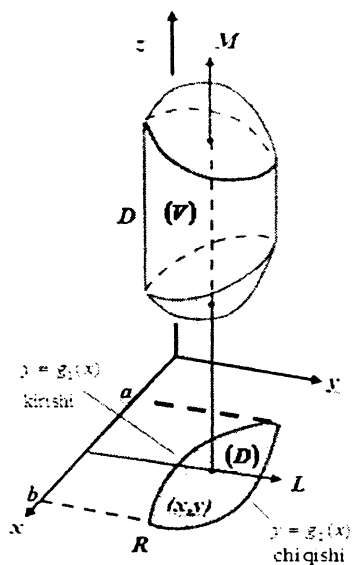
13.28-chizma.



13.29-chizma.

2 - qadam. Integrallashning z bo'yicha chegaralari. (D) dagi biror (x, y) nuqtadan Oz o'qqa parallel to'g'ri chiziqni o'tkazish. z ortganda M to'g'ri chiziq (V) ga $f_1(x, y)$ sirt bo'ylab kiradi va $f_2(x, y)$ sirt bo'yicha undan chiqadi. Ular z bo'yicha integrallash chegaralari bo'ladi (13.29-chizma).

3 - qadam. Integrallashning y bo'yicha chegaralari. (x, y) nuqtadan Oy o'qqa parallel L to'g'ri chiziqni o'tkazish. y ortganda L to'g'ri chiziq $y = g_1(x)$ bo'ylab (D) ga kiradi va $y = g_2(x)$ bo'ylab (D) dan chiqadi. Ular y bo'yicha integrallash chegaralari bo'ladi (13.30-chizma).



13.30-chizma.

4 - qadam. Integral-lashning x bo'yicha chegaralari. x bo'yicha integ-rallash chegaralari (D) da Oy o'qqa parallel o'tuvchi bar-cha to'g'ri chiziqlar ichidan tanlanadi (keltirilgan chiz-mada ular $x=a$ va $x=b$ to'g'ri chiziqlar). Ular x bo'yicha integrallash che-garalari bo'ladi. Bu holda

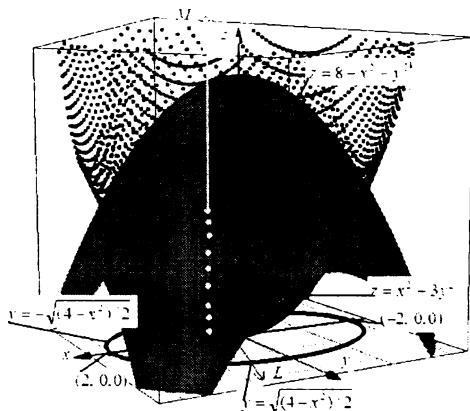
$$\iiint_{(V)} F(x, y, z) dV = \int_{x=a}^{x=b} dx \int_{y=g_1(x)}^{y=g_2(x)} dy \int_{z=f_1(x, y)}^{z=f_2(x, y)} F(x, y, z) dz$$

bo'ladi.

Uch karrali integralni olti xil usul bilan takroriy integralga keltirish mumkin. Bundan tashqari, (V) sohaga bog'liq holda berilgan integralni bir nechta takroriy integrallar yig'indisi ko'rinishida ham yozish mumkin.

13.20.1. Misol. Ushbu $J = \iiint_{(V)} (x+2) dx dy dz$ uch karrali integral hisoblansin, bunda (V) $z = x^2 + 3y^2$ va $z = 8 - x^2 - y^2$ sirtlar bilan chegaralangan soha.

Yechilishi. 1 - qadam. (V) soha pastdan $z = x^2 + 3y^2$ va yuqoridan $z = 8 - x^2 - y^2$ chegaralovchi sirtlar bilan chegaralangan. (V) sohani uning xOy tekislikdagi (vertikal proyeksiyasi) (D) sohasini topish uchun berilgan sirtlarning kesishish chizig'ini topamiz. $z > 0$ da sirtlarning kesishish chizig'i



13.31-chizma.

$x^2 + 3y^2 = 8 - x^2 - y^2$ yoki $x^2 + 2y^2 = 4$ ellipsoidan iborat ekan. (D) soha $x^2 + 2y^2 = 4$ ellips bilan chegaralangan (13.31-chizma).

2 - **qadam.** (D) dagi biror (x, y) nuqtadan Oz o'qqa parallel to'g'ri chiziqni o'tkazish. z ortganda M to'g'ri chiziq (V) ga $z = x^2 + 3y^2$ sirt bo'ylab kiradi va $z = 8 - x^2 - y^2$ sirt bo'yicha undan chiqadi. Ular z bo'yicha integrallash chegaralari bo'ladi.

3 - **qadam.** (x, y) nuqtadan Oy o'qqa parallel L to'g'ri chiziqni o'tkazish. y ortganda L to'g'ri chiziq $y = -\sqrt{(4-x^2)}/2$ bo'ylab (D) ga kiradi va $y = \sqrt{(4-x^2)}/2$ bo'ylab (D) dan chiqadi. Ular y bo'yicha integrallash chegaralari bo'ladi.

4-**qadam.** x bo'yicha integ-rallash chegara-lari (D) da Oy o'qqa parallel o'tuvchi barcha to'g'ri chiziqlar ichidan tanlana-di (keltirilgan chizmada ular $x = -2$, $(-2, 0, 0)$ va $x = 2$, $(2, 0, 0)$ to'g'ri chiziqlar).

Ular x bo'yicha integrallash chegaralari bo'ladi.

Endi berilgan uch karrali integralni takroriy integralga keltirib hisoblaymiz:

$$\begin{aligned} J &= \iiint_{(V)} (x+2) dx dy dz = \int_{-2}^2 dx \int_{-\sqrt{(4-x^2)}/2}^{\sqrt{(4-x^2)}/2} dy \int_{x^2+3y^2}^{8-x^2-y^2} (x+2) dz = \\ &= \int_{-2}^2 dx \int_{-\sqrt{(4-x^2)}/2}^{\sqrt{(4-x^2)}/2} (8-2x^2-4y^2)(x+2) dy = \\ &= \int_{-2}^2 (x+2) \left[(8-2x^2)y - \frac{4}{3}y^3 \right] \Big|_{y=-\sqrt{(4-x^2)}/2}^{y=\sqrt{(4-x^2)}/2} dx = \\ &= \frac{4\sqrt{2}}{3} \int_{-2}^2 (x+2)(4-x^2)^{3/2} dx = 16\pi\sqrt{2}. \end{aligned}$$

Misolni Maple tizimidan foydalanib, yechish:

➤ **with(student);**

➤ **Tripleint((x+2),z=x^2+3*y^2..8-x^2-y^2,y=-sqrt((4-x^2)/2)..sqrt((4-x^2)/2),x=-2..2);**

$$\int_{-2}^2 \int_{-1/2 \sqrt{8-2x^2}}^{1/2 \sqrt{8-2x^2}} \int_{x^2+3y^2}^{8-x^2-y^2} x+2 \, dz \, dy \, dx$$

> value(Tripleint((x+2),z=x^2+3*y^2..8-x^2-y^2,y=-sqrt((4-x^2)/2)..sqrt((4-x^2)/2),x=-2..2));

>

$$16\sqrt{2}\pi.$$

61- §. Uch karrali integrallarda o'zgaruvchilarni almashtirish

13.21. Uch karrali integrallarda o'zgaruvchilarni almashtirish. Uch karrali integrallarda ham o'zgaruvchilarni almashtirish ikki karrali integrallarda o'zgaruvchilarni almashtirish tartibida amalga oshiriladi.

Faraz qilaylik, fazoda $(Oxyz)$ dekart koordinatalar sistemasida yopiq (V) soha, $(Ouvw)$ dekart koordinatalar sistemasida yopiq (G) sohalar berilgan bo'lib, ular mos ravishda, bo'lakli silliq (S) va (σ) sirtlar bilan chegaralangan bo'lsin (13.32-chizma). Bu sohalar o'zaro ushbu

$$x = g(u, v, w), y = h(u, v, w), z = k(u, v, w) \quad (13.21.1)$$

sistema orqali bog'langan bo'lib, bu sistema quyidagi shartlarni qanoatlantirsin:

- 1) (13.21.1) – o'zaro bir qiymatli akslantirish bo'lsin;
- 2) $x = g(u, v, w), y = h(u, v, w), z = k(u, v, w)$ funksiyalar (G) sohada uzluksiz birinchi tartibli xususiy hosilalarga ega;
- 3) (13.21.1) akslantirishning yakobiani (G) sohada

$$J = \frac{D(x, y, z)}{D(u, v, w)} \neq 0$$

bo'lsin. (I') va (G) sohalar kublanuvchi bo'lsin.

13.21.2. Teorema. Agar $f(x, y, z)$ funksiya (I') sohada chegaralangan va bu sohaning chekli sondagi nol hajmli sirtlarida uzilishga ega bo'lib, qolgan barcha nuqtalarida uzluksiz bo'lsa, (13.21.1) akslantirish yuqoridagi 1)-3) shartlarni qanoatlantirsa, u holda ushbu

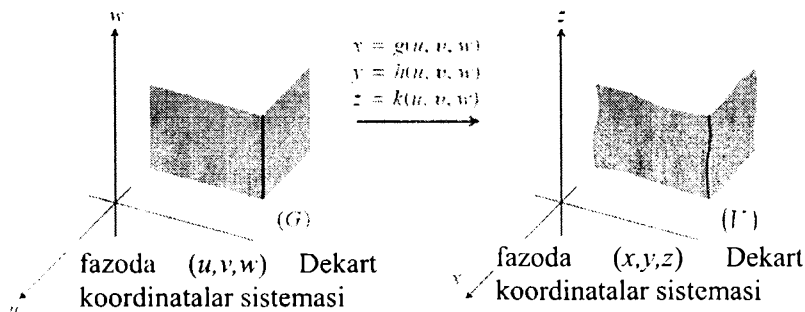
$$\iiint_{(V)} f(x, y, z) dV = \iiint_{(G)} f(g(u, v, w), h(u, v, w), k(u, v, w)) \left| \frac{D(x, y, z)}{D(u, v, w)} \right| du dv dw \quad (13.21.3)$$

formula o'rinli.

(13.21.3) formula uch karrali integrallarda o'zgaruvchilarni almashtirish formulasi deyiladi.

(13.21.3) formuladagi u, v, w lar x, y, z sistemadagi nuqtalarning egri chiziqli koordinatalari deyiladi.

$u = \text{const}, v = \text{const}, w = \text{const}$ sirtlar (x, y, z) fazoda egri chiziqli koordinat sirtlari deyiladi. Egri chiziqli koordinatalarning ikkitasi o'zgarish qiymat qabul qilib, faqat bittasi o'zgaruvchi bo'lsa, u holda bu egri chiziqni *koordinat chizig'i* deyiladi.



13.32-chizma.

Yuqoridagi mulohazalar (V) va (G) sohalar orasidagi qat'iy bir qiymatli moslikka asoslangan.

13.21.4. Misol. Ushbu $\int_0^3 dz \int_0^4 dy \int_{x=y/2}^{x=\frac{y}{2}+1} \left(\frac{2x-y}{2} + \frac{z}{3} \right) dx$ uch karrali integralni

$$u = \frac{2x-y}{2}, \quad v = \frac{y}{2}, \quad w = \frac{z}{3} \quad (13.21.5)$$

almashtirish yordamida (V) fazoda integrallash chegaralarini aniqlang va integralni hisoblang.

Yechilishi. (V) sohaning (xyz) fazoda integrallash chegaralari chizmada quyidagicha tasvirlanadi. Uch karrali integralda oʻzgaruvchilarni almashtirish formulasidagi yakobianni hisoblash uchun (13.21.5) sistemani x, y va z larga nisbatan yechamiz:

$$x = u + v, y = 2v, z = 3w \quad (13.21.6)$$

(x, y, z) koordinatalar tekisligida (V) sohaning chegaralari	(Δ) sohaning chegaralari uchun (u, v, w) tekisligidagi mosliklar	(u, v, w) koordinatalar tekisligidagi (Δ) sohaning chegaralari
$x = \frac{y}{2}$	$u + v = \frac{2v}{2} = v$	$u = 0$
$x = \frac{y}{2} + 1$	$u + v = \frac{2v}{2} + 1 = v + 1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$w = 1$
$z = 0$	$3w = 0$	
$z = 3$	$3w = 3$	

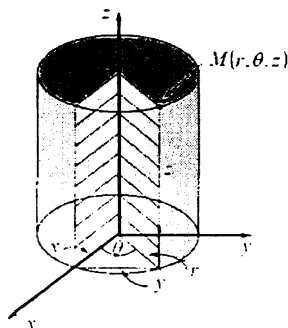
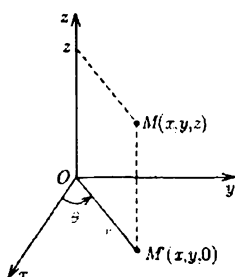
(21.21.6) tenglamalardan almashtirishning yakobianini hisoblaymiz:

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6.$$

(13.21.3) formulaga asosan, uch karrali integralni hisoblaymiz:

$$\begin{aligned}
& \int_0^3 dz \int_0^4 dy \int_{x=y/2}^{x=y/2+1} \left(\frac{2x-y}{2} + \frac{z}{3} \right) dx = \int_0^1 dw \int_0^2 dv \int_0^1 (u+w) J(u,v,w) du = \\
& = 6 \int_0^1 dw \int_0^2 \left(\frac{u^2}{2} + wu \right) \Big|_{u=0}^{u=1} dv = 6 \int_0^1 dw \int_0^2 \left(\frac{1}{2} + w \right) dv = 6 \int_0^1 \left(\frac{v}{2} + wv \right) \Big|_{v=0}^{v=2} dw = \\
& = 6 \int_0^1 (1+2w) dw = 6 [w + w^2] \Big|_{w=0}^{w=1} = 12.
\end{aligned}$$

13.22. (x, y, z) Dekart koordinatalar sistemasidan (r, θ, z) silindrik koordinatalar sistemasiga o'tish. Ma'lumki, (r, θ, z) -silindrik koordinatalar sistemasiga o'tilganda



13.33-chizma.

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z, \quad (0 \leq r < +\infty, 0 \leq \theta \leq 2\pi, -\infty < z < \infty) \quad (13.22.1)$$

$$J = \frac{D(x, y, z)}{D(r, \theta, z)} = r \quad (13.22.2)$$

formulalardan foydalaniladi. Bu (13.22.1)-(13.22.2) almashtirish natijasida berilgan uch karrali integral ushbu

$$J = \iiint_{(V)} f(x, y, z) dV = \iiint_{(G)} f(r \cos \theta, r \sin \theta, z) \cdot r \cdot dr d\theta dz, \quad (13.22.3)$$

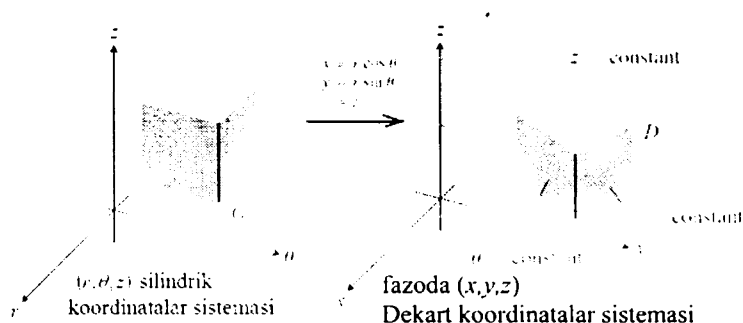
ko'rinishga keltiriladi. bunda (G) soha (V) sohaning yangi (r, θ, z) sistemasidagi aksidan iborat. (V) sohadagi $M(x, y, z)$ nuqta uchta (r, θ, z) sonlar juftligi bilan bir qiymatli aniqlanadi. $M(x, y, z)$ nuqtaning

tekislikdagi proyeksiyasi $M'(x, y, 0)$ bo'lib hisoblanadi (13.33-chizma), $z = M$ nuqtaning applikasi.

(r, θ, z) uchlik M nuqtaning *egri chiziqli koordinatasi* deyiladi.

(13.22.1) formula $0 \leq r < +\infty, 0 \leq \theta \leq 2\pi, -\infty < z < \infty$ sohani x, y, z tekisligiga akslantiradi. $r = 0, z = z$ to'g'ri chiziqlar bitta $(0; 0; z)$ nuqtaga akslantiriladi, bu bilan yuqorida 13.33-chizma aytilgan bir qiymatli moslik buziladi.

Bundan keyingi mulohazalarimizda muhim bo'lgan ba'zi tushunchalarni keltirib o'tamiz:



13.34-chizma.

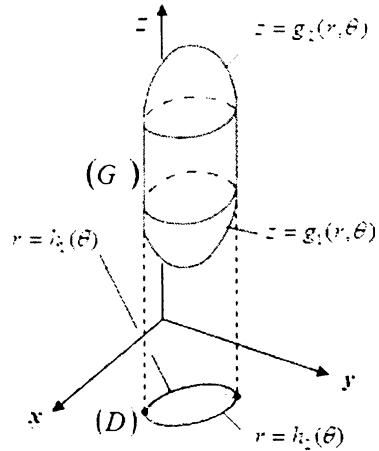
- a) $r = \text{const} - (r, \theta, z)$ koordinatalar sistemasida o'qi Oz va radiusi r bo'lgan doiraviy silindrik sirt tenglamasini ifoda qiladi (13.34-chizma);
- b) $\theta = \text{const} - (x, y)$ tekislik bilan φ burchak tashkil etib, $0z$ o'qidan o'tuvchi yarim tekislik tenglamasidan iborat (22.2-chizma);
- d) $z = \text{const} - 0z$ o'qiga perpendi-kulyar $((x, y)$ tekislikka parallel) tekislik tenglamasini ifoda qiladi (13.34-chizma).

13.23. Uch karrali integralni silindrik koordinatalarida integrallash algoritmi. Ushbu

$\iiint_{(G)} f(r, \theta, z) dV$ uch karrali integralni

lazodagi (G) sohada silindrik (r, φ, z) koordinatalarda, dastlab z ga nisbatan, so'ngra r ga nisbatan va nihoyat, θ ga nisbatan integrallash orqali hisoblash quyidagi qadamlarni o'z ichiga oladi.

1 - qadam. Chizish. (G) sohani va uning xOy tekislikka proyeksiyasi (D) ni chizish. So'ngra (G) ni va (D) ni chegaralovchi sirtlar va egri chiziqlarni belgilash (13.35-chizma).



13.35-chizma.

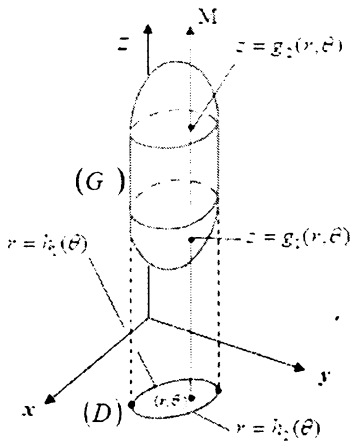
2 - qadam. Integrallashning z bo'yicha chegaralari. (D) dagi biror (r, θ) nuqtadan Oz o'qqa parallel M to'g'ri chiziqni o'tkazish. Bunda z ortganda M to'g'ri chiziq (G) ga $z = g_1(r, \theta)$ sirt bo'ylab kiradi va undan $z = g_2(r, \theta)$ sirt bo'ylab chiqadi. Ular z bo'yicha integrallash chegaralaridan iborat bo'ladi (13.36-chizma).

3 - qadam. Integrallashning r bo'yicha chegaralari. Koordinatalar boshidan chiqib, (r, θ) nuqtadan o'tuvchi L nurni chizamiz. Bu nur $r = h_1(\theta)$ bo'ylab (D) ga kiradi va $r = h_2(\theta)$ bo'ylab undan chiqadi. Ular r bo'yicha integrallash chegaralaridan iborat (13.37-chizma).

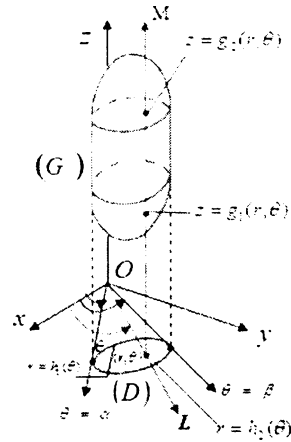
4- qadam. Integrallashning θ bo'yicha chegaralari. L nur (D) orqali o'tganda θ burchak Ox o'qning musbat yo'nalishida $\theta = \alpha$ dan $\theta = \beta$ gacha o'zgaradi. Ular θ bo'yicha integrallash chegaralari bo'ladi (13.38-chizma). Unda

$$\iiint_{(G)} f(r, \theta, z) dV = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{g_1(r, \theta)}^{g_2(r, \theta)} f(r, \theta, z) dz r dr d\theta \quad (13.23.1)$$

o'rinli bo'ladi.



13.37-chizma.



13.38-chizma.

13.23.2. Misol. Silindrik koordinatalar sistemasiga o'tib uch karrali integralni hisoblang:
$$\iiint_{(V)} \sqrt{x^2 + y^2} z dx dy dz, \text{ bunda } (V) \text{ soha}$$

$0 \leq x \leq 2, 0 \leq y \leq \sqrt{2x - x^2}, 0 \leq z \leq 1$ tengsizliklar bilan berilgan.

Yechilishi. 1 - qadam. Chizish. (V) sohani va uning xOy tekislikka proyeksiyasi (D) ni chizamiz (13.39-chizma). (V) sohani chegaralovchi sirtlar: $0 \leq x \leq 2, 0 \leq y \leq \sqrt{2x - x^2}, 0 \leq z \leq 1$. (D) sohani chegaralovchi egri chiziqlar: $0 \leq x \leq 2, 0 \leq y \leq \sqrt{2x - x^2}$.

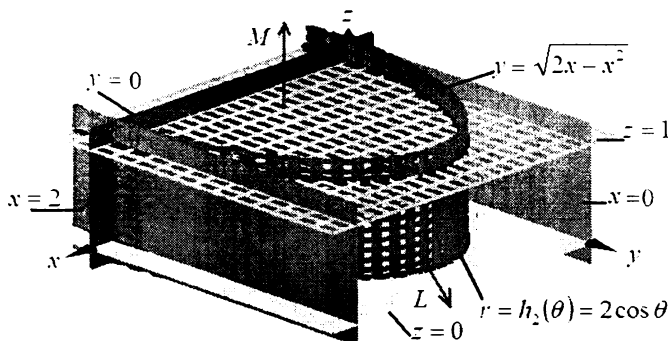
2 - qadam. Integrallashning z bo'yicha chegaralari. (D) dagi biror (r, θ) nuqtadan Oz o'qqa parallel M to'g'ri chiziqni o'tkazamiz. Bunda z ortganda M to'g'ri chiziq (V) ga $z=0$ sirt bo'ylab kiradi va undan $z=1$ sirt bo'ylab chiqadi. Ular z bo'yicha integrallash chegaralaridan iborat bo'ladi: $0 \leq z \leq 1$.

3 - qadam. Integrallashning r bo'yicha chegaralari. Koordinatalar boshidan chiqib (r, θ) nuqtadan o'tuvchi L nurni chizamiz. Bu nur

$r = h_1(\theta) = 0$ bo'ylab (D) ga kiradi va $r = h_2(\theta) = 2\cos\theta$ bo'ylab undan chiqadi. Ular r bo'yicha integrallash chegaralari bo'ladi: $0 \leq r \leq 2\cos\varphi$.

4- qadam. Integrallashning θ bo'yicha chegaralari. L nur (D) orqali o'tganda θ burchak Ox o'qning musbat yo'nalishida $\theta = 0$ dan $\theta = \frac{\pi}{2}$ gacha o'zgaradi. Ular θ bo'yicha integrallash chegaralari bo'ladi: $0 \leq \varphi \leq \frac{\pi}{2}$. (13.23.1) formulaga asosan, uch karrali integralni hisoblaymiz:

$$\begin{aligned} \iiint_{(V)} \sqrt{x^2 + y^2} z dx dy dz &= \iiint_{(r)} r^2 z dr d\varphi dz = \int_0^{\pi/2} d\varphi \int_0^{2\cos\varphi} r^2 dr \int_0^1 z dz = \frac{1}{2} \int_0^{\pi/2} d\varphi \int_0^{2\cos\varphi} r^2 dr = \\ &= \frac{4}{3} \int_0^{\pi/2} \cos^3 \varphi d\varphi = \frac{4}{3} \int_0^{\pi/2} (1 - \sin^2 \varphi) d(\sin \varphi) = \frac{4}{3} \left(\sin \varphi \Big|_0^{\pi/2} - \frac{\sin^3 \varphi}{3} \Big|_0^{\pi/2} \right) = \frac{4}{3} \left(1 - \frac{1}{3} \right) = \frac{8}{9}. \end{aligned}$$



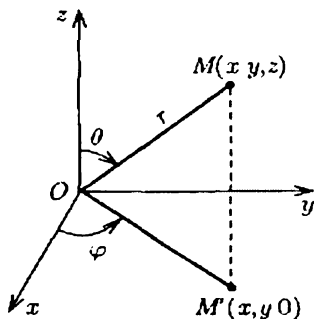
13.39-chizma.

13.24. (x, y, z) Dekart koordinatalar sistemasidan (r, θ, φ)

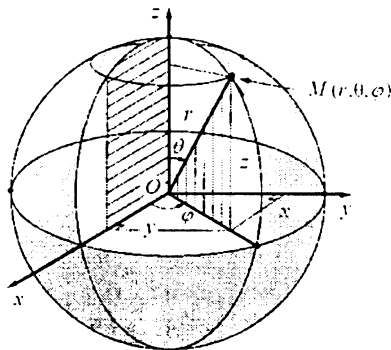
sferik koordinatalar sistemasiga o'tish. $M(x, y, z)$ nuqta xyz fazoning ixtiyoriy nuqtasi bo'lsin. $M'(x, y, 0)$ nuqta M nuqtaning xOy tekislikdagi proyeksiyasi bo'lsin (13.40-chizma). M nuqta uchta (r, θ, φ) sonlar juftligi bilan bir qiymati aniqlanadi, bunda r - koordinatalar boshi O nuqtadan M nuqttagacha bo'lgan masofa, θ - OM nur va Oz o'q orasidagi burchak, φ - (x, y) tekislikdagi $M'(x, y, 0)$ nuqtaning qutb

burchagi. (r, θ, φ) uchlik M nuqtaning sferik koordinatalari deyiladi (13.41-chizma).

To'g'ri burchakli (x, y, z) sistemadan sferik (r, θ, φ) koordinatalar sistemasiga o'tish, ushbu



13.40-chizma.



13.41-chizma.

$$\begin{aligned} x &= r \sin \theta \cos \varphi, \quad y = r \sin \theta \sin \varphi, \quad z = r \cos \theta \\ (0 \leq r < +\infty, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \varphi \leq 2\pi) \end{aligned} \quad (13.24.1)$$

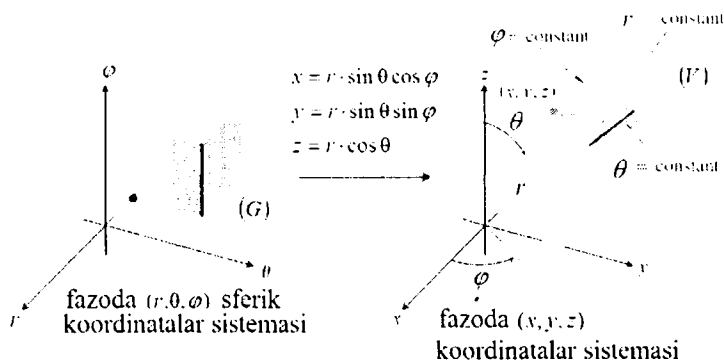
formulalar orqali amalga oshiriladi (ba'zi hollarda θ burchak deb OM va OM' nurlar orasidagi burchak olinadi, bu burchak $z > 0$ bo'lganda musbat ishora bilan, $z < 0$ bo'lganda esa manfiy ishora bilan olinadi, bu holda $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, (13.24.1) formulada $\sin \theta$ ni $\cos \theta$ ga, $\cos \theta$ ni $\sin \theta$ ga almashtirish kerak bo'ladi).

Bu holda ham (V) soha bilan (G) soha o'rtasidagi bir qiymatli moslikning buzilishini ko'ramiz, masalan, (13.24.1) akslantirish yordamida (r, θ, φ) fazodagi $r = 0$ tekislik koordinat boshi $x = y = z = 0$ ga, $\varphi = 0(\pi)$, $r = r$ to'g'ri chiziqlar bitta $x = y = 0$, $z = r$ nuqtaga akslantiriladi.

Sferik koordinatalar almashtirishda koordinatalar sirtlari quyidagi uchta sirtlar oilasini tashkil qiladi (13.42-chizma):

- a) $r = \text{const}$ — markazi koordinatalar boshida bo'lgan konsentrik sirtlar; b) $\theta = \text{const}$ — o'qidan o'tuvchi yarim tekislik;

d) $\varphi = \text{const}$ - o'qi z o'qidan iborat bo'lgan aylanma konuslardan tashkil topgan sirtlar oilasidan iborat.



13.42-chizma.

(13.24.1) akslantirishning yakobiani

$$J = \frac{D(x, y, z)}{D(r, \theta, \varphi)} = r^2 \sin \theta.$$

(θ burchak ikkinchi usul bilan tanlanganda, yakobian $r^2 \cos \theta$ ga teng bo'ladi).

Ba'zi hollarda (x, y, z) to'g'ri burchakli koordinatlar sistemasidan sferik koordinatlar sistemasiga o'tishda umumlashgan sferik koordinatalar deb ataluvchi formulalar orqali amalga oshiriladi:

$$x - x_0 = ar^n \sin^\alpha \theta \cos^\beta \varphi, y - y_0 = br^n \sin^\alpha \theta \sin^\beta \varphi, z - z_0 = cr^n \cos^\alpha \theta \quad (13.24.2)$$

($0 \leq r < \infty, 0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi$).

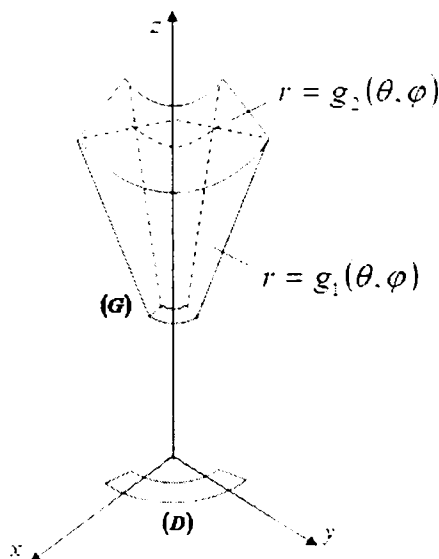
bunda $x_0, y_0, z_0, a, b, c, n, \alpha, \beta$ - biror sonlar (13.24.2) akslantirishning yakobiani

$$J = \frac{D(x, y, z)}{D(r, \theta, \varphi)} = abc n \alpha \beta r^{3n-1} \sin^{2\alpha-1} \theta \cos^{\alpha-1} \theta \sin^{\beta-1} \varphi \cos^{\beta-1} \varphi.$$

13.25. Uch karrali integralni sferik koordinatalarida integrallash algoritmi. Ushbu $\iiint_{(G)} f(r, \varphi, \theta) dG$ integralni, (G) soha

bo'yicha, sferik koordinatalarda, dastlab r bo'yicha, so'ngra θ

bo'yicha va nihoyat, φ bo'yicha integrallash yordamida hisoblash quyidagi qadamlarni o'z ichiga oladi.



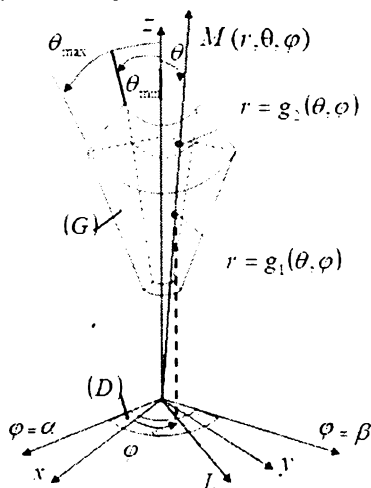
13.43-chizma.

1- qadam. *Chizish.* (G) sohani va uning xOy tekislikdagi (D) proyeksiyasini chizish, (G) ni chegaralovchi sirtlarni belgilash (aniqlash) (13.43-chizma).

2- qadam. *Integrallashning r bo'yicha chegaralari.* Koordinatalar boshidan (G) soha orqali Oz o'qining musbat yo'nalishi bilan θ burchak hosil qiluvchi M nurni o'tkazish. Shuningdek, M nurning xOy tekislikka proyeksiyasini (L proyeksiya deb ataladigan) o'tkazish. L nur Ox o'qining musbat yo'nalishi bilan φ burchak hosil qiladi. r ortganda M nur (G) sohaga $r = g_1(\theta, \varphi)$ bo'ylab kiradi va undan $r = g_2(\theta, \varphi)$ bo'ylab chiqadi. Ular integrallashning r bo'yicha chegaralari bo'ladi (13.44-chizma).

3- qadam. *Integrallashning θ bo'yicha chegaralari.* Ixtiyoriy berilgan φ uchun, M nur Oz o'qining musbat yo'nalishi bilan hosil

qladigan θ burchak $\theta = \theta_{\min}$ dan $\theta = \theta_{\max}$ gacha o'zgaradi. Shular integrallashning θ bo'yicha chegaralari bo'ladi.



13.44-chizma.

4- qadam. Integrallashning φ bo'yicha chegaralari. L nur (D) ustida o'tganda φ burchak α dan β gacha o'zgaradi. Ular integrallashning φ bo'yicha chegaralari bo'ladi.

Demak, uch karrali integral sferik koordinatalar sistemasida quyidagi

$$\iiint_{(G)} f(r, \theta, \varphi) dG = \int_{\alpha}^{\beta} \int_{\theta_{\min}}^{\theta_{\max}} \int_{r_{g_1}(\theta, \varphi)}^{r_{g_2}(\theta, \varphi)} f(r, \theta, \varphi) r^2 \sin \theta dr d\varphi d\theta \quad (13.25.1)$$

ko'rinishda bo'ladi.

13.25.2. Misol. Ushbu $\iiint_{(V)} (9 - x^2 - y^2) dx dy dz$ uch karrali

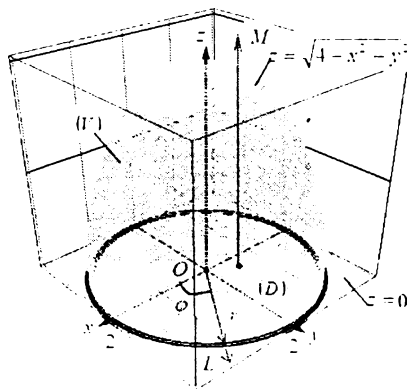
integralni sferik koordinatalar sistemasiga o'tib hisoblang, bunda (V) soha $x^2 + y^2 + z^2 \leq 4$ yarim shar, $z \geq 0$.

Yechilishi. 1- qadam. Chizish. $(x, y, z) \leftrightarrow (r, \varphi, \theta)$ koordinatalar sistemasiga o'tamiz:

$$f(r \cos \varphi \cos \theta, r \sin \varphi \cos \theta, r \sin \theta) = 9 - r^2 \cos^2 \theta, \quad I = r^2 \cos \theta.$$

(G) sohani va uning xOy tekislikdagi (D) proyeksiyasi markazi $O(0,0)$ nuqtada radiusi $R=2$ bo'lgan doirani $x^2 + y^2 \leq 4$ chizamiz (13.45-chizma). (G)sohani chegaralovchi sirt yarim sfera: $z = \sqrt{4 - x^2 - y^2}$.

2- qadam. *Integrallashning r bo'yicha chegaralari.* Koordinatalar boshidan (G) soha orqali Oz o'qining musbat yo'nalishi bilan θ burchak hosil qiluvchi M nurni o'tkazamiz. Shuningdek, M nurning xOy tekislikka proyeksiyasini (L proyeksiya deb ataladigan) o'tkazish. L nur Ox o'qining musbat yo'nalishi bilan φ burchak hosil qiladi. r ortganda M nur (G) sohaga $r = g_1(\theta, \varphi) = 0$ bo'ylab kiradi va undan $r = g_2(\theta, \varphi) = 2$ bo'ylab chiqadi. Ular integrallashning r bo'yicha chegaralari bo'ladi: $0 \leq r \leq 2$.



13.45-chizma.

3- qadam. *Integrallashning θ bo'yicha chegaralari.* Ixtiyoriy berilgan φ uchun, M nur Oz o'qining musbat yo'nalishi bilan hosil qiladigan θ burchak $\theta=0$ dan $\theta = \frac{\pi}{2}$ gacha o'zgaradi. Shular integrallashning θ bo'yicha chegaralari bo'ladi: $0 \leq \theta \leq \frac{\pi}{2}$.

4- qadam. *Integrallashning φ bo'yicha chegaralari.* L nur (D) ustidan o'tganda φ burchak $\alpha=0$ dan $\beta=2\pi$ gacha o'zgaradi. Ular integrallashning φ bo'yicha chegaralari bo'ladi: $0 \leq \varphi \leq 2\pi$.

Demak, berilgan uch karrali integral sferik koordinatalar sistemasida quyidagi

$$\iiint_{(G)} (9 - x^2 - y^2) dx dy dz = \iiint_{(G)} (9 - r^2 \cos^2 \theta) r^2 \cos \theta d\varphi dr d\theta.$$

bo'lmishda bo'ladi. Endi (13.25.1) formulaga asosan, integralni hisoblaymiz:

$$\begin{aligned}
 & \iiint_{(V)} (9r^2 \cos \theta - r^4 \cos^3 \theta) d\varphi dr d\theta = \\
 & \int_0^{2\pi} d\varphi \int_0^{\pi/2} d\theta \int_0^2 (9r^2 \cos \theta - r^4 \cos^3 \theta) dr = \varphi \Big|_0^{2\pi} \int_0^{\pi/2} \left(9 \frac{r^3}{3} \Big|_0^2 \cos \theta - \frac{r^5}{5} \Big|_0^2 \cos^3 \theta \right) d\theta = \\
 & 2\pi \left(3 \cdot 8 \cdot \int_0^{\pi/2} \cos \theta d\theta - \frac{32}{5} \int_0^{\pi/2} (1 - \sin^2 \theta) d(\sin \theta) \right) = \\
 & = 2\pi \left(24 \sin \theta \Big|_0^{\pi/2} - \frac{32}{5} \left(\sin \theta \Big|_0^{\pi/2} - \frac{\sin^3 \theta}{3} \Big|_0^{\pi/2} \right) \right) = \\
 & = 2\pi \left(24 - \frac{32}{5} \left[\left(\sin \frac{\pi}{2} - 0 \right) - \frac{1}{3} \left(\sin^3 \frac{\pi}{2} - \sin^3 0 \right) \right] \right) = \\
 & = 2\pi \left[24 - \frac{32}{5} \left(1 - \frac{1}{3} \right) \right] = 2\pi \left(24 - \frac{32}{5} \cdot \frac{2}{3} \right) = 2\pi \frac{24 \cdot 15 - 64}{15} = \frac{292}{15} \pi.
 \end{aligned}$$

62- §. Uch karrali integralning ba'zi bir tatbiqlari

13.26. Uch karrali integral yordamida jismning hajmini hisoblash. (x, y, z) fazoda kublanuvchi (V) jismning hajmi

$$V = \iiint_{(V)} dx dy dz \quad (13.26.1)$$

formula orqali ifoda qilinadi.

Agar (V) soha ushbu $(V) = \{(x, y, z) \in R^3 : (x, y) \in (D), 0 \leq z \leq f(x, y)\}$ (bunda $(D) - (x, y)$ tekislikdagi kvadratlanuvchi soha, $f(x, y)$ funksiya esa, (D) sohada uzluksiz) ko'rinishda bo'lsa, u holda (13.26.1) uch karrali integralni takroriy integrallarga keltirish natijasida:

$$V = \iint_{(D)} dx dy \int_0^{f(x,y)} dz = \iint_{(D)} \left[z \Big|_0^{f(x,y)} \right] dx dy = \iint_{(D)} f(x, y) dx dy \quad (13.26.2)$$

formula hosil bo'ladi.

Agar (V) soha $x = a$ va $x = b$ tekisliklar orasida bo'lib, (V) sohani $x = \text{const}$ tekislik bilan kesganda, har bir kesim yuzi $S(x)$ ga teng bo'lgan

kvadratlanuvchi (D) sohadan iborat bo'lsa, u holda (13.26.2) formulaning ko'rinishi

$$V = \int_a^b dx \iint_{(D)} dydz = \int_a^b S(x)dx \quad (13.26.3)$$

shaklga keladi.

(13.26.1) tenglikda yangi ξ, η, ζ o'zgaruvchilarga (13.26.3) formula orqali o'tganda (V) sohaning hajmi V uchun ushbu

$$V = \iiint_{(V)} \left| \frac{D(x, y, z)}{D(\xi, \eta, \zeta)} \right| d\xi d\eta d\zeta$$

formulaga ega bo'lamiz, bunda to'g'ri burchakli koordinatalardagi elementar hajm egri chiziqli koordinatalarda

$$dV = \left| \frac{D(x, y, z)}{D(\xi, \eta, \zeta)} \right| d\xi d\eta d\zeta$$

ko'rinishga ega bo'ladi.

13.26.4. Mmisol. $z = 4 - x^2 - y^2$ va $z = x^2 + y^2$ sirtlar bilan chegaralan-gan (V) jismning hajmini hisoblang.

Yechilishi. (V) sohada $f(x, y, z) = 1$ bo'lib, (V) jismning hajmini (13.26.1) formula bilan hisoblaymiz. Integrallash chegaralarini topishni qadam bilan bajaramiz.

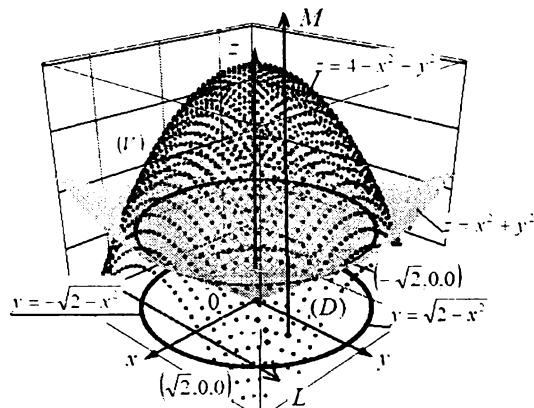
1 - qadam. Chizish. (V) jism 13.46-chizmada tasvirlangan. Sirtlarning kesish chizig'ini topish uchun $\begin{cases} z = 4 - x^2 - y^2 \\ z = x^2 + y^2 \end{cases}$ sistemani yechamiz:

$$\begin{cases} z = 4 - x^2 - y^2 \\ z = x^2 + y^2 \end{cases} \Rightarrow x^2 + y^2 = 4 - x^2 - y^2 \text{ yoki } 2x^2 + 2y^2 = 4 \text{ yoki}$$

$x^2 + y^2 = 2$ aylanadan iborat ekan. (V) jismning xOy tekislikdagi proyeksiyasi $x^2 + y^2 = 2$ aylana bilan chegaralangan (D) soha. (D) soha yuqoridan $y = \sqrt{2 - x^2}$ bilan, quyidan $y = -\sqrt{2 - x^2}$ esa, chiziqlar bilan chegaralangan.

2 - qadam. z o'zgaruvchining integrallash chegaralarini aniqlash. (D) dagi biror (x, y) nuqtadan Oz o'qqa parallel M to'g'ri

chiziqni o'tkazamiz. Bunda z ortqanda M to'g'ri chiziq (V) ga $x^2 + y^2$ sirt bo'ylab kiradi va undan $z = 4 - x^2 - y^2$ sirt bo'ylab chiqadi. Ular z bo'yicha integrallash chegaralaridan iborat bo'ladi, ya'ni $x^2 + y^2 \leq z \leq 4 - x^2 - y^2$.



13.46-chizma.

3 - qadam. y o'zgaruvchining integrallash chegaralarini aniqlash. (x, y) nuqtadan chiquvchi L to'g'ri chiziq Oy o'qqa parallel yo'nalishda bo'lib, (D) sohada y o'zgaruvchi $y = -\sqrt{2 - x^2}$ dan $y = \sqrt{2 - x^2}$ chiziqqacha o'zgaradi, ya'ni $-\sqrt{2 - x^2} \leq y \leq \sqrt{2 - x^2}$.

4 - qadam. x o'zgaruvchining o'z integrallash chegaralarini aniqlash. (D) sohadagi x o'zgaruvchi $x = -\sqrt{2}(-\sqrt{2}, 0, 0)$ dan $x = \sqrt{2}(\sqrt{2}, 0, 0)$ gacha o'zgaradi. Endi (V) jismning hajmini hisoblaymiz.

$$\begin{aligned} V &= \iiint_{(V)} dz dy dx = \int_{-\sqrt{2}}^{\sqrt{2}} dx \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{x^2+y^2}^{4-x^2-y^2} dz = \int_{-\sqrt{2}}^{\sqrt{2}} dx \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} (4 - 2x^2 - 2y^2) dy = \\ &= 2 \int_{-\sqrt{2}}^{\sqrt{2}} \left[(2 - x^2)y - \frac{1}{3}y^3 \right]_{y=-\sqrt{2-x^2}}^{y=\sqrt{2-x^2}} dx = 4 \int_{-\sqrt{2}}^{\sqrt{2}} \left((2 - x^2)^{3/2} - \frac{1}{3}(2 - x^2)^{3/2} \right) dx = \\ &= \frac{8}{3} \int_{-\sqrt{2}}^{\sqrt{2}} (2 - x^2)^{3/2} dx = 4\pi. \end{aligned}$$

Misolni Maply tizimidan foydalanib, yechish:

> with(student);

> Tripleint(1,z=x^2+y^2..4-x^2-y^2,y=-sqrt(2-x^2)..sqrt(2-x^2),x=-sqrt(2)..sqrt(2));

$$\int_{-2}^2 \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{x^2+y^2}^{4-x^2-y^2} 1 \, dz \, dy \, dx$$

> value(Tripleint(1,z=x^2+y^2..4-x^2-y^2,y=-sqrt(2-x^2)..sqrt(2-x^2),x=-sqrt(2)..sqrt(2)));

>

$$4\pi$$

13.27. Uch karrali integrallar yordamida mexanikaga oid masalalarni yechish. Agar (V) Oxyz fazoda zichligi uzluksiz $\rho(x, y, z)$ funksiyadan iborat bo'lgan (kublanuvchi) jism bo'lsa, u holda ushbu

$$\iiint_{(V)} \rho(x, y, z) dx dy dz$$

uch karrali integral (V) jismning massasi m ni ifoda qiladi, ya'ni

$$m = \iiint_{(V)} \rho(x, y, z) dx dy dz.$$

(V) jismning koordinatlar o'qlariga nisbatan inersiya momentlari:

$$J_z = \iiint_{(V)} (x^2 + y^2) \rho(M) dv, \quad J_y = \iiint_{(V)} (x^2 + z^2) \rho(M) dv,$$

$$J_x = \iiint_{(V)} (y^2 + z^2) \rho(M) dv.$$

Koordinatlar boshiga nisbatan inersiya momenti

$$J_0 = \iiint_{(V)} (x^2 + y^2 + z^2) dv$$

formulalar orqali topiladi. (V) jism og'irlik markazining koordinatalari

$$x_c = \frac{\iiint_{(V)} x \rho(M) dv}{m}; \quad y_c = \frac{\iiint_{(V)} y \rho(M) dv}{m}; \quad z_c = \frac{\iiint_{(V)} z \rho(M) dv}{m}$$

formulalar orqali ifoda qilinadi.

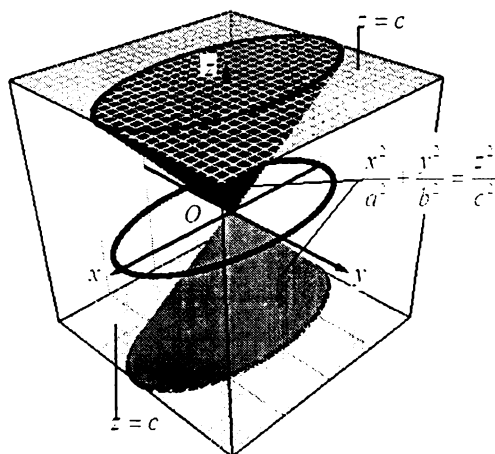
Agar $\rho(x, y, z) = \text{const}$ bo'lsa,

$$x_c = \frac{\iiint_{(V)} x dv}{V}; \quad y_c = \frac{\iiint_{(V)} y dv}{V}; \quad z_c = \frac{\iiint_{(V)} z dv}{V}$$

bunda $V - (V)$ jismning hajmi.

13.27.1. Misol. Ushbu $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$, $z = c$ sirtlar bilan chegaralangan bir jinsli jism og'irlik markazining koordinatlarini toping.

Yechilishi. Berilgan bir jinsli jism $z = c \sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}}$ kanonik sirtning qismi, $z = c$ tekislikning qismidan tashkil topgan bo'lib, u Oz o'qiga simmetrik (27.1-chizma). Uning og'irlik markazi $M_0(x_0, y_0, z_0)$ bo'lib, bu Oz o'qda bo'ladi, bunda $x_0 = 0, y_0 = 0$ teng.



13.47-chizma.

Bu holda jism og'irlik markazining koordinatasi ushbu

$$z_0 = \frac{\iiint_{(V)} z \cdot \rho(x, y, z) dx dy dz}{V}$$

formula orqali topiladi, bunda shartga ko'ra, $\rho(x, y, z) = 1$ deb olingani uchun

$z_0 = \frac{1}{V} \cdot \iiint_{(V)} z dx dy dz$, bunda $V = (V)$ jism hajmining sonli

qiymati: $V = \iiint_{(V)} dx dy dz$, bunga umumlashgan silindrik almashtirish

formulasini (ya'ni

$$x = a\rho \cos^{\alpha} \varphi, y = b\rho \sin^{\alpha} \varphi, z = z (0 \leq \varphi \leq 2\pi, \rho \geq 0) \text{ da } \alpha = 1)$$

qo'llaymiz.

$$V = ab \int_0^{2\pi} d\varphi \int_0^1 \rho d\rho \int_{c\rho}^c dz = 2\pi abc \int_0^1 \rho(1-\rho) d\rho = \frac{\pi}{3} abc.$$

Shunday qilib

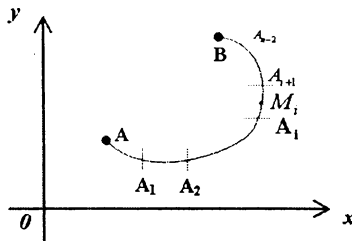
$$z_0 = \frac{3}{\pi} \int_0^{2\pi} d\varphi \int_0^1 \rho d\rho \int_{c\rho}^c dz = 3c \int_0^1 \rho(1-\rho^2) d\rho = \frac{3}{4} c.$$

Demak, $M_0 \left(0, 0, \frac{3}{4} c \right)$.

57-§. Birinchi tur egri chiziqli integralning ta'rif

14.1. Birinchi tur egri chiziqli integral va unga olib keladigan mexanik masala. Egri chiziqli integral tushunchasiga olib keladigan ushbu mexanik masalani qaraymiz.

Tekislikda sodda uzluksiz to'g'rilanuvchi (hozircha ochiq) (AB) egri chiziq berilgan bo'lsin (14.1-chizma). Bu egri chiziq bo'ylab biror massa tarqalgan bo'lib, uning $\forall M$ nuqtasidagi chiziqli zichligi $\rho(M)$ bo'lsin. (AB) egri chiziq bo'ylab tarqalgan massa m ni topish talab qilingan bo'lsin. Massani topish uchun (AB) egri chiziqni $A_0, A_1, \dots, A_{n-1}$ ($A_0 = A, A_{n-1} = B$) nuqtalar yordamida bo'laklarga bo'lamiz (bo'lishni B dan A ga qarab ham olib borish mumkin edi, bu umumiy mulohazaga halaqit bermaydi). (AB) egri chiziqning $A_i \tilde{A}_{i+1}$ yoyidan ixtiyoriy M_i nuqtani olamiz. Bu nuqtaning chiziqli zichligi $\rho(M_i)$. Faraz qilaylik, $A_i \tilde{A}_{i+1}$ yoyining hamma nuqtasidagi chiziqli zichlik $\rho(M_i)$ bo'lsin. $A_i \tilde{A}_{i+1}$ yoyining uzunligini σ_i , $A_i \tilde{A}_{i+1}$ yoy bo'ylab tarqalgan massani esa, m_i deb belgilaymiz. U holda $A_i \tilde{A}_{i+1}$ dagi massa $m_i \approx \rho(M_i) \sigma_i$ bo'ladi. (AB) egri chiziq bo'ylab tarqalgan butun massa



$$m \approx \sum_{i=0}^{n-1} \rho(M_i) \sigma_i.$$

14.1-chizma.

Agar $\max_{1 \leq i \leq n} \sigma_i = \lambda$ deb belgilab yuqoridagi yig'indida $\lambda \rightarrow 0$ da limitga o'tsak, talab qilingan massani topish uchun aniq formulaga ega bo'lamiz, ya'ni

$$m = \lim_{\lambda \rightarrow 0} \sum_{i=0}^{n-1} \rho(M_i) \sigma_i. \quad (14.1.1)$$

14.2. Birinchi tur egri chiziqli integralning ta'rifi. Tekislikda xOy koordinatlar sistemasi tanlagan bo'lsin. Tekislikda biror silliq yoki bo'lakli silliq¹ (AB) chiziq berilgan. Bu egri chiziqda ikki yo'nalishdan birini musbat yo'nalish, ikkinchisini manfiy yo'nalish deb qabul qilaylik.

$f(x, y)$ funksiya (AB) egri chiziqda aniqlangan va chegaralangan bo'lsin. (AB) egri chiziqni yuqoridagi singari $A = A_0, A_1, A_2, \dots, A_{n-1}, A_n = B$ nuqtalar bilan n ta bo'lakka bo'lamiz. Bu $A_0, A_1, \dots, A_n$ nuqtalar sistemasi $\overset{\cup}{AB}$ yoyning bo'linishi deb ataladi va u

$$P = \{A_0, A_1, A_2, \dots, A_n\}$$

kabi belgilanadi.

$\overset{\cup}{A_i A_{i+1}}$ yoy uzunligini Δs_i ($i = 0, 1, 2, \dots, n-1$) bilan, $\max\{\Delta s_i\} = \lambda_p$ deb belgilaymiz.

(AB) egri chiziqning ixtiyoriy $P = \{A_0, A_1, A_2, \dots, A_n\}$ bo'lishini va uning har bir $\overset{\cup}{A_i A_{i+1}}$ yoyidan ixtiyoriy $M_i(\xi_i, \eta_i)$ ($i = 0, 1, 2, \dots, n-1$) nuqtani olib, berilgan funksiyaning bu nuqtadagi $f(\xi_i, \eta_i)$ qiymatini $\overset{\cup}{A_i A_{i+1}}$ yoyining uzunligi Δs_i ga ko'paytirib, quyidagi

$$\sigma_p = \sum_{i=0}^{n-1} f(\xi_i, \eta_i) \Delta s_i$$

yig'indini tuzamiz.

$(\bar{AB})$ egri chiziqning shunday

$$P_1, P_2, \dots, P_m, \dots \quad (14.2.1)$$

bo'linishlar ketma-ketligini qaraylikki, ularning mos diametrlaridan tuzilgan

$$\lambda_{P_1}, \lambda_{P_2}, \dots, \lambda_{P_m}, \dots$$

ketma-ketlik nolga intilsin: $\lambda_{P_m} \rightarrow 0$. (14.2.1) bo'linishlarga mos kelgan ushbu

¹ Agar egri chiziq tenglamasi $x = \varphi(t)$, $y = \psi(t)$, $\alpha \leq t \leq \beta$ ko'rinishida berilgan bo'lib, $\varphi(t)$, $\psi(t)$ funksiyalar uzluksiz va uzluksiz bir vaqtda nolga ega bo'lmaydigan $\varphi'(t)$ va $\psi'(t)$ hosilalaraga ega bo'lsa, u holda bu egri chiziq silliq chiziq deyiladi.

Числи соизиди сизлик бу нақлардан тuzилган узлуksiz эгри чизикка бу такис-сизлик дейилади.

$$\sigma_{P_1}, \sigma_{P_2}, \dots, \sigma_{P_n} \dots \quad (14.2.2)$$

yig'indilar ketma-ketligini tuzamiz. Bu ketma-ketlikning har bir hadi $M_i(\xi_i, \eta_i)$ nuqtalarga bog'liq.

14.2.3. Ta'rif. Agar (AB) egri chiziqlarning (14.2.1) ko'rinishidagi har qanday bo'linishlar ketma-ketligi $\{P_n\}$ olinganda ham, unga mos kelgan (14.2.2) ko'rinishdagi yig'indilar $\{\sigma_n\}$ ketma-ketligi $M_i(\xi_i, \eta_i)$ nuqtalarning tanlab olinishiga bog'liq bo'lmagan holda hamma vaqt bitta chekli J songa intilsa, bu son σ_P yig'indining limiti deb ataladi va u

$$\lim_{\lambda_P \rightarrow 0} \sigma_P = \lim_{\lambda_P \rightarrow 0} \sum_{i=0}^{n-1} f(\xi_i, \eta_i) \Delta s_i = J$$

kabi belgilanadi. Bu holda $f(x, y)$ funksiya (AB) egri chiziq bo'yicha integrallanuvchi deyiladi. Bu J limitga $f(x, y)$ funksiyaning (AB) egri chiziq bo'yicha birinchi tur egri chiziqli integrali deb ataladi va u

$$J = \int_{(AB)} f(x, y) ds \quad (14.2.4)$$

kabi belgilanadi.

Shunday qilib, kiritilgan egri chiziqli integral tushunchasining o'ziga xosligi qaralayotgan ikki argumentli $f(x, y)$ -funksiyaning berilish sohasi tekislikdagi (AB) egri chiziq ekanligidadir. Qolgan boshqa hamma mulohazalar (bo'linishlarning olinishi, bo'linish oraliqlaridan ixtiyoriy nuqtani tanlab olib integral yig'indi tuzish, limitga o'tish) 7-bob, 7.2-bandda kiritilgan aniq integral tushunchasi kabi.

Demak, yuqorida keltirilgan (14.1.1) ifodani, ya'ni egri chiziq bo'ylab tarqalgan m massani ham quyidagicha

$$m = \int_{(AB)} \rho(M) ds \quad (14.2.5)$$

vozish mumkin bo'lar ekan. Ta'kidlaymizki, yuqorida keltirilgan ta'rifda yo'nalish hech qanday rol o'ynamaydi. Masalan, egri chiziq vopiq bo'lmasdan egri chiziq yo'nalishini A dan B ga qarab olingandagi egri chiziqli integral bilan, yo'nalish B dan A ga qarab olingandagi egri chiziqli integralning qiymati o'zaro teng bo'ladi, ya'ni

$$\int_{(AB)} f'(x, y) ds = \int_{(B, A)} f(x, y) ds$$

Shunday qilib birinchi tur egri chiziqli integral egri chiziqda olingan yo'nalishga bog'liq bo'lmaydi.

Xuddi shunday, egri chiziq fazoda berilganda, egri chiziqli integral tushunchasi kiritiladi:

$$\int_{(AB)} f(M) ds = \int_{(AB)} f(x, y, z) ds.$$

64-§. Birinchi tur egri chiziqli integralning mavjudlik sharti va uni aniq integralga keltirish

14.3. Birinchi tur egri chiziqli integralning mavjudlik sharti.

Faraz qilaylik (AB) sodda to'g'rilanuvchi egri chiziqda ixtiyoriy ravishda yo'nalish (ikki yo'nalishdan biri) aniqlangan bo'lsin. Egri chiziqdagi M nuqtaning holati A boshlang'ich nuqtadan hisoblanuvchi $s = \overset{\circ}{AB}$ uzunligi bilan aniqlansin. U holda (AB) egri chiziq parametrik tenglamasi ushbu

$$x = x(s), \quad y = y(s) \quad (0 \leq s \leq S),$$

ko'rinishda bo'lsin, bunda $S, \overset{\circ}{AB}$ ning uzunligi ko'rinishda ifodalanadi, egri chiziqda berilgan $f(x, y)$ funksiya esa, s ning murakkab $f(x(s), y(s))$ funksiyasiga keltiriladi. (AB) egri chiziqning ixtiyoriy $P = \{A_0, A_1, \dots, A_n\}$ bo'linishni va har bir $A_i A_{i+1}$ dan $\forall M_i(\xi_i, \eta_i)$ nuqtani olaylik. Har bir A_i nuqtaga mos kelgan AA_i yoyining uzunligi s_i , har bir M_i nuqtaga mos keladigan AA_i yoyning uzunligini s'_i deb belgilaylik. U holda, $A_i A_{i+1}$ yoyining uzunligi $s_{i+1} - s_i = \Delta s_i$ bo'ladi. Natijada P bo'linishga nisbatan tuzilgan integral yig'indi

$$\sigma_P = \sum_{i=0}^{n-1} f(\xi_i, \eta_i) \cdot \Delta s_i,$$

ushbu

$$\sigma_P(f) = \sum_{i=0}^{n-1} f(x(s'_i), y(s'_i)) \Delta s_i$$

ko'rinishga keladi, bunda $\xi_i = x(s'_i), \eta_i = y(s'_i)$. Bu yig'indining $[0, S]$

oraliqda $f(x(s), y(s))$ funksiyaning Riman integral yig'indisi ekanligiga ishonch hosil qilish qiyin emas.

Agar $f(x, y)$ funksiya (AB) egri chiziqda uzluksiz bo'lsa, $f(x(s), y(s))$ funksiya ham $[0, S]$ da uzluksiz va integrallanuvchi bo'ladi, ya'ni

$$\lim_{\lambda_P \rightarrow 0} \sigma_P = \lim_{\lambda_P \rightarrow 0} \sum_{i=0}^{n-1} f(x(s'_i), y(s'_i)) \cdot \Delta s_i = \int_0^S f(x(s), y(s)) ds$$

mavjud bo'ladi.

Shunday qilib quyidagi teorema kelamiz.

14.3.1. Teorema. Agar $f(x, y)$ funksiya (AB) egri chiziqda uzluksiz bo'lsa, u holda bu funksiyaning (AB) egri chiziqli bo'yicha birinchi tur egri chiziqli integrali mavjud va ushbu

$$\int_{(AB)} f(x, y) ds = \int_0^S f(x(s), y(s)) ds \quad (14.3.2)$$

tenglik o'rinli bo'ladi.

14.4. Birinchi tur egri chiziqli integralning aniq integralga keltirish.

1. (AB) egri chiziqli o'zining parametrik $x = \varphi(t), y = \psi(t)$ ($t_0 \leq t \leq T$) tenglamasi bilan berilgan bo'lsin, bunda $\varphi(t)$ va $\psi(t)$ funksiyalar uzluksiz va uzluksiz $\varphi'(t), \psi'(t)$ hosilalarga ega. Bundan tashqari egri chiziqli karrali nuqtalariga ega emas deb faraz qilamiz. Bu holda egri chiziqli to'g'rilanuvchi bo'ladi. $S = \overline{AM} = s(t)$ yoyning o'sishi parametr t ning o'sishiga mos kelsa, u holda, ma'lumki, o'zgaruvchan yoy differensial $ds = \sqrt{[\varphi'(t)]^2 + [\psi'(t)]^2} dt$ bo'ladi. (14.3.2) da s o'zgaruvchini t o'zgaruvchiga almashtirish bilan ushbu egri chiziqli integralni

$$\int_{(AB)} f(x, y) ds = \int_{t_0}^T f(\varphi(t), \psi(t)) \sqrt{[\varphi'(t)]^2 + [\psi'(t)]^2} dt \quad (14.4.1)$$

hisoblash formulasiga keltiramiz.

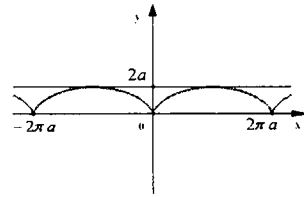
14.4.2. Misol. Ushbu $J = \int_{(K)} y ds$ birinchi tur egri chiziqli integral

hisoblansin, bunda $(K): x = a(t - \sin t), y = a(1 - \cos t)$ ($0 \leq t \leq 2\pi$) sikloidaning birinchi arkasi.

Yechilishi. Misolning berilishiga ko'ra, $f(x, y) = y$.

$$\begin{aligned} ds &= \sqrt{x_t'^2 + y_t'^2} dt = \sqrt{a^2(1 - \cos t)^2 + a^2 \sin^2 t} dt = \\ &= a\sqrt{2(1 - \cos t)} dt = \sqrt{2}a\sqrt{1 - \cos t} dt = 2a \sin \frac{t}{2} dt \end{aligned}$$

(14.4.1) formulaga asosan (14.2 - chizma).



14.2-chizma.

$$\begin{aligned} J &= \int_0^{2\pi} a(1 - \cos t) 2a \sin \frac{t}{2} dt = a^2 \int_0^{2\pi} \sin^3 \frac{t}{2} dt = 8a^2 \int_0^{\pi} \sin^3 x dx = \\ &= 8a^2 \left(-\cos x + \frac{\cos^3 x}{3} \right) \Big|_0^{\pi} = 16a^2 \left(1 - \frac{1}{3} \right) = \frac{32}{3} a^2. \end{aligned}$$

2. Agar xususiyl holda (AB) egri chiziq tenglamasi $y = y(x)$ ($a \leq x \leq b$) ko'rinishda bo'lib, $y(x)$ – uzluksiz va uzluksiz $y'(x)$ hosilaga ega bo'lsa, u holda, o'zgaruvchan yoy differensial $ds = \sqrt{1 + (y')^2} dx$ bo'ladi va birinchi tur egri chiziq integral mavjud va u ushbu

$$\int_{(AB)} f(x, y) ds = \int_a^b f(x, y(x)) \sqrt{1 + y_x'^2} dx \quad (14.4.3)$$

formula orqali hisoblanadi.

Xuddi shunday, (AB) egri chiziq tenglamasi $x = x(y)$ ($c \leq y \leq d$) ko'rinishda bo'lib, $x(y)$ – uzluksiz va uzluksiz $x'(y)$ hosilaga ega bo'lsa, u holda, o'zgaruvchan yoy differensial $ds = \sqrt{1 + (x_y')^2} dy$ bo'ladi va birinchi tur egri chiziq integral mavjud bo'ladi va u ushbu

$$\int_{(AB)} f(x, y) ds = \int_c^d f(x(y), y) \sqrt{1 + x_y'^2} dy$$

formula orqali hisoblanadi.

14.4.4. Misol. Ushbu $\int_{(AB)} y ds$ egri chiziqli integralni hisoblang,

bunda $(AB) - y^2 = 2x$ parabola yoyining $(0; 0)$ va $(2; 2)$ nuqtalarni birlashtiruvchi qismi.

Yechilishi. Berilgan $(A\bar{B})$ egri chiziqning tenglamasi $y = \sqrt{2x}$, $y' = \frac{1}{\sqrt{2x}}$. (14.4.3) formulaga asosan, $ds = \sqrt{1 + y'^2} dx = \sqrt{1 + \frac{1}{2x}} dx$,

$$\int_{(A\bar{B})} y ds = \int_0^2 \sqrt{2x} \cdot \sqrt{1 + \frac{1}{2x}} dx = \int_0^2 \sqrt{2x+1} dx = \frac{1}{3} [2x+1]^{3/2} \Big|_0^2 = \frac{1}{3} (5 \cdot \sqrt{5} - 1)$$

3. (AB) egri chiziq qutb koordinatalar sistemasida ushbu $r = r(\theta)$ ($\theta_1 \leq \theta \leq \theta_2$) tenglama berilgan bo'lib, $r(\theta)$ funksiya $[\theta_1, \theta_2]$ da uzluksiz va uzluksiz hosilaga ega bo'lsin. Agar $f(x, y)$ funksiya shu (AB) egri chiziqda berilgan va uzluksiz bo'lsa, u holda egri chiziqli integral mavjud bo'ladi va u ushbu

$$\int_{(AB)} f(x, y) ds = \int_{\theta_1}^{\theta_2} f(r(\theta)\cos\theta, r(\theta)\sin\theta) \sqrt{r^2 + r'^2} d\theta \quad (14.4.5)$$

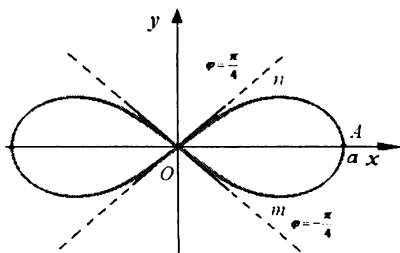
formula bilan hisoblanadi.

14.4.6. Misol. Ushbu $\int_{(K)} (x+y) ds$ egri chiziqli integralni hisoblang,

bunda $(K) - r^2 = a^2 \cos^2 \varphi$ Bernulli lemniskatasining o'ng yaprog'i.

Yechilishi. Lemniskataning o'ng yaprog'i $\varphi = -\frac{\pi}{4}$ va $\varphi = \frac{\pi}{4}$ nurlar orasida joylashgan bo'ladi (14.3-chizma). $(OmAnO)$ egri chiziqda

$$r = a\sqrt{\cos 2\varphi}, \quad r'_\varphi = -\frac{a \sin 2\varphi}{\sqrt{\cos 2\varphi}}, \quad ds = \sqrt{r^2 + r_\varphi'^2} d\varphi = \frac{a}{\sqrt{\cos 2\varphi}} d\varphi.$$



14.3-chizma.

Bernulli lemniskatasining parametrik tenglamasi:

$$x = a\sqrt{\cos 2\varphi} \cdot \cos \varphi, \quad y = a\sqrt{\cos 2\varphi} \cdot \sin \varphi.$$

(14.4.5) formulaga asosan

$$\begin{aligned} \int_{(K)} (x + y) ds &= a^2 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\sqrt{\cos 2\varphi} \cdot \cos \varphi + \sqrt{\cos 2\varphi} \cdot \sin \varphi \right) \cdot \frac{d\varphi}{\sqrt{\cos 2\varphi}} = \\ &= a^2 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\cos \varphi + \sin \varphi) d\varphi = 2 \cdot a^2 \int_0^{\frac{\pi}{4}} \cos \varphi d\varphi = 2a^2 \sin \varphi \Big|_0^{\frac{\pi}{4}} = 2a^2 \cdot \frac{\sqrt{2}}{2} = \sqrt{2}a^2. \end{aligned}$$

4. Agar $(K) = (AB)$ egrichiziq $\varphi = \varphi(\rho)$ ($\rho_1 \leq \rho \leq \rho_2$) tenglama bilan berilgan bo'lsa, u holda egrichizikli integral mavjud bo'ladi va u ushbu

$$\int_{(K)} f(x, y) ds = \int_{\rho_1}^{\rho_2} f(\rho \cos \varphi, \rho \sin \varphi) \sqrt{1 + \rho'^2} d\rho$$

formula bilan hisoblanadi.

14.5. Birinchi tur egri chiziqli integralning xossalari. Ushbu $x = \varphi(t), y = \psi(t)$ ($t_0 \leq t \leq T$) sistema bilan aniqlangan (AB) egri chiziqda $f(x, y)$ uzluksiz funksiya berilgan bo'lsin.

Biz yuqorida ko'rdikki birinchi tur egrichizikli integrallari Riman integraliga keltirilar ekan, shu tufayli egrichizikli integrallar ham Riman integrali kabi hossalari ega bo'ladi. Shuning uchun egrichizikli integrallarning asosiy xossalarini isbotsiz keltiramiz.

1^o. Agar $AB = AC + CB$ bo'lsa u holda

$$\int_{(AB)} f(x, y) ds = \int_{(AC)} f(x, y) ds + \int_{(CB)} f(x, y) ds$$

bo'ladi.

2^o. Ushbu

$$\int_{(AB)} C f(x, y) ds = C \int_{(AB)} f(x, y) ds \quad (C = \text{const})$$

tenglik o'rinli.

3^o. (AB) egri chiziqda uzluksiz $f(x, y)$ bilan birgalikda uzluksiz $g(x, y)$ ham berilgan bo'lsin, u holda

$$\int_{(AB)} [f(x, y) \pm g(x, y)] ds = \int_{(AB)} f(x, y) ds \pm \int_{(AB)} g(x, y) ds$$

formula o'rinli bo'ladi.

4^o. Agar $\forall (x, y) \in (AB)$ da $f(x, y) \geq 0$ bo'lsa, u holda

$$\int_{(AB)} f(x, y) dx \geq 0$$

bo'ladi.

5^o. Agar $f(x, y)$ funksiya (AB) egrichiziqda integrallanuvchi bo'lsa, $|f(x, y)|$ funksiya ham (AB) da integrallanuvchi va

$$\left| \int_{(AB)} f(x, y) ds \right| \leq \int_{(AB)} |f(x, y)| ds$$

tengsizlik o'rinli.

6^o, Agar $f(x, y)$ funksiya (AB) egrichiziqda uzluksiz bo'lsa, u holda shunday $(\bar{x}, \bar{y}) \in (AB)$ nuqta topiladiki,

$$\int_{(AB)} f(x, y) ds = f(\bar{x}, \bar{y}) S$$

bo'ladi, bunda $S = (AB)$ yoy ning uzunligi.

65-§. Birinchi tur egri chiziqli integrallarning ba'zi bir tatbiqlari

Birinchi tur egri chiziqli integral yordamida yoy uzunligini jismning massasini, og'irlik markazlarini, statik va inersiya momentlarini topish mumkin.

14.6. Yoy uzunligini va jismning massasini topish. Tekislikda sodda to'g'irlanuvchi (AB) egrichiziq berilgan bo'lib, unda $f(x, y)$ uzluksiz funksiya aniqlangan bo'lsin. Agar xususiy holda, $f(x, y)$ uzluksiz funksiya aniqlangan bo'lsin. Agar xususiy holda $f(x, y) = 1$ deb faraz qilsak, u holda (14.2.3) ga asosan.

$$\int_{(AB)} f(x, y) ds = \lim_{\Delta p \rightarrow 0} \sum_{i=0}^{n-1} f(\xi_i, \eta_i) \Delta s_i = \lim_{\Delta p \rightarrow 0} \sum_{i=0}^{n-1} \Delta s_i = S.$$

Demak,

$$S = \int_{(AB)} ds. \quad (14.6.1)$$

14.6.2. Misol. Ushbu $x(t) = a \cos^3 t$, $y(t) = a \sin^3 t$, $(0 \leq t \leq 2\pi)$ sistema bilan berilgan (AB) egri chiziqli (astroida) yoy ning uzunligi topilsin.

Yechilishi. Astroida koordinata o'qlariga nisbatan simmetrik bo'lgani uchun (14.6.1) formula bo'yicha topamiz:

$$\begin{aligned} S = \int_{(AB)} ds &= 4 \int_0^{\pi/2} \sqrt{x'^2(t) + y'^2(t)} dt = \int_0^{\pi/2} \sqrt{(-3a \cos^2 t \sin t)^2 + (3a \sin^2 t \cos t)^2} dt = \\ &= 4 \int_0^{\pi/2} \sqrt{\frac{9a^2}{4} \sin^2 2t} dt = 6a \int_0^{\pi/2} \sin 2t dt = 6a. \end{aligned}$$

Tekislikda uzluksiz sodda egri chiziqli berilgan bo'lib, bu chiziqli bo'ylab biror massa tarqalgan bo'lsin. Uning chiziqli zichligi $\rho(M)$ ($\forall M \in (AB)$). Biz yuqorida ko'rgan edikki, bu m massa (14.2.4) formula orqali topilar edi, ya'ni

$$m = \int_{(AB)} \rho(M) ds = \int_{(AB)} \rho(x, y) ds.$$

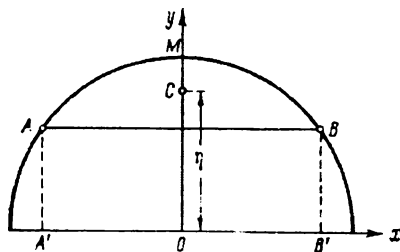
14.7. Egri chiziqli statik va inersiya momentlari hamda og'irlik markazini topish. m - massaga ega bo'lgan M material nuqtaning biror o'qqa nisbatan K_y statik momenti deb uning massasi m -ni y o'qiga bo'lgan masofa d ga ko'paytmasiga aytiladi:

$$K_y = md$$

n -ta material nuqtalar berilgan bo'lib, ularning massalari mos ravishda $m_1, m_2, \dots, m_n$ bo'lib, o'qgacha bo'lgan masofalari esa $d_1, d_2, \dots, d_n$ bo'lsin. U holda

$$K_y = \sum_{i=1}^n m_i d_i$$

(AB) egri chiziqli bo'ylab biror massa tarqalgan bo'lsin. Bu egri chiziqli x o'qiga nisbatan statik momentini topamiz. Bunda, (AB) egri chiziqli bir jismli deb faraz qilamiz, ya'ni uning



14.4-chizma.

chiziqli zichligi ρ ni (bir birlik uzunlikdagi tarqalgan massani) o'zgarmas deb faraz qilamiz, soddalik uchun $\rho = 1$ deb olamiz. Bu

hulllarda egri chiziqning istalgan bo'lagidagi massa uning uzunligi bilan o'lchanadi.

(AB) egri chiziqning ds elementar bo'lagini olaylik. Bunday massa ham ds ga teng, ya'ni $m = ds$. ds - ni material nuqta deb qarash va bu nuqtadan Ox o'qigacha bo'lgan masofani y deb belgilasak, u holda bu material nuqtaning statik momenti

$$dK_y = yds, K_y = \int_0^s \rho y ds = \int_{(AB)} y ds = \int_{(AB)} y ds,$$

bunda $y(s) = y$, $x(s) = x$, $\rho = 1$ deb olingan.

Xuddi shunday

$$K_x = \int_0^s \rho x ds = \int_{(AB)} x ds.$$

(AB) egri chiziqning markazi $G(\xi, \eta)$ nuqtada joylashgan bo'lsin (14.4-chizma). U holda (AB) egri chiziq massasi $S = m$ shu $G(\xi, \eta)$ nuqtada joylashgan bo'ladi, demak,

$$m\xi = S\xi = K_y, \quad S\eta = K_x = \int_0^s y ds,$$

$$\xi = \frac{K_y}{S} = \frac{\int_0^s x ds}{S} : \eta = \frac{K_x}{S} = \frac{\int_0^s y ds}{S}$$

bundan

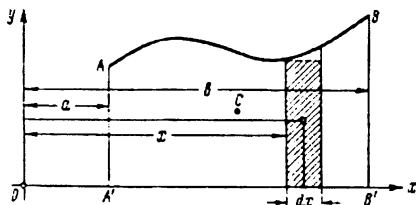
$$\eta S = \int_0^s y ds, \quad 2\pi\eta S = 2\pi \int_0^s y ds.$$

Shunday qilib ushbu teorema kelamiz.

14.7.1. Teorema (Guldin). (AB) egri chiziqni kesib o'tmaydigan o'q atrofida aylantirish natijasida hosil bo'lgan sirtning yuzi Q uning markazi chizgan aylana uzunligini (AB) egri chiziqning yoy uzunligiga ko'paytmasiga teng:

$$Q = 2\pi \int_0^s y ds = 2\pi\eta S.$$

14.8. Tekis figuraning statik momenti va og'irlik markazini topish. Tekis $AA'B'B$ figurani aytaylik, u yuqoridan tenglamasi $y = f(x)$ ko'rinishda bo'lgan AB egri chiziq bilan chegaralangan (14.5-chizma). Bu tekis figura bo'ylab biror massa tekis tarqalgan bo'lsin, uning sirt zichligi



14.5-chizma.

(bir birlik yuzaga tarqalgan massa) ρ o'zgarmas bo'lsin. Umumiylikni buzmaslik uchun $\rho = 1$ deb olamiz. Bu holda tekis figuraning istalgan qismidagi massa uning yuzasi bilan o'lchanadi. Bu figuraning koordinata o'qlariga nisbatan statik momentlari K_x va K_y larni topish uchun figuraning biror elementar qismini olamiz va uni taqribiy to'g'ri turtburchak deb qaraymiz. Undagi tarqalgan massa, uning yuzasiga teng bo'ladi, ya'ni ydx . dK_x , dK_y statik momentlarni topish uchun massa elementar to'g'ri turtburchakning og'irlik markazi S nuqtaning koordinatalari mos ravishda $\frac{1}{2}y$, $\left(x + \frac{1}{2}dx\right)$ ga teng. $\left(x + \frac{1}{2}dx\right)$ ifodani x bilan almashtirish mumkin, chunki uni ydx massaga ko'paytirilganda, ikkinchi tartibli cheksiz kichik ifoda hosil bo'ladi.

Shunday qilib

$$dK_x = \frac{1}{2}y^2 dx, \quad dK_y = xy dx$$

bunda

$$K_x = \frac{1}{2} \int_a^b y^2 dx, \quad K_y = \int_a^b xy dx. \quad (14.8.1)$$

Endi tekis figura og'irli markazi $G(\xi, \eta)$ ning koordinatalarini topamiz:

Tekis figuraning yuzini P deb belgilaymiz, u holda tekis figura buylab tarqalgan massa ham P ga teng bo'ladi.

$$\xi = \frac{K_y}{P} = \frac{a}{P}, \eta = \frac{K_x}{P} = \frac{1}{2} \frac{\int_a^b y^2 dx}{P}, \pi \eta P = \pi \int_a^b y^2 dx.$$

(14.8.2)

Shunday qilib quyidagi teoremani hosil qilamiz.

14.8.3. Teorema (Guldinning ikkinchi teoremasi). Tekis figurani uni kesib o'tmaydigan o'q atrofida aylantirish natijasida hosil bo'lgan aylanma jismning hajmi uning og'irlik markazi chizgan aylana uzunligini tekis figura yuzi P ga ko'paytmasiga teng, ya'ni $V = P \cdot 2\pi\eta$.

(14.8.1) va (14.8.2) formulalar soha yuqoridan $y_2 = y_2(x)$ va quyidan $y_1 = y_1(x)$ ($a \leq x \leq b$) egri chiziqlar bilan chegaralangan egri chiziqli trapetsiya shaklidagi soha bo'lgan holda ham tarqatiladi.

Misol uchun

$$K_x = \frac{1}{2} \int_a^b (y_2^2 - y_1^2) dx, \quad K_y = \int_a^b x(y_2 - y_1) dx,$$

$$\xi = \frac{K_y}{P} = \frac{\int_a^b x(y_2 - y_1) dx}{P}, \quad \eta = \frac{K_x}{P} = \frac{\frac{1}{2} \int_a^b (y_2^2 - y_1^2) dx}{P}.$$

14.8.4. Misol. $y^2 = 2px$ parabola, Ox o'qi bilan hamda x ga mos keluvchi ordinata bilan chegaralangan figuraning statik momentlari K_x , K_y va og'irlik markazining koordinatalari topilsin.

Yechilishi. $y = \sqrt{2px}$ bo'lgani uchun

$$K_x = \frac{1}{2} 2p \int_0^x x dx = \frac{1}{2} px^2, \quad K_y = \sqrt{2p} \int_0^x x^{\frac{3}{2}} dx = \frac{2\sqrt{2p}}{5} x^{\frac{5}{2}}.$$

Ikkinchi tomondan

$$P = \sqrt{2p} \int_0^x x^{\frac{1}{2}} dx = \frac{2\sqrt{2p}}{5} x^{\frac{3}{2}}, \quad \xi = \frac{3}{5} x, \quad \eta = \frac{3}{8} \sqrt{2px} = \frac{3}{8} y.$$

66-§. Ikkinchi tur egri chiziqli integrallar

14.9. Ikkinchi tur egri chiziqli integralning ta'rif. $OxOy$ koordinatalar sistemasida to'g'rilanuvchi sodda (AB) egri chiziq (hozircha ochiq) berilgan bo'lib, unda biror $f(x, y)$ funksiya aniqlangan

bo'lsin. Egri chiziqni $A_i (i=0,1,2,\dots,n)$ nuqtalar yordamida bo'laklarga bo'lamiz, ya'ni uning ixtiyoriy $P = \{A_0, A_1, \dots, A_n\}$ bo'linishini qaraymiz. Bu bo'linishning har bir $A_i, \tilde{A}_{i+1} (i=0, \dots, n-1)$ yoyidan ixtiyoriy $M_i(\xi_i, \eta_i) (i=0, \dots, n-1)$ nuqtani olib, berilgan funksiyaning shu nuqtadagi qiymati $f(\xi_i, \eta_i)$ ga $A_i, \tilde{A}_{i+1}$ yoyining $Ox(Oy)$ o'qidagi proyeksiyasiga ko'paytirib, ushbu

$$\sigma' = \sum_{i=0}^{n-1} f(\xi_i, \eta_i) \Delta x_i \left(\sigma'' = \sum_{i=0}^{n-1} f(\xi_i, \eta_i) \Delta y_i \right)$$

yig'indini tuzamiz. (AB) egri chiziqning shunday

$$P_1, P_2, \dots, P_m, \dots \quad (14.9.1)$$

bo'linishlar ketma-ketligini qaraylikki, ularning diametrlaridan tashkil topgan mos $\lambda_{P_1}, \lambda_{P_2}, \dots, \lambda_{P_m}, \dots$ ketma-ketlik nolga intilsin: $\lambda_{P_m} \rightarrow 0$, bu bo'linishlarga mos ushbu

$$\sigma'_1, \sigma'_2, \dots, \sigma'_m (\sigma''_1, \sigma''_2, \dots, \sigma''_m, \dots)$$

yig'indilar ketma-ketligini tuzamiz.

14.9.2. Ta'rif. (AB) egri chiziqning har qanday (14.9.1) ko'rinishdagi bo'linishlar ketma-ketligi $\{P_m\}$ olinganda ham, unga mos yig'indilar ketma-ketligi $\{\sigma'_m\}(\{\sigma''_m\}) M_i(\xi_i, \eta_i)$ nuqtalarning $(M_i(\xi_i, \eta_i) \in A_i, \tilde{A}_{i+1})$ tanlab olinishiga bog'liq bo'lmagan holda hamma vaqt bitta J' songa (J'' songa) intilsa, bu son $\sigma'(\sigma'')$ yig'indining limiti deb ataladi va

$$\lim_{\lambda_P \rightarrow 0} \sigma' = \lim_{\lambda_P \rightarrow 0} \sum_{i=0}^{n-1} f(\xi_i, \eta_i) \cdot \Delta x_i = J'$$

$$\left(\lim_{\lambda_P \rightarrow 0} \sigma'' = \lim_{\lambda_P \rightarrow 0} \sum_{i=0}^{n-1} f(\xi_i, \eta_i) \Delta y_i = J'' \right)$$

kabi belgilanadi. Bu ta'rifni quyidagicha keltirish ham mumkin.

14.9.3. Ta'rif. Agar $\forall \varepsilon > 0$ son olinganda ham, shunday $\delta > 0$ son topilsaki, (AB) egri chiziqning diametri $\lambda_P < \delta$ bo'lgan har qanday P bo'linish uchun tuzilgan $\sigma'(\sigma'')$ yig'indi uchun ixtiyoriy $M_i(\xi_i, \eta_i)$ nuqtalarda $(M_i \in A_i, \tilde{A}_{i+1}, i=0, 1, \dots, n-1) |\sigma' - J'| < \varepsilon (|\sigma'' - J''| < \varepsilon)$ tengsizlik bajarilsa, J' son (J'' son) $\sigma'(\sigma'')$ yig'indining $\lambda_P \rightarrow 0$ dagi limiti deb ataladi.

Yuqoridagi keltirilgan ta'riflar o'zaro ekvivalent ta'riflardir.

14.9.4. Ta'rif. Agar $\lambda_p \rightarrow 0$ da $\sigma'(\sigma'')$ yig'indi chekli limitga ega bo'lsa, u holda $f(x, y)$ funksiya (AB) egri chiziq bo'yicha integrallanuvchi deyiladi. Bu limitga $f(x, y)$ funksiyaning (AB) egri chiziq bo'yicha ikkinchi tur egri chiziqli integrali deb ataladi va u

$$\iint_{(AB)} f(x, y) dx \left(\int_{(AB)} f(x, y) dy \right)$$

kabi belgilanadi.

Demak,

$$\begin{aligned} \int_{(AB)} f(x, y) dx &= \lim_{\lambda_p \rightarrow 0} \sigma' = \lim_{\lambda_p \rightarrow 0} \sum_{i=0}^{n-1} f(x_i, y) \Delta x_i \\ \left(\int_{(AB)} f(x, y) dy &= \lim_{\lambda_r \rightarrow 0} \sigma'' = \lim_{\lambda_r \rightarrow 0} \sum_{i=0}^{n-1} f(x, y) \Delta y_i \right). \end{aligned}$$

Umumiy holda, agar (AB) egri chiziqda ikkita $P(x, y)$ va $Q(x, y)$ funksiyalar berilgan bo'lib, $\int_{(AB)} P(x, y) dx$, $\int_{(AB)} Q(x, y) dy$ integrallar, ularning ikkinchi tur egri chiziqli integrali bo'lsa, u holda ushbu

$$\int_{(AB)} P(x, y) dx + \int_{(AB)} Q(x, y) dy$$

yig'indi umumiy holda *ikkinchi tur egri chiziqli integral* deb ataladi va

$$\int_{(AB)} P(x, y) dx + Q(x, y) dy$$

kabi belgilanadi.

Demak,

$$\int_{(AB)} P(x, y) dx + Q(x, y) dy = \int_{(AB)} P(x, y) dx + \int_{(AB)} Q(x, y) dy.$$

Birinchi va ikkinchi tur egri chiziqli integrallarning ta'riflarini solishtirib ko'rganimizda, ular orasida muhim farq borligini ko'ramiz, ya'ni birinchi tur integral yig'indini tuzishda funksiyaning $f(M_i)$ qiymatini A, A_{i+1} yoyniing uzluksizligi $\sigma_i = \Delta s_i$ ga ko'paytiriladi, ikkinchi turda esa, funksiyaning $f(M_i)$ qiymatini A, A_{i+1} yoyniing $Ox(Oy)$ o'qdagi proyeksiyasi $\Delta x_i(\Delta y_i)$ ga ko'paytiriladi.

Ravshanki, birinchi tur egri chiziqli integralda $A, \bar{A}_{-1}$ yoyning uzunligi (AB) egri chiziqlarning yo'nalishiga bog'liq bo'lmaydi, ikkinchi tur egri chiziqli integralda yoyning proyeksiyasi egri chiziqlarning yo'nalishiga bog'liq bo'ladi, ya'ni (AB) egri chiziqlarning yo'nalishi teskariga o'zgarganda $A, \bar{A}_{-1}$ yoyning o'qdagi proyeksiyasi teskari ishoraga o'zgaradi. Shunday qilib ikkinchi tur egri chiziqli integrallarda

$$\int_{(BA)} f(x, y) dx = - \int_{(AB)} f(x, y) dx \left(\int_{(BA)} f(x, y) dy = - \int_{(AB)} f(x, y) dy \right)$$

bo'ladi, bunda o'ng tomondagi integralning mavjudligidan chap tomondagi integralning mavjudligi va aksincha chap tomondagi integralning mavjudligidan o'ng tomondagi integralning mavjudligi kelib chiqadi.

Agar (AB) fazoviy egri chiziq bo'lib, bu chiziqda $f(x, y, z)$ funksiya berilganda ikkinchi tur egri chiziqli integralning ta'rif xuddi yuqoridagi singari ta'riflanadi va ular

$$\int_{(AB)} f(x, y, z) dx, \int_{(AB)} f(x, y, z) dy, \int_{(AB)} f(x, y, z) dz$$

kabi belgilanadi.

Umumiy holda (AB) egri chiziqda $P(x, y, z), Q(x, y, z), R(x, y, z)$ funksiyalar berilgan bo'lib,

$$\int_{(AB)} P(x, y, z) dx, \int_{(AB)} Q(x, y, z) dy, \int_{(AB)} R(x, y, z) dz$$

integrallar mavjud bo'lsa, u holda ushbu

$$\int_{(AB)} P(x, y, z) dx + \int_{(AB)} Q(x, y, z) dy + \int_{(AB)} R(x, y, z) dz$$

yig'indi ikkinchi tur egri chiziqli integralning *umumiy ko'rinishi* deb ataladi va u

$$\int_{(AB)} P(x, y, z) dx + Q(x, y, z) dy + R(x, y, z) dz$$

kabi belgilanadi. Demak,

$$\begin{aligned} & \int_{(AB)} P(x, y, z) dx + \int_{(AB)} Q(x, y, z) dy + \int_{(AB)} R(x, y, z) dz = \\ & = \int_{(AB)} P(x, y, z) dx + Q(x, y, z) dy + R(x, y, z) dz. \end{aligned}$$

14.9.5. Eslatma. Agar (AB) egri chiziq Ox o'qiga (Oy o'qiga) perpendikulyar to'g'ri chiziq kesmasidan iborat bo'lib, $f(x, y)$ funksiya shu chiziqda berilgan bo'lsa, u holda

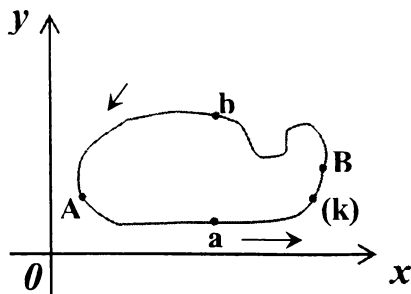
$$\int_{(AB)} f(x, y) dx = 0 \quad \left(\int_{(AB)} f(x, y) dy = 0 \right)$$

bo'ladi. Bu tenglik bevosita ikkinchi tur egri chiziqli integralning ta'rifidan kelib chiqadi.

Faraz qilaylik (AB) - sodda yopiq chiziq bo'lsin, ya'ni A va B nuqtalar ustma-ust tushsin. Bundan keyin yopiq chiziqni (k) deb belgilaymiz. Bu sodda yopiq chiziqda ham ikki yo'nalish bo'ladi. Bularning birini musbat, ikkinchisini manfiy yo'nalish deb qabul qilamiz. Kuzatuvchi yopiq chiziq bo'ylab harakat qilganda, yopiq chiziq bilan chegaralangan soha kuzatuvchiga nisbatan chapda qolgan holga musbat yo'nalish, o'ngda qolgan holga esa, manfiy yo'nalish deb qabul qilamiz.

(k) -sodda yopiq chiziqda $f(x, y)$ funksiya berilgan bo'lsin. Bu chiziqda ixtiyoriy ikkita turli A va B nuqtalarni olamiz. Natijada yopiq chiziq (AaB) va (BbA) chiziqlarga ajraladi (14.6-chizma).

$$\text{Agar } \int_{(AaB)} f(x, y) dx \text{ va } \int_{(BbA)} f(x, y) dx$$



14.6-chizma

integrallar mavjud bo'lsa, u holda

$$\int_{(AaB)} f(x, y) dx + \int_{(BbA)} f(x, y) dx$$

integral $f(x, y)$ funksiyaning (k) yopiq chiziq bo'yicha ikkinchi tur egri chiziqli integral deyiladi va u $\int_{(k)} f(x, y) dx$ kabi belgilanadi.

Demak,

$$\int_{(k)} f(x, y) dx = \int_{(AaB)} f(x, y) dx + \int_{(BbA)} f(x, y) dx.$$

huddi shunday $\int_{(k)} f(x, y) dy$ integral ham ta'riflanadi, umumiy holda

$$\int_{(A)} P(x, y) dx + Q(x, y) dy$$

integralning ham ta'rif yuqoridagilar singari beriladi.

Agar (AB) egri chiziq o'zining $x = x(y)$ ($c \leq y \leq d$) oshkor tenglamasi bilan berilgan bo'lib, $x(y)$ funksiya uzluksiz va uzluksiz $x'(y)$ hosilaga ega bo'lsa, u holda

$$\int_{(AB)} P(x, y) dx + Q(x, y) dy = (R) \int_c^d [P(x(y), y) \cdot x'_y + Q(x(y), y)] dy$$

formula o'rinli bo'ladi.

14.10. Ikkinchi tur egrichizikli integralning mavjudligi va uni hisoblash.

Fazar qilaylik, (AB) egri chiziq ushbu

$$\begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases} (\alpha \leq t \leq \beta) \quad (14.10.1)$$

parametrik tenglamasi bilan berilgan bo'lib, bunda $\varphi(t), \psi(t)$ funksiyalar $[\alpha, \beta]$ da uzluksiz va uzluksiz $\varphi'(t), \psi'(t)$ hosilalarga ega, hamda $(\varphi(\alpha), \psi(\alpha)) = A$ va $(\varphi(\beta), \psi(\beta)) = B$ bo'lsin.

t parametrlar α dan β ga qarab o'zgarganda $(x, y) = (\varphi(t), \psi(t))$ nuqta A dan B ga qarab (AB) egri chiziqni chizsin. $f(x, y)$ funksiya (AB) egri chiziqda berilgan va uzluksiz bo'lsin. Shu shartlar bajarilganda ushbu

$$\int_{(AB)} f(x, y) dx = (R) \int_{\alpha}^{\beta} f(\varphi(t), \psi(t)) \varphi'(t) dt, \quad (14.10.2)$$

$$\int_{(AB)} f(x, y) dy = (R) \int_{\alpha}^{\beta} f(\varphi(t), \psi(t)) \psi'(t) dt \quad (14.10.3)$$

tengliklar o'rinli bo'ladi.

Isboti. $[\alpha, \beta]$ oraliqning ixtiyoriy $P = \{t_0, t_1, \dots, t_n\}$ ($t_0 < t_1 < t_2, \dots, t_{n-1} < t_n$) bo'linishini qaraylik. Bu bo'linish A_i ($i = 0, 1, \dots, n$) nuqtasining o'rnini t_i parametrlarning qiymati bilan aniqlansin. A_i, A_{i+1} yoyidagi M_i nuqtaning o'rnini esa, parametrlarning t_i qiymati $t_i \leq \tau_i \leq t_{i+1}$ bilan aniqlansin. Bu bo'linishga mos kelgan

$$\sigma = \sum_{i=0}^{n-1} f(\xi_i, \eta_i) \Delta x_i$$

integral yig'indini tuzamiz. Agar $\Delta x_i = \varphi(t_{i+1}) - \varphi(t_i) = \int_{t_i}^{t_{i+1}} \varphi'(t) dt$ ni e'tiborga

olajak, yuqoridagi yig'indini ushbu

$$\sigma = \sum_{k=0}^{n-1} f(\varphi(\tau_i), \psi(\tau_i)) \int_{t_i}^{t_{i+1}} \varphi'(t) dt$$

ko'rinishda yozish mumkin. Ikkinchi tomondan (14.10.2) ning o'ng tomonidagi integral mavjud, chunki integral ostidagi funksiya uzluksiz funksilarning superpozitsiyasi sifatida uzluksiz. Uni quyidagi integrallar yig'indisi shaklida tasvirlash mumkin:

$$J = \int_{\alpha}^{\beta} f(\varphi(t), \psi(t)) \varphi'(t) dt = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} f(\varphi(t), \psi'(t)) \varphi'(t) dt.$$

Bu yerdan

$$\sigma - J = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} [f(\varphi(\tau_i), \psi(\tau_i)) - f(\varphi(t), \psi(t))] \varphi'(t) dt.$$

o berilgan bo'lsin. Δt_i larni shunday kichik qilib olaylikki, $[t_i, t_{i+1}]$ dagi $f(\varphi(t), \psi(t))$ uzluksiz funksiyaning tebranishi berilgan ε dan kichik bo'lsin. $\varphi'(t)$ funksiya uzluksiz bo'lgani uchun $|\varphi'(t)| \leq L$ bo'ladi. U holda

$$|\sigma - J| < \varepsilon \cdot L \cdot |\beta - \alpha|.$$

Bundan, $\lambda_p = \max |\Delta t_i| \rightarrow 0$ da $\lim_{\lambda_p \rightarrow 0} \sigma = J$.

Shunday qilib, $\int_{(AB)} f(x, y) dx$ egri chiziqning mavjudligi va (14.10.2) tenglikning o'rinli ekanligi isbot bo'ldi. Xuddi shunday qilib, $\int_{(AB)} f(x, y) dy$ integralning mavjudligi va (14.10.3) tenglikning o'rinli ekanligi isbot qilinadi. ■

Umumiy holda $P(x, y), Q(x, y)$ funksiyalar (AB) egri chiziqda berilgan va uzluksiz bo'lsa hamda (14.10.1) dagi $\varphi(t), \psi(t)$ funksiyalar yuqoridagi shartlarni qanoatlantirsa, u holda

$$\int_{(AB)} P(x, y) dx + Q(x, y) dy = (R) \int_{\alpha}^{\beta} [P(\varphi(t), \psi(t)) \varphi'(t) + Q(\varphi(t), \psi(t)) \psi'(t)] dt \quad (14.10.3)$$

tenglik o'rinli bo'ladi.

Agar (AB) egri chiziq o'zining $y = y(x)$ ($a \leq x \leq b$) tenglamasi bilan berilgan bo'lib, $y(x)$ funksiya uzluksiz va uzluksiz $y'(x)$ ga ega bo'lsa, u holda x ni parametr deb qarash, u holda

$$\int_{(AB)} P(x, y) dx + Q(x, y) dy = (R) \int_a^b [P(x, y(x)) + Q(x, y(x)) y'(x)] dx. \quad (14.10.4)$$

Formulani hosil qilamiz. Agar (AB) egrichiziq o'zining $x = x(y)$ ($c \leq y \leq d$) oshkor tenglamasi bilan berilgan bo'lib, $x(y)$ funksiya uzluksiz va uzluksiz $x'(y)$ hosilaga ega bo'lsa, u holda

$$\int_{(AB)} P(x, y) dx + Q(x, y) dy = (R) \int_c^d [P(x(y), y) \cdot x'_y + Q(x(y), y)] dy.$$

14.10.4. Misol. $\int_{(\gamma)} xy dx$, bu yerda (γ) – egri chiziq $y = \sin x$ sinusoida chiziqning $(0,0)$ hamda $(\pi;0)$ nuqtalar orasidagi qismi.

Yechilishi. Ushbu $\int_{(\gamma)} P(x, y) dx = \int_a^b P(x, f(x)) dx$ formulaga asosan,

$$\int_{(\gamma)} xy dx = \int_0^{\pi} x \sin x dx = -x \cos x \Big|_0^{\pi} + \int_0^{\pi} \cos x dx = \pi.$$

14.10.5. Misol. Ushbu $\int_{(AB)} x^2 dx + xy dy$ integralni hisoblang, bunda

$(AB): x = \cos t, y = \sin t \left(0 \leq t \leq \frac{\pi}{2} \right)$ aylananing to'rtidan biri, $t=0$ ga A nuqta, $t = \frac{\pi}{2}$ ga B nuqta mos keladi.

Yechilishi. (14.10.3) formulaga asosan,

$$\int_{(AB)} x^2 dx + xy dy = \int_0^{\frac{\pi}{2}} (-\cos^2 t \cdot \sin t + \cos^2 t \sin t) dt = 0.$$

14.11. Birinchi va ikkinchi tur egri chiziqlar orasidagi bog'lanish. (AB) sodda silliq egri chiziq, o'zining ushbu $x = x(s), y = y(s)$ ($0 \leq s \leq S$) tenglamasi bilan berilgan bo'lib, bunda $x(s)$ va $y(s)$ funksiyalar uzluksiz. $x'(s), y'(s)$ hosilalarga ega bo'lsin. Ravshanki, bu egri chiziq, uzining har bir nuqtasida urinmaga ega.

Urinmaning α_x va α_y o'qlar bilan yoy o'sishi tomoniga qarab yo'nalishi orasidagi burchaklarni mos raavishda α va β deyilsa, unda $x'(s) = \cos\alpha, y'(s) = \cos\beta$ bo'ladi.

$f(x, y)$ funksiya (AB) egri chiziqda berilgan va uzluksiz bo'lsin. U holda $\int_{(AB)} f(x, y) dx$ integral mavjud bo'ladi va u (14.10.2) formula orqali hisoblanadi:

$$\begin{aligned} \int_{(AB)} f(x, y) dx &= (R) \int_0^S f(x(s), y(s)) x'(s) ds = \int_0^S f(x(s), y(s)) \cos\alpha ds = \\ &= \int_{(AB)} f(x, y) \cos\alpha ds. \end{aligned} \quad (14.11.1)$$

Xuddi shunday

$$\begin{aligned} \int_{(AB)} f(x, y) dy &= (R) \int_0^S f(x(s), y(s)) y'(s) ds = \int_0^S f(x(s), y(s)) \cos\beta ds = \\ &= \int_{(AB)} f(x, y) \cos\beta ds. \end{aligned} \quad (14.11.2)$$

Umumiy holda, agar (AB) egri chiziqda $P(x, y), Q(x, y)$ uzluksiz funksiyalar berilganda

$$\int_{(AB)} P(x, y) dx + Q(x, y) dy = \int_{(AB)} (P \cos\alpha + Q \cos\beta) ds \quad (14.11.3)$$

formula o'rinli bo'ladi. (14.11.1), (14.11.2) va (14.11.3) formulalar birinchi va ikkinchi tur egri chiziqli integrallar orasidagi bog'lanishlarni ifodalovchi formulalar bo'lib hisoblanadi.

Agar (AB) egri chiziq fazoda berilganda (14.11.3) ga o'xshash formulaning ko'rinishi ushbu

$$\int_{(AB)} P dx + Q dy + R dz = \int_{(AB)} (P \cos\alpha + Q \cos\beta + R \cos\gamma) ds$$

ko'rinishda bo'ladi, bunda P, Q, R funksiyalar (AB) da uzluksiz, $\cos\alpha, \cos\beta$ va $\cos\gamma$ lar urinmaning yo'naltiruvchi kosinuslari bo'lib, uning yo'nalishi egri chiziq yo'nalishiga mos deb faraz qilinadi.

14.12. Grin formulasi. a) Yuqoridan $y = Y(x)$, quyidan $y = y_0(x) (a \leq x \leq b)$ egri chiziqlar bilan yon tomonlardan esa, OY o'qqa parallel (PS) va (RQ) to'g'ri chiziq kesmalari bilan chegaralangan (D) -egri chiziqli trapesiyani qaraylik (14.7-chizma). Shu (D) sohada $P(x, y)$ uzluksiz va uzluksiz $\frac{\partial P}{\partial y}$ xususiy hosilaga ega bo'lgan funksiya berilgan bo'lsin. Shu shartlarda ushbu

$$\iint_{(D)} \frac{\partial P}{\partial y} dx dy = - \int_{(K)} P(x, y) dx \quad (14.12.1)$$

formula o'rinli bo'ladi, bunda (K) – (D) sohaning chegarasi.

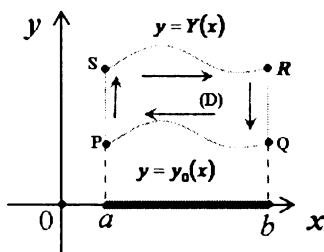
b) Xuddi shunday, (D) yuqoridan (SR) quyidan (PQ) to'g'ri chiziq kesmalari bilan, yon tomonlaridan esa, $x = x_0(y)$, $x = x_1(y) (c \leq y \leq d)$ egri chiziqlar bilan chegaralangan soha bo'lib (14.8- chizma), unda $Q(x, y)$ uzluksiz va uzluksiz $\frac{\partial Q}{\partial x}$ xususiy hosilaga ega bo'lgan funksiya bo'lganda, ushbu

$$\iint_{(D)} \frac{\partial Q}{\partial x} dx dy = \int_{(K)} Q(x, y) dy \quad (14.12.2)$$

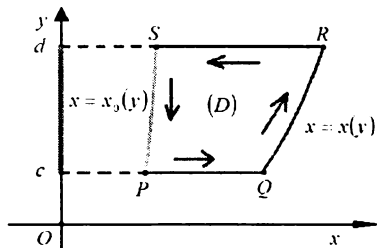
formula o'rinli bo'ladi. Endi (D) soha yuqorida qaralgan ikkala holning ham shartlarini qanoatlantirsa (12.1-12.2 chizmalar), bu sohada $P(x, y)$ va $Q(x, y)$ funksiyalar yuqoridagi shartlarni qanoatlantirsa, u holda (14.12.1), (14.12.2) formulalar o'rinli bo'ladi. (14.12.2) formuladan (14.12.1) formulani ayirsak, quyidagi

$$\iint_{(D)} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \int_{(K)} P dx + Q dy \quad (14.12.3)$$

formulaga ega bo'lamiz. (14.12.3) formulaga *Grin formulasi* deyiladi.



14.7-chizma.



14.8-chizma.

Ushbu $\iint_{(D)} \frac{\partial P}{\partial y} dx dy$ integralni hisoblash talab qilingan bo'lsin. Buning uchun soha egri chiziqli soha bo'lganda ikki karrali integralni hisoblash formulasidan foydalanamiz, ya'ni

$$\iint_{(D)} \frac{\partial P}{\partial y} dx dy = \int_a^b dx \int_{y_0}^{Y(x)} \frac{\partial P}{\partial y} dy = \int_a^b P(x, Y(x)) dx - \int_a^b P(x, y_0(x)) dx.$$

Keyingi tenglikning o'ng tomonidagi integrallarning ikkalasini ham (14.10.4) formulaga asosan, egri chiziqli integralga aylantiramiz:

$$\int_a^b P(x, Y(x)) dx = \int_{(SR)} P(x, y) dx, \quad \int_a^b P(x, y_0(x)) dx = \int_{(PQ)} P(x, y) dx.$$

Natijada,

$$\iint_{(D)} \frac{\partial P}{\partial y} dx dy = \int_{(SR)} P(x, y) dx + \int_{(PQ)} P(x, y) dx,$$

bu tenglikning o'ng tomoniga

$$\int_{(PS)} P(x, y) dx = 0, \quad \int_{(RQ)} P(x, y) dx = 0$$

(PS va RQ kesmalar Ox o'qiga perpendikulyar bo'lgani uchun yuqoridagi integrallar nolga teng) integrallarni qo'shamiz:

$$\iint_{(D)} \frac{\partial P}{\partial y} dx dy = \int_{(SR)} P(x, y) dx + \int_{(QP)} P(x, y) dx + \int_{(PS)} P(x, y) dx + \int_{(RQ)} P(x, y) dx.$$

Natijada keyingi tenglamaning o'ng tomoni (D) sohaning chegarasi (k) yopiq chiziqni ifoda qiladi, lekin yo'nalish manfiy yo'nalishda olingan. Shuning uchun

$$\iint_{(D)} \frac{\partial P}{\partial y} dx dy = - \int_{(k)} P(x, y) dx.$$

Xudi shunday

$$\iint_{(D)} \frac{\partial Q}{\partial x} dx dy = \int_{(k)} Q(x, y) dy$$

formula ham isbot qilinadi, bunda $Q(x, y)$ funksiya (D) sohada uzluksiz va uzluksiz $\frac{\partial Q}{\partial x}$ xususiy xosilaga ega, (D) soha esa, yuqoridan (SR) , qo'yidan (PQ) Ox o'qqa parallel bo'lgan to'g'ri chiziq kesmalari bilan, yon tomonlardan esa, $x = x_0(y), x = x(y) (c \leq y \leq d)$ egri chiziqlar bilan chegralangan soha.

14.12.5. Eslatma. Grin formulasi (D) soha bo'yicha olingan ikki karrali integral bilan, sohaning chegarasi bo'yicha olingan ikkinchi tur egri chiziqli integrallarni bog'lovchi formula ekan. Biz yuqorida Grin formulasini maxsus ko'rinishdagi (D) sohalar (egri chiziqli trapeziyalar) uchun isbotladik.

Aslida bu formula ancha keng sinfdagi sohalar uchun ham o'rinli bo'ladi, ya'ni (D) sohani chekli sondagi egri chiziqli trapeziya yig'indisi sifatida tasvirlash bilan isbot qilinadi.

14.12.6. Misol. Grin formulasi yordamida ushbu

$$\int_{(k)} (x - y) dx + (x + y) dy$$

egri chiziqli integralni hisoblang, bunda (k) - aylana: $x^2 + y^2 = R^2$.

Yechilishi. Bu holda, $P(x, y) = x - y$, $Q(x, y) = x + y$, $\frac{\partial P}{\partial y} = -1$, $\frac{\partial Q}{\partial x} = 1$ funksiyalar $x^2 + y^2 \leq R^2$ yopiq doirada uzluksiz bo'lgani uchun berilgan integralga Grin formulasini qo'llash mumkin:

$$\int_{(k)} (x - y) dx + (x + y) dy = \iint_{(D)} [1 - (-1)] dx dy = 2 \iint_{(D)} dx dy = 2 \cdot \pi R^2.$$

14.13. Grin formulasining ba'zi bir qo'llanishlari.

1. Grin formulasi yordamida yassi shaklning yuzini egri chiziqli integrallar orqali hisoblash. Haqiqatdan ham, agar (14.12.3) formuladagi $P(x, y)$ va $Q(x, y)$ funksiyalarni $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ ayirma 1 ga teng bo'ladigan qilib tanlasak, u holda ikki karrali integral (D) sohaning yuzini ifodalaydi, natijada (D) sohaning yuzi D ushbu

$$D = \int_{(k)} P dx + Q dy$$

formula orqali, ya'ni egri chiziqli integral orqali ifoda qilinadi. Misol uchun

1) $Q = x, P = 0$ deb olinsa, $D = \int_{(k)} x dy.$

2) $Q = 0, P = -y$ deb olinsa, $D = - \int_{(k)} y dx.$

3) $Q = \frac{1}{2}x, P = -\frac{1}{2}y$ deb olinsa,

$$D = \frac{1}{2} \int_{(k)} x dy - y dx \quad (14.13.1)$$

formulaga ega bo'lamiz.

14.13.2. Misol. Yarim o'qlari a va b bo'lgan ellipsning yuzi topilsin.

Yechilishi. Ellipsning parametrik tenglamasi $x = a \cos t, y = b \sin t (0 \leq t \leq 2\pi)$ dan foydalanib (14.13.1) formula bo'yicha topamiz:

$$D = \frac{1}{2} \int_{(k)} x dy - y dx = \frac{1}{2} \int_0^{2\pi} (a \cos t \cdot b \cos t + b \sin t \cdot a \sin t) dt = \frac{1}{2} \int_0^{2\pi} ab dt = \pi ab.$$

2. Egri chiziqli integralning integrallash yo'liga bog'liq bo'lmashlik sharti. Chegaralangan bog'lamli $(D) ((D) \subset R^2)$ sohada $P(x, y)$ va $Q(x, y)$ funksiyalar berilgan bo'lib, ular uzluksiz va uzluksiz $\frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$ xususiy hosilalarga ega bo'lsin.

14.13.3-teorema. (D) sohaga qarashli har qanday (k) yopiq egri chiziq bo'yicha olingan integral

$$\int_{(k)} Pdx + Qdy = 0 \quad (14.13.4)$$

bo'lishi uchun

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad (14.13.5)$$

shartning bajarilishi zarur va yetarlidir.

Isboti. Zaruriy shart. (k) yopiq chiziq bilan chegaralangan sohani $(D_1) \cap (D_1) \subset (D)$ bilan belgilaymiz. U holda Grin formulasi o'rinli bo'ladi:

$$\int_{(k)} Pdx + Qdy = \iint_{(D_1)} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy.$$

(14.13.4) tenglik o'rinli bo'lsin. Bu tenglik Grin formulasiga asosan

$$\iint_{(D_1)} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = 0. \quad (14.13.6)$$

tenglikka teng kuchli. Integral ostidagi $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ funksiya uzluksiz

bo'lgani uchun integralni soha bo'yicha differensiallasak, (D_1) sohada $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$ ekanligi kelib chiqadi.

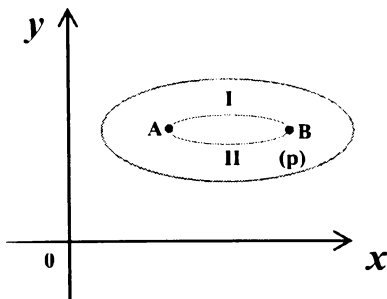
Yetarliligi. (14.13.5) shart o'rinli bo'lsin. U holda yana Grin formulasiga asosan, (14.13.6) dan (14.13.4) kelib chiqadi.

14.13.7. Teorema. $P(x,y)$ va $Q(x,y)$ funksiyalar bir bog'lamli yopiq (D) sohada yuqoridagi shartlarni qanoatlantirganda

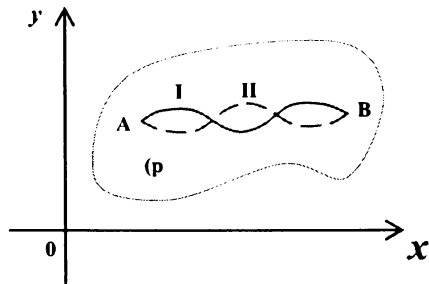
$$\int_{(AB)} Pdx + Qdy \quad ((AB) \subset (D)) \quad (14.13.8)$$

integralning integrallash yo'liga (A) va (B) nuqtalarni birlashtiruvchi egri chiziqqa) bog'liq bo'lmasligi uchun (14.13.5) shartining bajarilishi zarur va yetarli.

Isboti. Zarurligi. (14.14.8) egri chiziqli integral integrallash yo'liga bog'liq bo'lmasin. (D) sohadan ixtiyoriy yopiq egri chizig'ini olaylik. A va B nuqtalar uning ixtiyoriy nuqtalari bo'lsin.



14.10-chizma.



14.11-chizma.

Farazimizga asosan,

$$\int_{(AIB)} Pdx + Qdy = \int_{(AII B)} Pdx + Qdy. \quad (14.14.9)$$

Bundan

$$\int_{(k)} Pdx + Qdy = \int_{(AIB)} Pdx + Qdy + \int_{(BII A)} Pdx + Qdy = 0 \quad (14.13.10)$$

Demak, 14.14.3-teoremaga asosan, (14.13.5) shart bajariladi.

Yetarliligi. (14.13.5) shart (D) sohada o'rinli bo'lsin. (D) sohadan olingan A va B nuqtalarni ixtiyoriy sodda (AIB) va (AII B) egri chiziqlar bilan birlashtiramiz. U holda har doim (14.13.9) ning o'rinli bo'lishini ko'rsatamiz. (AIB) va (AII B) egri chiziqlar A va B nuqtalardan boshqa umumiy nuqtalarga ega emas. Bu holda (k) = (AIBII A) sodda yopiq chiziq bo'ladi. Bundan esa, (14.13.9) tenglikning o'rinli ekanligi kelib chiqadi.

Agar (AIB) va (AII B) egri chiziqlar chekli sondagi nuqtalarda kesishsa, u holda (k) egri chiziq sodda chiziq bo'lmaydi. A nuqtadan boshlab (k) chiziqning yo'nalishi bo'ylab (AIB) va (AII B) chiziqlarning birinchi kesishish nuqtasini c_1 deb belgilaymiz. Natijada sodda yopiq chiziq hosil bo'ladi, uni (k_1) deb belgilaymiz. (k_1) yopiq chiziq bo'yicha olingan (14.13.10) integral nolga teng bo'ladi. Bu (k_1) yopiq chiziqni fikran olib tashlaymiz. Yana (AIB) va (AII B) egri chiziqning kesishish nuqtasini c_2 bilan belgilaymiz. Natijada (k_2) sodda yopiq chiziq hosil bo'ladi. (k_2) yopiq chiziq bo'yicha olingan (14.13.10) integral ham 14.13.3-teoremaga asosan nolga teng bo'ladi. Chekli sondagi qadamdan

keyin (k) chiziq $(k_1), (k_2), \dots, (k_n)$ sodda yopiq chiziqlarga ajraladi, qaysiki bu sodda yopiq chiziqlarning har birida (14.13.10) integral nolga teng bo'ladi. Shunday qilib (k) chiziq bo'yicha olingan integral nolga teng bo'ladi.

3. $Pdx + Qdy$ ifodaning biror $F(x, y)$ funksiyaning to'liq differensial bo'lishlik sharti. Ushbu

$$Pdx + Qdy \quad (14.13.11)$$

ifoda bizga biror $F(x, y)$ funksiyaning to'liq differensial bo'lishligini eslatadi, ya'ni

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy,$$

bunda $\frac{\partial F}{\partial x} = P, \frac{\partial F}{\partial y} = Q$ desak, (14.13.11) ifoda $F(x, y)$ funksiyaning to'liq differensialini ifoda qiladi, lekin (14.13.11) ifoda har doim ham biror $F(x, y)$ funksiyaning to'liq differensialini ifoda qilavermaydi. Bu ifoda biror $F(x, y)$ funksiyaning to'liq differensialini ifoda qilish uchun qanday shart bajarilishi kerak degan savolning qo'yilishi tabiiy. Bu savolga quyidagi teorema javob beradi.

14.13.12. Teorema. $P(x, y)$ va $Q(x, y)$ funksiyalar yuqoridagi shartlarni qanoatlantirganda (14.13.11) ifoda biror $F(x, y)$ funksiyaning to'liq differensialini ifodalashligi uchun (14.13.5) shartning bajarilishi zarur va yetarli.

Isbot. Zarurligi. (14.13.11) ifoda biror $F(x, y)$ funksiyaning to'liq differensialini ifodalasin: $dF = Pdx + Qdy$.

Bundan $\frac{\partial F}{\partial x} = P, \frac{\partial F}{\partial y} = Q$ ekanligi kelib chiqadi. Shartga ko'ra $\frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$ uzluksiz xususiy hosilalar mavjud bo'lganligi uchun yuqoridagi tenglikning birinchisidan u bo'yicha, ikkinchisidan x bo'yicha xususiy hosilalar olamiz:

$$\frac{\partial P}{\partial y} = \frac{\partial^2 F}{\partial x \partial y}, \quad \frac{\partial Q}{\partial x} = \frac{\partial^2 F}{\partial y \partial x}.$$

$P, \frac{\partial Q}{\partial x}$ lar uzluksiz bo'lganligi uchun $\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}, \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ ekanligi

kelib chiqadi.

Yetarililigi. (14.13.5) shart bajarilgan bo'lsin. U holda **14.13.7-**teorema asosan (14.13.8) integral integrallash yo'liga bog'liq emas, ya'ni

$$\int_{(A|B)} Pdx + Qdy = \int_{(A|B)} Pdx + Qdy.$$

Bu holda (14.13.8) integral $A(x_0, y_0)$ va $B(x_1, y_1)$ nuqtalarning berilishi bilan bir qiymatli aniqlanadi. Shuning uchun uni

$$\int_A^B Pdx + Qdy = \int_{(x_0, y_0)}^{(x_1, y_1)} Pdx + Qdy$$

deb belgilash mumkin, bu yerda integrallash yo'lining boshlang'ich va oxirgi nuqtasi ko'rsatilgan bo'lib, yo'lining o'zi ko'rsatilmagan, lekin unga qanday yo'l bilan integrallashning farqi yo'q.

Agar $A(x_0, y_0)$ nuqtani belgilab, B nuqtani (D) sohadagi $\forall M(x, y)$ nuqta bilan almashtirsak, hosil bo'lgan integral M nuqtaning funksiyasi bo'ladi, ya'ni uning koordinatalari x va y larga bog'liq bo'ladi:

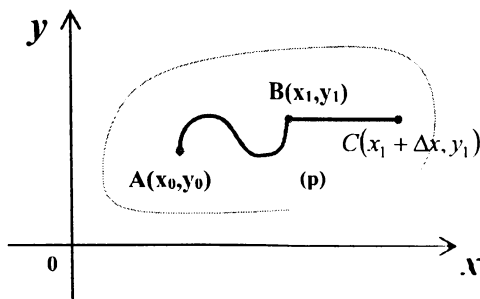
$$F(x, y) = \int_{(x_0, y_0)}^{(x, y)} Pdx + Qdy$$

Endi, $F(x, y)$ funksiyaning (14.13.11) ifodaning to'liq differensial bo'lishligini ko'rsatamiz: buning uchun (D) sohada ixtiyoriy $B(x_1, y_1)$ nuqtani olib, x_1 ga Δx ortirma beramiz, natijada $C(x_1 + \Delta x, y_1)$ nuqta hosil bo'ladi (Δx ortirmaning shunday qilib o'zgarishi).

$C(x_1 + \Delta x, y_1) \in (D)$ bo'lsin.

$$F(x_1, y_1) = \int_{(x_0, y_0)}^{(x_1, y_1)} Pdx + Qdy,$$

$$F(x_1 + \Delta x, y_1) = \int_{(x_0, y_0)}^{(x_1 + \Delta x, y_1)} Pdx + Qdy.$$



14.9-chizma.

Bundan

$$F(x_1 + \Delta x, y_1) - F(x_1, y_1) = \int_{(BC')} P dx + Q dy = \int_{(BC)} P(x, y) dx, \text{ chunki,}$$

$(BC) \perp OY$, $\int_{(BC)} Q dy = 0$. Keyingi $\int_{(BC)} P(x, y) dx$ integralni oddiy integralga keltiramiz:

$$F(x_1 + \Delta x, y_1) - F(x_1, y_1) = \int_{x_1}^{x_1 + \Delta x} P(x, y_1) dx.$$

Bu integralga o'rta qiymat haqidagi teoremani qo'llab,

$$\frac{F(x_1 + \Delta x, y_1) - F(x_1, y_1)}{\Delta x} = P(x_1 + \theta \Delta x, y_1) \quad (0 < \theta < 1)$$

ni hosil qilamiz. Bundan $P(x, y)$ funksiyaning uzluksizligini e'tiborga olgan holda, $\Delta x \rightarrow 0$ da limitga o'tsak, $\frac{\partial F(x_1, y_1)}{\partial x_1} = P(x_1, y_1)$ ni hosil qilamiz. Xuddi shunday

$$\frac{\partial F(x_1, y_1)}{\partial y_1} = Q(x_1, y_1)$$

ekanligi ham isbot qilinadi.

(D) sohada (x_1, y_1) nuqtaning ixtiyoriyligini e'tiborga olsak, u holda (D) sohaning hamma nuqtalarida

$$\frac{\partial F}{\partial x} = P, \quad \frac{\partial F}{\partial y} = Q$$

tengliklarning o'rinli ekanligi kelib chiqadi.

Yuqoridagi **14.13.3**, **14.13.7**, **14.13.12**- teoremlarni taqqoslash natijasida quyidagi natijaga kelamiz:

14.13.13. Natija. $P(x, y)$ va $Q(x, y)$ funksiyalar bir bog'lamli yoyiq (D) sohada yuqoridagi shartlarni qanoatlantirganda

$$\int_{(AB)} P dx + Q dy$$

integralning integrallash yo'liga bog'liq bo'lmazligi, $\int_{(k)} P dx + Q dy$

integralning nolga teng bo'lishi ((k)-yopiq chiziq), $P dx + Q dy$ ifodaning biror ikki o'zgaruvchili funksiyaning to'liq differensial bo'lishi uchun (14.13.5) shartning bajarilishi zarur va yetarli.

14.14. Egri chiziqli integrallarning fizikaga qo'llanishi. Biror material nuqta $\vec{F}$ kuch ta'siri ostida uzluksiz tekis BC chizig'i bo'ylab B dan C ga qarab harakatlanayotgan bo'lsin. $\vec{F}$ kuchni o'zgaruvchan (BC chiziqdagi nuqtaning holatiga bog'liq) deb faraz qilamiz (14.11-chizma). Material nuqtani B holatdan C holatga o'tkazishda $\vec{F}$ kuchni bajargan ishini hisoblaymiz. Buning uchun (BC) chiziqning ixtiyoriy

$$P = \{B = M_0, M_1, M_2, \dots, M_n = C\}$$

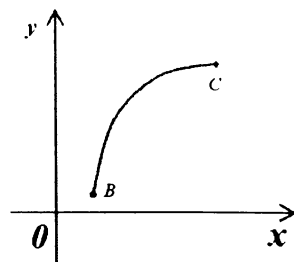
bo'lishini qaraymiz. Bu bo'linishning $M_i M_{i+1}$ bo'lagida $\vec{F}$ kuchni taqribiy ravishda o'zgarmas $\vec{F}_i \approx \vec{F}(M_i)$ qiymatga teng deb faraz qilib, nuqtaning $M_i \overset{\circ}{M}_{i+1}$ yoy bo'ylab harakatini esa, $\overline{M_i M_{i+1}}$ kesma bo'ylab harakatga almashtirsak, u holda o'zgarmas $\vec{F}(M_i)$ kuchning $M_i M_{i+1}$ kesma bo'ylab bajargan ishini taqribiy ravishda $\vec{F}$ o'zgaruvchan kuchning $M_i M_{i+1}$ yoy bo'ylab bajargan ishi deb qabul qilish mumkin:

$$A_i \approx \vec{F}(M_i) \cdot \overline{M_i M_{i+1}}$$

Bu taqribiy tenglikning o'ng tomoni $\vec{F}(M_i)$ va $\overline{M_i M_{i+1}}$ vektoring skalyar ko'paytmasini ifodalaydi. Ravshanki, skalyar ko'paytma

$$\vec{F}(M_i) = \{P(M_i), Q(M_i)\}, \overline{M_i M_{i+1}} = \{\Delta x_i, \Delta y_i\}$$

vektorlarning mos ravishda koordinatalari ko'paytmasining yig'indisiga teng:



14.11-chizma.

$$A_i \approx P(M_i)\Delta x_i + Q(M_i)\Delta y_i.$$

Bundan, (BC) egri chiziq bo'ylab bajarilgan ish

$$A \approx \sum_{i=1}^n [P(M_i)\Delta x_i + Q(M_i)\Delta y_i] = \sum_{i=1}^n \left[P(M_i)\Delta x_i + \sum_{i=1}^n Q(M_i)\Delta y_i \right] \quad (14.14.1)$$

Bajarilgan ishning aniq qiymati (14.14.1) tenglikda

$\lambda_p = \max \left| \overline{M_i \overset{\circ}{M}_{i+1}} \right| \rightarrow 0$ da limitga o'tish bilan topiladi. Ikkinchi tomondan

(14.14.1) yig'indi (BC) egri chiziqda berilgan $P(x, y)$ va $Q(x, y)$ funksiyalarning integral yig'indilarining yig'indisini ifoda qiladi. Bu

yig'indilarning $\lambda_p \rightarrow 0$ dagi limiti ikkinchi tur egri chiziqli integraldan iborat.

Demak, bajarilgan ish

$$A = \int_{(BC)} P(x, y) dx + Q(x, y) dy$$

formula bilan topilar ekan, bunda P va Q lar $\vec{F}$ kuchning (koordinata o'qlardagi proyeksiyalari) koordinatalari. Agar berilgan masalani fazoda qarasak, u holda ish

$$A = \int_{(BC)} P dx + Q dy + R dz$$

formula orqali topiladi. .

**68-§. Sirtning tomoni va uning yuzini ikki karrali
integral orqali hisoblash**

15.1. Sirtning parametrik shaklida tasvirlanishi. Ma'lumki, egri chiziqlarning fazodagi parametrik tenglamasi ushbu

$$x = \varphi(t), \quad y = \psi(t), \quad z = \chi(t) \quad (\alpha \leq t \leq \beta)$$

ko'rinishda bo'ladi, bunda egri chiziqdagi nuqtalarning o'zini parametr t ning qiymatlari bilan aniqlanadi. Tenglamasi oshkor

$$z = f(x, y) \quad (15.1.1)$$

ko'rinishda berilgan (S) sirtidagi nuqtalarning o'zini esa, ikkita x va y parametrlarning (x, y) juft qiymatlar bilan aniqlanadi. Umumiy holda (S) sirtning ixtiyoriy u va v parametrlar orqali tenglamasi ushbu

$$x = \varphi(u, v), \quad y = \psi(u, v), \quad z = g(u, v), \quad (u, v) \in (\Delta) \quad (15.1.2)$$

ko'rinishda bo'lib, bunda φ, ψ, g - lar (Δ) sohada uzluksiz va uzluksiz birinchi tartibli xususiy hosilalarga ega, (Δ) - chegaralangan, yopiq soha, uning chegarasi o'zaro kesishmaydigan bo'lakli siliq chiziqdan iborat, hamda (Δ) sohaning har xil (u, v) ichki nuqtasiga (S) sirtning har xil (x, y, z) nuqtasi mos keladi, (Δ) ning har bir chegara nuqtasiga (S) sirtning bitta chegara nuqtasi mos keladi deb faraz qilinadi. (Δ) sohaning chegara nuqtalariga mos keladigan (S) sirtning nuqtalar to'plami sirtning chegarasini tashkil qiladi. Shunday qilib, (15.1.2) sistema yordamida (S) sirt nuqtalari bilan (Δ) soha nuqtalari o'rasida o'zaro bir qiymatli moslik o'rnatiladi. Bu holda sirtni sodda sirt deyiladi. u va v parametrlar (S) sirt nuqtalarining egrichiziqli kordinalari deyiladi. (S) sirtning (15.1.2) ko'rinishdagi parametrik tenglamasi uning oshkor (15.1.1) ko'rinishdagi tenglamasiga qandaydir ma'noda, ya'ni biror nuqtaning kichik atrofida teng kuchli ekanligini ko'rsatamiz. Buning uchun (S) sirtning biror $M_0(x_0, y_0, z_0)$ nuqtasini olib, u nuqta parametrlarning bitta

$u = u_0, v = v_0$ juft qiymatlariga mos kelsin deb faraz qilamiz.

(15.1.2) sitemaning ushbu

$$\begin{pmatrix} \varphi_u' & \psi_u' & g_u' \\ \varphi_v' & \psi_v' & g_v' \end{pmatrix} \quad (15.1.3)$$

funksiyaonal matritsasini qaraymiz.

Parametrlarning (u_0, v_0) qiymati bilan aniqlangan $M_0(x_0, y_0, z_0)$ nuqtada (15.1.3) matritsaning hech bo'lmaganda bitta determenanti noldan farqli bo'lsin.

Masalan,

$$\begin{vmatrix} \varphi_u' & \psi_u' \\ \varphi_v' & \psi_v' \end{vmatrix} \neq 0.$$

Bu holda M_0 nuqtani oddiy (maxsus emas) nuqta deyiladi. Agarda (15.1.3) matritsaning hamma determinanti M_0 nuqtada birdaniga nolga teng bo'lsa, M_0 nuqta *maxsus nuqta* deyiladi. Maxsus nuqtaga ega bo'lmagan sodda sirtni silliq sirt deb ataymiz.

(15.1.2) ning birinchi ikkitasini ushbu

$$\varphi(u, v) - x = 0, \quad \psi(u, v) - y = 0$$

ko'rinishda yozib olamiz. Oshkormas funksiyaning mavjudligi haqidagi – teoremaga asosan, u, v, x, y o'zgaruvchili ikkita tenglamalar sistemasidan u, v lar x va y larning bir qiymatli funksiyasi sifatida aniqlanadi:

$$u = u(x, y), \quad v = v(x, y).$$

bu funksiyalar o'zlarining xususiy xosilalari bilan birgalikda uzluksiz. u va v larning topilgan $u(x, y), v(x, y)$ ifodalarini (15.1.2) ning uchunchisiga olib borib qo'ysak, natijada sirtning $M_0(x_0, y_0, z_0)$ nuqta atrofida oshkor $z = g(v(x, y), u(x, y)) = f(x, y)$ tenglamasi hosil bo'ladi, bunda $f(x, y)$ ham uzluksiz. uzluksiz xususiy hosilalarga ega bo'ladi. M_0 nuqta maxsus nuqta bo'lsa, sirt tenglamasini, bu nuqta atrofida oshkor shaklda ifodalab bo'lmaydi. $M(x, y, z)$ nuqta (15.1.2) sirtning oddiy ichki nuqtasi bo'lsin. Yuqorida ko'rdikki, bu nuqtaning biror kichik atrofida sirt tenglamasi oshkor shaklda ifoda qilinadi va sirt $M(x, y, z)$ nuqtada o'rinma tekislikga ega bo'ladi. Ma'lumki bu o'rinma tekislik tenglamasi ushbu

$$A(X-x) + B(Y-y) + C(Z-z) = 0 \quad (15.1.4)$$

ko'rinishda bo'ladi, bunda A, B, C hozircha no'malum koeffitsiyentlar. Agar (15.1.2) sirt tenglamasida v ning belgilangan M nuqtaga mos keladigan qiymatini belgilasak, natijada M nuqtadan o'tuvchi va sirtga butunlay yotuvchi $x = \varphi(u, v_0)$, $y = \psi(u, v_0)$, $z = g(u, v_0)$ chiziq hosil bo'ladi. v_0 ning qiymatlarini o'zgartirish natijasida M nuqtadan o'tuvchi "v" koordinat chiziqlari oilasi hosil bo'ladi. Bu chiziqqa M nuqtada o'tkazilgan o'rinma chiziq tenglamasi

$$\frac{X-x}{x'_u} = \frac{Y-y}{y'_u} = \frac{Z-z}{z'_u}$$

ko'rinishda bo'ladi. Xuddi shunday u ni belgilasak M nuqtadan o'tuvchi boshqa "u" koordinat chiziqlari oilasi hosil bo'ladi. Bu chiziqqa M nuqtada o'tkazilgan o'rinma chiziq tenglamasi ushbu

$$\frac{X-x}{x'_v} = \frac{Y-y}{y'_v} = \frac{Z-z}{z'_v}$$

ko'rinishda bo'ladi. Bu o'rinma chiziqlarning ikkalasi ham (15.1.4) tekislikda yotadi. U holda,

$$Ax'_u + By'_u + Cz'_u = 0,$$

$$Ax'_v + By'_v + Cz'_v = 0$$

shartlar o'rinli bo'ladi. Bu holda A, B, C koeffitsiyentlar, (15.1.3) matritsaning determinantlariga (qaralayotgan nuqtada hamma determinantlar birdaniga nolga teng emasligini takidalaymiz) proporsional bo'lishi kerak, ya'ni ularni bundan keyin bu determinantlarga teng deb olish mumkin, ya'ni

$$A = \begin{vmatrix} y'_u & z'_u \\ y'_v & z'_v \end{vmatrix}, B = \begin{vmatrix} z'_u & x'_u \\ z'_v & x'_v \end{vmatrix}, C = \begin{vmatrix} x'_u & y'_u \\ x'_v & y'_v \end{vmatrix} \quad (15.1.5)$$

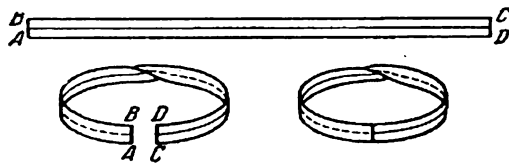
Ma'lumki, $M(x, y, z)$ nuqtaga o'tkazilgan normalning yo'naltiruvchi kosinuslari:

$$\cos \alpha = \frac{A}{\pm \sqrt{A^2 + B^2 + C^2}}, \cos \beta = \frac{B}{\pm \sqrt{A^2 + B^2 + C^2}}, \cos \gamma = \frac{C}{\pm \sqrt{A^2 + B^2 + C^2}}.$$

15.2. Sirtning tomoni. Ko'p hollarda sirt tomoni tushunchasini ichki (intuitiv) hissiyatlarimiz asosida aniqlashimiz mumkin. Masalan,

agar sirt biror jismni qoplab turgan bo'lsa, u holda sirtning jismga qaragan tomonini, sirtning ichki tomoni, jismning fazoga qaragan tomonini tashqi tomon deb qarash mumkin. Agar sirt tenglamasi $z = f(x, y)$ oshkor shaklda tasvirlangan bo'lsa, sirtning yuqori va pastki tomoni to'g'risida gapirish mumkin (bunda z o'qi vertikal yuqoriga qaralgan deb tushuniladi). Bu ichki hissiyatlarimiz yordamida sirtning tomoni ta'rifini berishga harakat qilamiz. (S)- siliq sirtning ixtiyoriy M_0 nuqtasini olib, undan (S)- sirtga $\vec{n}$ normal o'tkazamiz va uning biror yo'nalishini belgilaymiz. (S) sirtning M_0 nuqtasidan o'tuvchi va unga qaytib keluvchi, sirtning chegarasini kesib o'tmaydigan biror (K) yopiq egri chiziqni olib, M nuqtani bu yopiq (K) chiziq bo'ylab uzluksiz haraktlantiramiz va uning har bir holatiga normal o'tkazib, uning yo'nalishini belgilab boramiz, natijada, M nuqta yana M_0 nuqtaga qaytib kelganda, normalning yo'nalishi ikki yo'nalishdan biri, ya'ni oldingi belgilangan yo'nalishda yoki unga qarama-qarshi yo'nalishda bo'lishi mumkin.

Agar belgilangan M_0 nuqtadan o'tuvchi va (S) sirtning chegarasini kesib o'tmaydigan ixtiyoriy yopiq (K) egri chiziq bo'ylab xarakatlanayotgan M nuqta yana oldingi belgilangan M_0 ga qaytib kelganda normal o'zining yo'nalishini o'zgartirmasa, ya'ni yo'nalishi oldingi belgilangan yunalishga mos tushsa, u holda (S) sirt ikki tomonli sirt deyiladi. Ikki tomonli sirtga, tenglamasi $z = f(x, y)$ (bunda $f(x, y)$, $f'_x(x, y)$, $f'_y(x, y)$ – biror D sohada uzluksiz funksiyalar) shaklda berilgan sirtlar misol bo'laoladi.



15.1-chizma.

Agar (S) sirtning berilgan M_0 nuqtasidan o'tuvchi va (S) sirtning chegarasini kesib o'tmaydigan (K) yopiq chiziq bo'ylab

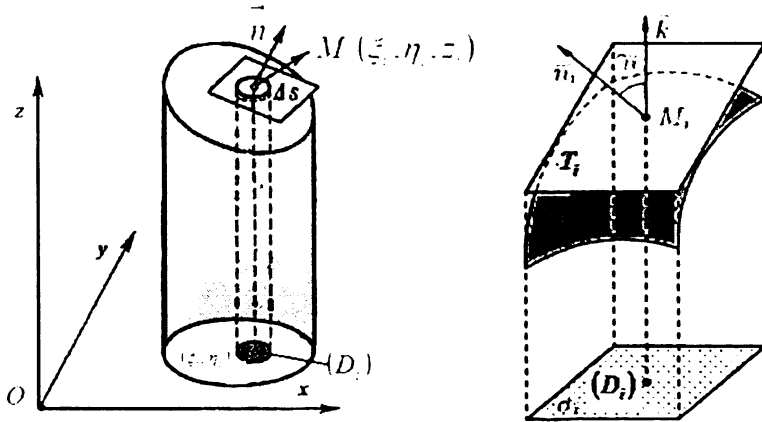
harakatlanayotgan M nuqta oldingi belgilangan M_0 nuqta holatiga kelganda, unga mos kelgan normalning yo'nalishi o'z yo'nalishini o'zgartirsa, u holda (S) sirt *bir tomonli sirt* deyiladi. Bir tomonli sirtga Mebius listi misol bo'la oladi (15.1-chizma). Mebius listini quyidagicha hosil qilish mumkin: ABSD qog'oz bo'lagini olib, uni shunday bo'raylikki, natijada A nuqta C nuqta, B nuqta esa D nuqta ustiga tushsin, so'ngra uni kleylaymiz. Mebius listida M_0 nuqtani olib unga o'tkazilgan normalning yo'nalishini belgilaymiz. Mebius listi bo'ylab M nuqtani va $\vec{n}$ normalni harakatlantiramiz. M nuqta M_0 nuqta holatiga kelganda $\vec{n}$ normalning yo'nalishi oldingi belgilangan yo'nalishga qarama – qarshi bo'ladi. Biz bundan keyin faqat ikki tomonli sirtni qaraymiz. Ikki tomonli sirtida bitta belgilangan nuqtadagi normalning tanlangan yo'nalishi, sirtning qolgan nuqtalardagi normalning yo'nalishini bir qiymatli aniqlaydi. Sirtning, unda ko'rsatilgan qoida bo'yicha normalning yo'nalishlari aniqlangan barcha nuqtalari to'plami sirtning *belgilangan tomoni (oriyentirlanuvchi sirt)* deyiladi, sirtning biror tomonini tanlash esa, oriyentatsiya deyiladi.

Bir tomonli sirt – *oriyentirlanmaydigan sirt* deb ataladi.

15.3. Sirt yuzi va uni ikki karrali integral orqali hisoblash.

Avvalo sirtning yuzi tushunchasini kiritamiz. (S) sirt o'zining oshkor $z = z(x, y)$ tenglamasi bilan berilgan bo'lib, uning xOy tekislikdagi proyeksiyasi (D) soha bo'lsin (15.2-chizma). (D) sohada $z(x, y)$ – uzluksiz va uzluksiz $z'_x(x, y), z'_y(x, y)$ xususiy hosilalarga ega bo'lgan funksiya aniqlangan bo'lsin. (S) sirtning yuzini topish uchun (D) sohaning ixtiyoriy P bo'linishini qaraymiz: $(D_1), (D_2), \dots, (D_n)$. Bu bo'linishning bo'luvchi chiziqlarini yo'naltiruvchilar sifatida qarab, ular orqali yasovchilari Oz o'qiga parallel bo'lgan silindrik sirtlarni o'tkazamiz. Bu silindrik sirtlar (S) sirtni $(S_1), (S_2), \dots, (S_n)$ bo'laklarga ajratadi. Har bir $(D_i) (i=1, 2, \dots, n)$ dan ixtiyoriy (ξ_i, η_i) nuqtani olib, (S) sirtida unga mos kelgan $(\xi_i, \eta_i, z_i) (z_i = z(\xi_i, \eta_i))$ nuqtani topamiz. (S) sirtning shu nuqtasiga o'rinma tekislik o'tkazamiz. Bu o'rinma tekislik bilan yuqorida aytilgan

silindirik sirtning keshishidan hosil bo'lgan o'rinma tekislik qismini (T_i) bilan, uning yuzini esa, τ_i deb belgilaymiz (15.2-chizma). Ma'lumki (D_i) soha (T_i) ning ortogonal preksiyasi bo'lib, uning yuzi



15.2-chizma.

$$D_i = T_i |\cos \gamma_i|, \quad T_i = \frac{D_i}{|\cos \gamma_i|} \quad (15.3.1)$$

formula bilan topiladi, bunda γ_i – (s) sirtga (ξ_i, η_i, z_i) ($z_i = z(\xi_i, \eta_i)$) nuqtada o'tkazilgan o'rinma tekislik normalining Oz o'qi bilan takshil etgan burchagi. Ravshanki, $\lambda_{p \rightarrow 0}$ da (S_i) ($i = 1, 2, \dots, n$) ning diametri ham nolga intiladi.

Agar $\lambda_{p \rightarrow 0}$ da $\sum_{i=1}^n T_i$ yig'indi chekli limitga ega bo'lsa, bu limit (S) sirtning yuzi deyiladi, ya'ni (S) sirtning yuzi

$$S = \lim_{\lambda_{p \rightarrow 0}} \sum_{i=1}^n T_i.$$

$z = z(x, y)$ funksiyaga yuqoridagi shartlar qo'yilganda

$$\cos \gamma_i = \frac{1}{\sqrt{1 + (z'_x(\xi_i, \eta_i))^2 + (z'_y(\xi_i, \eta_i))^2}}$$

bo'ladi. Unda (15.3.1) munosabatdan $\tau_i = \frac{D_i}{\cos \gamma_i}$.

Shunday qilib,

$$\sum_{i=1}^n T_i = \sum_{i=1}^n \frac{D_i}{\cos \gamma_i} = \sum_{i=1}^n \sqrt{1 + (z'_x(\xi_i, \eta_i))^2 + (z'_y(\xi_i, \eta_i))^2} D_i. \quad (15.3.2)$$

Ravshanki, (15.3.2) ning o'ng tomonidagi yig'indi

$$\sqrt{1 + (z'_x(x, y))^2 + (z'_y(x, y))^2}$$

funksiya uchun integral yig'indi bo'lib hisoblanadi.

Bu funksiya (D) sohada uzluksiz, demak, u integrallanuvchi.

Shuning uchun

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt{1 + (z'_x(\xi_i, \eta_i))^2 + (z'_y(\xi_i, \eta_i))^2} \cdot D_i = \iint_{(D)} \sqrt{1 + (z'_x(x, y))^2 + (z'_y(x, y))^2} dx dy.$$

Demak, (S) sirtning yuzi ushbu

$$S = \iint_{(D)} \sqrt{1 + (z'_x)^2 + (z'_y)^2} dx dy = \iint_{(D)} \frac{1}{|\cos \gamma|} dx dy \quad (15.3.3)$$

formula bilan hisoblanar ekan.

2. Umumiy holda sirt yuzi. (S) sirt tenglamasi parametrik shaklida:

$$x = x(u, v), y = y(u, v), z = z(u, v), (u, v) \in (\Delta).$$

berilgan bo'lib, $x(u, v)$, $y(u, v)$, $z(u, v)$ lar (Δ) sohada uzliksiz va uzliksiz hususiy hosilalarga ega hamda karrali va maxsus nuqtalarga ega emas deb faraz qilamiz.

Ixtiyoriy $M \in (S)$ nuqtani olamiz va shu nuqtada $C \neq 0$ bo'lsin deb faraz qilamiz. 15.1-bandda aytilganlarga va yuqoridagi (S) sirtga qo'yilgan shartlarda quyidagi tasdiqlar o'rinli bo'ladi:

(S) sirtning M nuqtani o'rab turuvchi shunday (S_1) qismi mavjud bo'lib, u quyidagi xossalarga ega bo'ladi:

1^o. (S_1) sirt tenglamasi oshkor $z = z(x, y)$ shalida ifoda qilinadi.

2^o. (S_1) ga sirt mos keluvchi uOv tekislikdagi (Δ) sohaning qismini (δ) deb belgilasak, u holda (δ) da $C \neq 0$ bo'ladi.

Agar A yoki B determinatlar noldan farqli bo'lsa, u holda $z = z(x, y)$ tenglama, $x = x(y, z)$ tenglamaga yoki $y = y(z, x)$ tenglamaga almashiladi. Bundan (S) sirt chekli sondagi (S_1) sirt bo'laklariga yoyiladi degan xulosani chiqarish mumkin. Endi yuqoridagi ta'rifga ko'ra (S_1)

sirt bo'lagining yuzini hisoblaymiz. Agar (S_1) sirt bo'lagining xOy tekislikdagi proyeksiyasini (d) deb belgilasak, u holda uning yuzi ((15.3.3) qarang).

$$S_1 = \iint_{(d)} \frac{dx dy}{|\cos \gamma|} \quad (15.3.4)$$

formula bilan hisoblanadi.

(d) sohaning nuqtalari (δ) sohaning nuqtalari bilan bir qiymatli moslikda ekanligini e'tiborga olgan holda va bundan tashqari tekis figuralarni almashtirish mavzusidagi talablar bajarilgan bo'lsin degan farazda, u holda (15.3.4) da u va v o'zgaruvchilarga o'tsak hamda

$$\frac{D(x, y)}{D(u, v)} = C, |\cos \gamma| = \frac{|C|}{\sqrt{A^2 + B^2 + C^2}}$$

ekanligini iborga olgan holda

$$S_1 = \iint_{(\delta)} \sqrt{A^2 + B^2 + C^2} du dv$$

ni topamiz. Endi (S) sirtning yuzi S ni uning (S_1) bo'laklari yuzlarining yig'indisi sifatida topamiz:

$$S = \iint_{(\Delta)} \sqrt{A^2 + B^2 + C^2} du dv. \quad (15.3.5)$$

(15.3.5) formulani boshqacha ko'rinishda ham yozish mumkin:

$$x_u'^2 + y_u'^2 + z_u'^2 = E, \quad x_v'^2 + y_v'^2 + z_v'^2 = G, \quad x_u'x_v' + y_u'y_v' + z_u'z_v' = F$$

belgilashlarni kiritsak, natijada $A^2 + B^2 + C^2 = EG - F^2$ ekanligiga ishonch hosil qilamiz. U holda (15.3.5) ko'rinishi

$$S = \iint_{(\Delta)} \sqrt{EG - F^2} du dv. \quad (15.3.6)$$

ko'rinishga keladi. E, G, F koeffitsiyentlarni Gauss koeffitsiyentlari deb yuritiladi.

69-§. Birinchi tur sirt integrali va uning fizikaga qo'llanishi

15.4. Birinchi tur sirt integralining ta'rif. $f(x, y, z)$ funksiya biror ikki tomonli silliq yoki bo'lakli siliq $(S) \subset R^3$ sirtida berilgan bo'lsin. Bu sirtni bo'lakli – silliq chiziqlar to'ri yordamida

$(S_1), (S_2), \dots, (S_n)$ bo'laklarga ajratamiz. Bu bo'linishni P bo'linish deb ataymiz. Bu bo'linishning $(S_i) (i=1, 2, \dots, n)$ bo'lagidan ixtiyoriy $M_i(x_i, y_i, z_i)$ nuqtani olib, bu nuqtadagi berilgan funksiyaning $f(x_i, y_i, z_i)$ qiymatiga (S_i) sirtning yuzi S_i ni ko'paytirib, quyidagi yig'indini tuzamiz:

$$\sigma = \sum_{i=1}^n f(x_i, y_i, z_i) S_i = \sum_{i=1}^n f(M_i) S_i$$

Bu yig'indiga $f(x, y, z)$ funksiyaning Riman integral yig'indisi deb ataladi. (s) sirtning shundan

$$P_1, P_2, \dots, P_n, \dots \quad (15.4.1)$$

bo'linishlar ketma ketligini qaraylikki, ularning mos diametrlaridan tashkil topgan $\lambda_{p_1}, \lambda_{p_2}, \dots, \lambda_{p_n}, \dots$ ketma – ketlik nolga intilsin: $\lambda_{p_n} \rightarrow 0$.

$\{P_m\}$ bo'linishlar ketma – ketligiga mos kelgan $f(x, y, z)$ funksiyaning integral yig'indilar ketma – ketligini tuzamiz:

$$\sigma_1, \sigma_2, \dots, \sigma_n, \dots$$

15.4.2. Ta'rif. Agar (s) sirtning har qanday (15.4.1) ko'rinishdagi bo'linishlar ketma – ketligi $\{P_m\}$ olingan ham unga mos kelgan integral yig'indilar ketma – ketligi $\{\sigma_m\}$ $M_i(x_i, y_i, z_i)$ nuqtalarni tanlash qonuniga bog'liq bo'lmagan holda, hamma vaqt bitta J limitga intilsa, u holda bu limit σ integral yig'indining limiti deb ataladi va u

$$\lim_{\lambda_p \rightarrow 0} \sigma = \lim_{\lambda_p \rightarrow 0} \sum_{i=1}^n f(x_i, y_i, z_i) S_i = J$$

kabi belgilanadi.

15.4.3. Ta'rif. Agar $\lambda_p \rightarrow 0$ da $f(x, y, z)$ funksiyaning integral yig'indisi σ chekli limitga ega bo'lsa, u holda $f(x, y, z)$ funksiya (s) sirt bo'yicha integrallanuvchi funksiya deb ataladi. Bu yig'indining chekli limiti J ga $f(x, y, z)$ funksiyaning *birinchi tur sirt integrali* deyiladi va u

$$\iint_{(s)} f(M) ds = \iint_{(s)} f(x, y, z) ds \quad (15.4.4)$$

kabi belgilanadi.

15.5. Birinchi sirt integralining mavjudlik sharti va uni hisoblash. (s) sirt o'zining (15.1.2) ko'rinishdagi parametrik tenglamasi bilan berilgan bo'lib, (s) -sodda siliq sirt bo'lsin, ya'ni φ, ψ, g funksiyalar

(Λ) sohada $((u,v) \in (\Lambda))$ uzluksiz va uzluksiz xususiy hosilalarga ega bo'lib maxsus nuqtalarga ega bo'lmasin.

15.5.1. Teorema. Agar $f(x,y,z)$ funksiya tenglamasi (15.1.2) ko'rishda berilgan (S) silliq sirtida uzluksiz bo'lsa, u holda (15.4.4) integral mavjud bo'ladi va ushbu

$$\iint_{(S)} f(x,y,z) ds = (R) \iint_{(\Lambda)} f(\varphi(u,v), \psi(u,v), g(u,v)) \sqrt{EG - F^2} dudv \quad (15.5.2)$$

tenglik o'rinli bo'ladi.

Isboti. (S) sirtning ixtiyoriy P bo'linishini: $(S_1), (S_2), \dots, (S_n)$ qaraymiz; (S) sirtning bu bo'linishiga (Δ) sohaning $(\Delta_1), (\Delta_2), \dots, (\Delta_n)$ bo'linishi mos keladi. Bu bo'linishning har bir $(S_i) (i=1, 2, \dots, n)$ bo'lagidan ixtiyoriy (x_i, y_i, z_i) nuqtani Δ_i dan unga mos kelgan (u_i, v_i) nuqtani olamiz; bunda

$$x_i = \varphi(u_i, v_i), y_i = \psi(u_i, v_i), z_i = g(u_i, v_i). \quad (15.5.3)$$

(15.4.4) integral uchun integral yig'indini tuzamiz.

$$\sigma = \sum_{i=1}^n f(x_i, y_i, z_i) S_i$$

(15.3.6) formulaga asosan, $S_i = \iint_{(\Lambda)} \sqrt{EG - F^2} dudv$. Bu integralga o'rta

qiymat haqidagi teoremani qo'llab

$$S_i = \left[\sqrt{EG - F^2} \right] \Big|_{u=\bar{u}_i, v=\bar{v}_i}^{u=\bar{u}_i, v=\bar{v}_i} \cdot \Delta_i$$

topamiz, bunda $(\bar{u}_i, \bar{v}_i) \in (\Delta_i)$ (15.5.3) ni va S_i ning bu qiymatini e'tiborga olib, yuqoridagi integral yig'indini quyidagicha yozamiz:

$$\sigma = \sum_{i=1}^n f(\varphi(u_i, v_i), \psi(u_i, v_i), g(u_i, v_i)) \left[\sqrt{EG - F^2} \right]_{u=\bar{u}_i, v=\bar{v}_i}^{u=\bar{u}_i, v=\bar{v}_i} \cdot \Delta_i.$$

Integral yig'indining bu ko'rinishiga qarab, (15.5.2) ning o'ng tomonidagi integral uchun integral yig'indining ko'rinishi

$$\sigma^* = \sum_{i=1}^n f(\varphi(u_i, v_i), \psi(u_i, v_i), g(u_i, v_i)) \left[\sqrt{EG - F^2} \right]_{u=\bar{u}_i, v=\bar{v}_i}^{u=\bar{u}_i, v=\bar{v}_i} \cdot \Delta_i$$

bo'lishini anglash qiyin emas, σ bilan σ^* yig'indi orasidagi farq quyidagicha: σ^* da $f(\varphi(u_i, v_i), \psi(u_i, v_i), g(u_i, v_i))$ murakkab funksiya va $\sqrt{EG - F^2}$ lar ixtiyoriy (u_i, v_i) nuqtada, σ yig'indida esa,

$f(\varphi(u_i, v_i), \psi(u_i, v_i), g(u_i, v_i))$ funksiya (u, v) nuqta $\sqrt{EG-F^2}$ ildiz esa, (u, v) (o'rta qiymat haqidagi teremani qo'llagandagi nuqta) nuqtada hisoblangan. Endi bu yig'inlarning ayirmasini qaraymiz:

$$\sigma - \sigma^* = \sum_{i=1}^n f(\varphi(u_i, v_i), \psi(u_i, v_i), g(u_i, v_i)) \left\{ \left[\sqrt{EG-F^2} \right]_{u=\bar{u}_i, v=\bar{v}_i} - \left[\sqrt{EG-F^2} \right]_{u=u_i, v=v_i} \right\} \Delta x_i.$$

Δ - ixtiyoriy kichik son bo'lsin. $\sqrt{EG-F^2}$ funksiyaning tekis uzluksizligidan (Δ_i) sohalarining istalgancha kichik diametrlari, ya'ni δ uchun

$$\left| \left[\sqrt{EG-F^2} \right]_{u=\bar{u}_i, v=\bar{v}_i} - \left[\sqrt{EG-F^2} \right]_{u=u_i, v=v_i} \right| < \varepsilon$$

bo'ladi, $f(x, y, z)$ funksiyaning uzluksizligidan

$$|f(x, y, z)| \leq M$$

o'rinli. Demak, $|\sigma - \sigma^*| < \varepsilon M$, bundan

$$\lim_{\Delta \rightarrow 0} (\sigma - \sigma^*) = 0.$$

Shu bilan teorema isbot bo'ladi.

Agar (S) sirt tenglamasi oshkor $z = z(x, y)$ shaklda berilgan bo'lsa, u holda (15.5.2) formulaning ko'rinishi

$$\iint_{(S)} f(x, y, z) ds = (R) \iint_{(D)} f(x, y) \sqrt{1 + (z'_x)^2 + (z'_y)^2} dx dy \quad (15.5.4)$$

shaklida bo'ladi, bunda (D) - (S) sirtning xOy tekislikdagi proyeksiya

$$\sqrt{1 + (z'_x)^2 + (z'_y)^2} = \frac{1}{|\cos \gamma|}.$$

(γ) - (S) sirtga o'tkazilgan normalning Oz o'qi bilan tashkil qilgan burchagi bo'lishini e'tiborga olib, (15.5.4) formulani

$$\iint_{(S)} f(x, y, z) ds = (R) \iint_{(D)} f(x, y, z(x, y)) \frac{dx dy}{|\cos \gamma|}$$

ko'rinishda yozish mumkin. Shunga o'xshash, (S) siliq sirt $x = x(y, z) \in (D_1)$ tenglama bilan berilganda (bu yerda (D_1) - (S) sirtning (y, z) tekislikdagi proyeksiyasi), sirt differensial $ds = \sqrt{1 + x_y'^2 + x_z'^2} dy dz$ ko'rinishda yoziladi. Shuning uchun sirt integralini quyidagi ko'rinishda yozish mumkin:

$$\iint_{(S)} f(x, y, z) ds = (R) \iint_{(D_1)} f(x(y, z), y, z) \sqrt{1 + x_y'^2 + x_z'^2} dz dy, \quad (15.5.5)$$

yoki

$$\iint_{(S)} f(x, y, z) ds = (R) \iint_{(D_1)} f(x(y), y, z) \cdot \frac{dy dz}{|\cos \alpha|}$$

(bu yerda α – (S) sirtga o'tkazilgan $\vec{n}$ normal vektorning Ox o'qi bilan

tashkil burchagi, ya'ni $\alpha = \left(\vec{Ox}, \vec{n} \right)$ (S) sirt tenglamasi

$y = y(x, z)$ ($(x, z) \in (D_2)$) tenglama bilan berilganda, sirt differensial

$ds = \sqrt{1 + y_x'^2 + y_z'^2} dx dz$ ko'rinishda yoziladi va

$$\iint_{(S)} f(x, y, z) ds = (R) \iint_{(D_2)} f(x, y(x, z), z) \sqrt{1 + y_x'^2 + y_z'^2} dx dz \quad (15.5.6)$$

yoki

$$\iint_{(S)} f(x, y, z) ds = (R) \iint_{(D_2)} f(x, y(x, z), z) \frac{dx dz}{|\cos \beta|}$$

formula o'rinli bo'ladi (bu yerda (D_2) – (S) sirtining (x, z) tekislikdagi

proyeksiyasi, $\beta = \left(\vec{Oy}, \vec{n} \right)$).

15.5.7. Eslatma. (15.5.2), (15.5.4), (15.5.5) va (15.5.6) formulalar bo'lakli sirtlar uchun ham o'rinli.

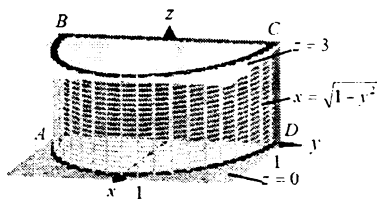
15.5.8. Misol. Ushbu

$$J = \iint_{(S)} x(y+1) ds$$

birinchi tur sirt integralini hisoblang, bunda (S) – $x = \sqrt{1-y^2}$ silindrik sirtning $z=0$ va $z=3$ tekisliklar orasidagi qismi.

Yechilishi. (S) sirtning yOz tekislikdagi proyeksiyasi (D) soha- ABCD to'g'ri to'rtburchakdan iborat, ya'ni

$$(D) = \{(y, z); -1 \leq y \leq 1, 0 \leq z \leq 3\}$$



15.3-chizma.

(15.3-chizma). Bu (D) sohada $x = \sqrt{1-y^2}$ funksiya uzluksiz va

$x'_y = \frac{y}{\sqrt{1-y^2}}$, $x'_z = 0$ uzluksiz xususiy hosilalarga ega. (S) sirtida berilgan

$f(x, y, z) = x(y+1)$ funksiya shu sirtida uzluksiz. Berilgan birinchi tur sirt integralini (15.5.5) formuladan foydalanib, hisoblaymiz:

$$\begin{aligned} J &= \iint_{(D)} \sqrt{1-y^2} (y+1) \sqrt{1+\frac{y^2}{1-y^2}} dydz = \iint_{(D)} (y+1) dydz = \\ &= \int_{-1}^1 (y+1) dy \int_0^3 dz = \frac{(y+1)^2}{2} \Big|_{-1}^1 \cdot z \Big|_0^3 = 6. \end{aligned}$$

Misolni Maple tizimidan foydalanib, yechish:

`int(int(y+1,z=0..3),y=-1..1);`

6

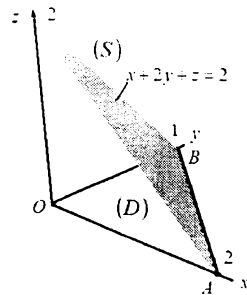
15.5.9.Misol. Ushbu

$$J = \iiint_{(S)} [2x + y + 3z] ds$$

birinchi tur sirt integralini hisoblang, bunda (S) sirt- $x + 2y + z = 2$ tekislikning birinchi oktantdagi qismi.

Yechilishi. (S) sirt tenglamasini z ga nisbatan yechamiz: $z = 2 - x - 2y$. (S) sirtning xOy ($z=0$) tekislikdagi proyeksiyasi (D) soha esa, ABO uchburchakdan iborat (15.4-chizma).

Bu (D) sohada z funksiya uzluksiz va



15.4-chizma.

$z'_x = -1$, $z'_y = -2$ uzluksiz xususiy hosilalarga ega. $f(x, y, z) = 2x + y + 3z$ funksiya (S) sirtida uzluksiz. (15.5.4) formuladan foydalanib, integralni hisoblaymiz:

$$\begin{aligned} J &= \iint_{(D)} [2x + y + 6 - 3x - 6y] \sqrt{6} dx dy = \sqrt{6} \iint_{(D)} [6 - x - 5y] \sqrt{6} dx dy = \\ &= \sqrt{6} \int_0^1 dy \int_0^{2-2y} (6 - x - 5y) dx = \sqrt{6} \int_0^1 \left[(6 - 5y)x - \frac{x^2}{2} \right] \Big|_{x=0}^{x=2-2y} dy = \end{aligned}$$

$$= 2\sqrt{6} \int_0^1 (5 - 9y + 4y^2) dy = \frac{11}{3} \sqrt{6}.$$

15.6. Birinchi tur sirt integralining fizikaga qo'llanishi. $\forall M(x, y, z)$ nuqtadagi sirt zichligi $\rho(x, y, z)$ bo'lgan material (S) sirt berilgan bo'lsin ($M \in (S)$). U holda, quyidagi formulalar o'rinli bo'ladi:

$$a) m = \iint_{(S)} \rho(x, y, z) ds$$

sirtning massasi;

$$b) M_{xy} = \iint_{(S)} z\rho(x, y, z) ds, M_{xz} = \iint_{(S)} y\rho(x, y, z) ds, M_{yz} = \iint_{(S)} x\rho(x, y, z) ds$$

xOy, xOz, yOz koordinat tekisliklariga nisbatan statik momentlar;

$$v) x_0 = \frac{M_{yz}}{m}, y_0 = \frac{M_{xz}}{m}, z_0 = \frac{M_{xy}}{m}$$

(S) sirtning og'irlik markasining koordinatalari;

$$g) J_x = \iint_{(S)} (y^2 + z^2) \rho(x, y, z) ds$$

(S) sirtning Ox o'qiga nisbatan inersiya momenti ;

$$d) J_{yz} = \iint_{(S)} x^2 \rho(x, y, z) ds$$

(S) sirtning yOz tekisligiga nisbatan inersiya momenti;

$$e) J_0 = \iint_{(S)} (x^2 + y^2 + z^2) \rho(x, y, z) ds$$

(S) sirtning koordinat boshiga nisbatan momenti;

j) $\vec{F} = \{F_x, F_y, F_z\}$ – massasi m_0 ga teng bo'lgan material $M_0(x_0, y_0, z_0)$ nuqtadagi material sirtning tortish kuchi, bunda

$$F_x = \gamma m_0 \iint_{(S)} \frac{x - x_0}{r^3} \rho(x, y, z) ds,$$

$$F_y = \gamma m_0 \iint_{(S)} \frac{y - y_0}{r^3} \rho(x, y, z) ds,$$

$$F_z = \gamma m_0 \iint_{(S)} \frac{z - z_0}{r^3} \rho(x, y, z) ds,$$

$\vec{r} = \{x - x_0, y - y_0, z - z_0\}$, $r = |\vec{r}|$, v – gravitatsion o'zgarmasi.

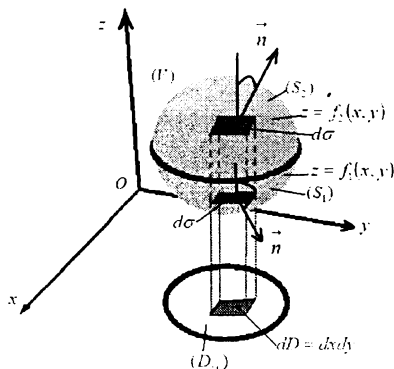
70-§. Ikkinchi tur sirt integrali va uning fizikaga qo'llanishi

15.7. Ikkinchi tur sirt integrali va uning ta'rif. Ikki

tomonli siliq yoki bo'lakli siliq (S) sirt berilgan bo'lib, uning ikki tomonidan birini belgilaymiz. Faraz qilaylik (S) sirt oshkor tenglamasi

$$z = z(x, y)$$

bilan berilgan bo'lsin. Bu sirtida $f(x, y, z)$ funksiya aniqlangan. Sirtning ixtiyoriy P bo'linishini va bo'lishning har bir (S_k) bo'lagidan ($k = 1, 2, \dots, n$) ixtiyoriy $M_k(\xi_k, \eta_k, \zeta_k)$ nuqtani olib, bu nuqtadagi berilgan funksiyaning $f(\xi_k, \eta_k, \zeta_k)$ qiymatni (S_k) ning xy tekislikdagi preksiyasi (D_k) ning yuzi D_k ga ko'paytirib ushbu



15.5-chizma.

$$\sigma = \sum_{k=1}^n f(\xi_k, \eta_k, \zeta_k) D_k$$

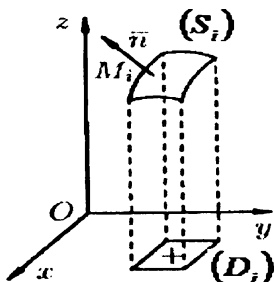
yig'indini tuzamiz. (S) sirtning shunday

$$P_1, P_2, \dots, P_m, \dots \quad (15.7.1)$$

bo'linishlar ketma – ketligini qaraylikki, ularning mos diametrlaridan tuzilgan $\lambda_{p1}, \lambda_{p2}, \dots, \lambda_{pn}, \dots$ ketma–ketlik nolga intilsin, ya'ni $\lambda_{pn} \rightarrow 0$. (15.7.1) ko'rinishdagi bo'linishlarga mos kelgan $f(x, y, z)$ funksiyaning integral yig'indilar ketma- ketligi

$$\sigma_1, \sigma_2, \dots, \sigma_m, \dots \quad (15.7.2)$$

ni tuzamiz,



15.6-chizma.

15.7.3. Ta'rif. Agar (s) sirtning har qanday (15.7.1) ko'rinishdagi bo'linishlar ketma – ketli $\{p_n\}$ olinganda ham, unga mos kelgan (15.7.2) integral yig'indilar $\{\sigma_n\}$ ketma – ketligi $M_i(\xi_k, \eta_k, \zeta_k)$ nuqtani tanlashga bog'liq bo'lmagan holda hamma vaqt bita chekli J limitga intilsa, u holda bu limit $f(x, y, z)$ ning *ikkinchi tur sirt integrali* (belgilangan tomonni bo'yicha) deyiladi va u

$$J = \iint_{(S)} f(x, y, z) dx dy$$

kabi belgilanadi. Bu holda $f(x, y, z)$ funksiya (S) sirtida integrallanuvchi deb ataladi.

Demak,

$$\lim_{\lambda_p \rightarrow 0} \sigma = \lim_{\lambda_p \rightarrow 0} \sum_{k=1}^n f(\xi_k, \eta_k, \zeta_k) D_k = \iint_{(S)} f(x, y, z) dx dy.$$

Ravshanki, $f(x, y, z)$ funksiyaning (s) sirtning bir tomoni bo'yicha olingan ikkinchi tur sirt integrali, sirtning ikkinchi tomoni bo'yicha olingan ikkinchi tur sirt integralidan faqat ishoraga farq qiladi. (s) sirtida aniqlangan $p(x, y, z)[Q(x, y, z)]$ funksiylarning sirtning belgilangan tomoni bo'yicha y va z (z va x) o'zgaruvchilar bo'yicha sirt integralining ta'rifi ham xuddi yuqoridagi singari beriladi:

$$\iint_{(S)} P(x, y, z) dy dz \quad \left(\iint_{(S)} Q(x, y, z) dz dx \right).$$

Umumiy holda, (S) sirtida $P(x, y, z), Q(x, y, z), R(x, y, z)$ funksiyalar berilgan bo'lib,

$$\iint_{(S)} P(x, y, z) dydz, \iint_{(S)} Q(x, y, z) dzdx, \iint_{(S)} R(x, y, z) dxdy$$

integrallar mavjud bo'lsa,

$$\iint_{(S)} P(x, y, z) dydz + \iint_{(S)} Q(x, y, z) dzdx + \iint_{(S)} R(x, y, z) dxdy$$

yig'indi umumiy holda ikkinchi tur sirt integrali deyiladi va u

$$\iint_{(S)} P(x, y, z) dydz + Q(x, y, z) dzdx + R(x, y, z) dxdy$$

kabi belgilanadi.

15.8. Ikkinchi tur sirt integralining mavjudlik sharti va uni karrali integralga keltirish. 1. (S) sirt o'zining oshkor ko'rinishdagi $z(x, y), (x, y) \in (D)$ tenglamasi bilan berilgan bo'lib, $z(x, y)$ funksiya chegaralangan yopiq (D) sohada uzluksiz va uzluksiz $z'_x(x, y), z'_y(x, y)$ xususiy hosilalarga ega bo'lsin.

15.8.1. Teorema. Agar $f(x, y, z)$ funksiya (S) sirtida berilgan va uzluksiz bo'lsa, u holda bu funksiya (S) sirt bo'yicha olingan ikkinchi tur sirt integrali mavjud bo'ladi va u ushbu

$$\iint_{(S)} f(x, y, z) dxdy = \iint_{(D)} f(x, y, z(x, y)) dxdy$$

formula bilan hisoblanadi.

Isboti. (S) sirtning ixtiyoriy P bo'linishi qaraylik. $(S_1), (S_2), \dots, (S_n)$. Bu sirt va uning bo'laklarining xOy tekislikdagi proyeksiyalari mos ravishda (D) va $(D_1), \dots, (D_n)$ lardan iborat bo'lsin. P bo'linishga nisbatan integral yig'indini tuzamiz:

$$\sigma = \sum_{k=1}^n f(\xi_k, \eta_k, \zeta_k) \cdot D_k \quad (15.8.2)$$

Agar (S) sirtning ustki tomoni bo'yicha qarayotgan bo'lsa, u holda barcha D_k lar musbat bo'ladi. $f(x, y, z)$ funksiya $z = z(x, y)$ sirtida berilgan bo'lganligi uchun

$$f(x, y, z) = f(x, y, z(x, y))$$

bo'ladi. Bundan, $\zeta_k = z(\xi_k, \eta_k) (k = 1, 2, \dots, n)$ bo'lishi kelib chiqadi.

Natijada, (15.8.2) integral yig'ini ushbu

$$\sigma = \sum_{k=1}^n f(\xi_k, \eta_k, z(\xi_k, \eta_k)) \cdot D_k$$

ko'rinishga keladi. Bu yig'ini $f(x, y, z)$ funksiyani ikkikarali integrali uchun integral yig'ini ekanligini ko'rish qiyin emas. Shartga ko'ra $f(x, y, z(x, y))$ funksiya (D) da uzluksiz bo'lganligi uchun

$$\lim_{\lambda_P \rightarrow 0} \sum_{k=1}^n f(\xi_k, \eta_k, z(\xi_k, \eta_k)) D_k$$

limit mavjud bo'ladi va ushbu

$$\lim_{\lambda_P \rightarrow 0} \sum_{k=1}^n f(x_k, y_k, z(\xi_k, \eta_k)) D_k = \iint_{(D)} f(x, y, z(x, y)) dx dy$$

tenglik o'rinli bo'ladi, ya'ni

$$\iint_{(S)} f(x, y, z) dx dy = \iint_{(D)} f(x, y, z(x, y)) dx dy \quad (15.8.3)$$

shu bilan teorema isbot bo'ladi. ■

Agar (S) sirtning pastki tomoni qaralayotgan bo'lsa, u holda har bir D_k lar manfiy bo'lib,

$$\iint_{(S)} f(x, y, z) dx dy = - \iint_{(D)} f(x, y, z(x, y)) dx dy$$

tenglik o'rinli bo'ladi.

Xuddi shuningdek, tegishli shartlarda ushbu

$$\iint_{(S)} f(x, y, z) dy dz, \quad \iint_{(S)} f(x, y, z) dz dx$$

integrallar ham mavjud bo'ladi va

$$\iint_{(S)} f(x, y, z) dy dz = \iint_{(D_1)} f((y, z), y, z) dy dz, \quad (15.8.4)$$

$$\iint_{(S)} f(x, y, z) dz dx = \iint_{(D_2)} f(x, y(z, x), z) dz dx \quad (15.8.5)$$

tengliklar o'rinli bo'ladi, bunda (D_1) va (D_2) lar $x = x(y, z)$ va $y = y(z, x)$ sirtlarning mos ravishda yOz va zOx tekisliklardagi proyeksiyalari.

15.8.6. Eslatma. Agar $f(x, y, z)$ funksiya yasovchilari Oz o'qiga parallel bo'lgan (s) - silindrik sirtida berilgan bo'lsa, u holda $\iint_{(s)} f(x, y, z) dx dy = 0$ bo'ladi.

Xuddi shunday tegishli shartlarda

$$\iint_{(S)} f(x, y, z) dy dz = 0, \iint_{(S)} f(x, y, z) dz dx = 0$$

bo'ladi.

15.8.7. Misol. Ushbu

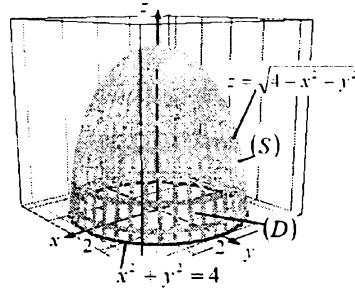
$$I = \iint_{(S)} (x+2)z^2 dx dy$$

sirt integralini hisoblang, bu yerda (S)

$$x^2 + y^2 + z^2 = 4 \quad (z \geq 0) \quad \text{yarim}$$

sferaning yuqori tashqi tomoni.

Yechilishi. Berilgan yarim sferaning xOy tekislikdagi proyeksiyasi $x^2 + y^2 = 4$ aylana bilan chegaralangan (D) doiradan iborat (15.7- chizma.).



15.7- chizma.

(S) yarim sferaning tenglamasini $z = \sqrt{4 - x^2 - y^2}$ ko'rinishda yozish mumkin. Endi ikkinchi tur sirt integralini (15.8.3) formulaga asosan, ikki karrali integralga keltirib hisoblaymiz:

$$\begin{aligned} I &= \iint_{(S)} (x+2)z^2 dx dy = \iint_{(D)} (x+2)(4-x^2-y^2) dx dy = \\ &= \int_0^2 dr \int_0^{2\pi} (4-r^2)(r \cos \varphi + 2) d\varphi = 4\pi \int_0^2 (4r-r^3) dr = 16\pi. \end{aligned}$$

Misolni Maple tizimidan foydalanib, yechish:

```
> int(int((4-r^2)*r*(r*cos(x)+2),x=0..2*Pi),r=0..2);
```

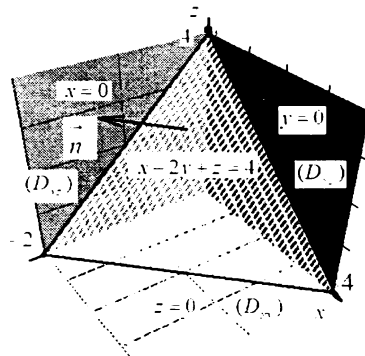
16 π

15.8.8. Misol. Ushbu

$$I = \iint_{(S)} x dy dz - z dz dx + 6 dx dy$$

ikkinchi tur sirt integralini hisoblang, bu yerda (S) - ushbu $x-2y+z=4$, $x=0$, $y=0$, $z=0$, $z=0$ tekisliklar bilan chegaralangan piramidaning yuqori tashqi sirti.

Yechilishi. Tekislikning joylashishi 15.8- chizmadan ko'rinib turibdi. Tekislikning $\vec{n}$ normal vektori Oy o'q bilan o'tmas burchak, Ox , Oz o'qlar bilan esa, o'tkir burchaklar



15.8- chizma.

tashkil yetadi. Bu tekislikning $\vec{n} = \{1, -2, 1\}$ normal vektorining yo'naltiruvchi kosinuslari quyidagicha topiladi:

$$|\vec{n}| = 2 = \sqrt{6}, \cos \alpha = \frac{1}{\sqrt{6}} > 0, \cos \beta = \frac{-2}{\sqrt{6}} < 0, \cos \gamma = \frac{1}{\sqrt{6}} > 0.$$

Endi ikkinchi tur sirt integralini (15.8.3)- (15.8.5) formulalarga asosan, ikki karrali integralga keltirib hisoblaymiz, bunda birinchi, uchinchi integrallarni "plyus", ikkinchi integralni "minus" ishora bilan yozamiz:

$$\begin{aligned} I &= \iint_{(D_4)} (2y - z + 4) dy dz + \iint_{(D_1)} z dz dx + \iint_{(D_2)} 6 dx dy = \\ &= \int_{-2}^0 dy \int_0^{4-2y} (2y - z + 4) dz + \int_0^4 dx \int_0^{4-x} z dz + 6 \cdot \frac{1}{2} \cdot 2 \cdot 4 = \\ &= \frac{-8}{3} + \frac{32}{3} + 24 = 32. \end{aligned}$$

15.9. Birinchi va ikkinchi tur sirt integrallari orasidagi bog'lanish. Biz yuqorida ko'rdikki (S) sirtida $f(x, y, z)$, $P(x, y, z)$, $Q(x, y, z)$ va $R(x, y, z)$ funksiyalar tegishli shartlarni qanoatlantirganda

$$\begin{aligned} \iint_{(S)} f(x, y, z) \cos \alpha ds &= \iint_{(D_1)} f(x(y, z), y, z) dy dz, \\ \iint_{(S)} f(x, y, z) \cos \beta ds &= \iint_{(D_2)} f(x, y(z, x), z) dz dx, \\ \iint_{(S)} f(x, y, z) \cos \gamma ds &= \iint_{(D_3)} f(x, y, z(x, y)) dx dy. \end{aligned}$$

formula o'rinli bo'ladi.

Umumiy holda (15.8.3), (15.8.4), (15.8.5) formulalarni e'tiborga olgan holda

$$\begin{aligned} \iint_{(S)} [P(x, y, z) \cos \alpha + Q(x, y, z) \cos \beta + R(x, y, z) \cos \gamma] ds = \\ = \iint_{(S)} P(x, y, z) dy dz + Q(x, y, z) dz dx + R(x, y, z) dx dy \end{aligned} \quad (15.9.1)$$

formulaning o'rinli bo'lishiga ishonch hosil qilamiz. (15.9.1) formulaga birinchi tur sirt integrali bilan ikkinchi tur sirt integrali orasidagi bog'lanishni beruvchi formula deb yuritiladi.

Ma'lumki, (S) sirt umumiy holda o'zining parametrik tenglamasi $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ ($(u, v) \in (\Delta)$) bilan berilganda, ikkinchi tur sirt integrali

$$\iint_{(S)} P dy dz + Q dz dx + R dx dy = \iint_{(\Delta)} (AP + BQ + CR) du dv \quad (15.9.2)$$

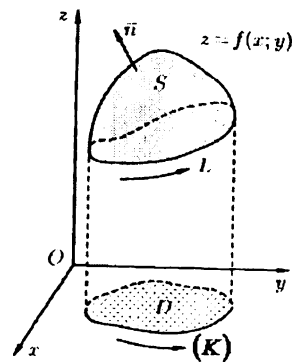
formula bilan hisoblanadi. Bu tenglamaning o'ng tomonida P, Q, R funksiyalarda x, y, z larning o'rniga $x(u, v)$, $y(u, v)$, $z(u, v)$ lar yozilgan.

15.9.3. Eslatma. Yuqorida tasdiqlar bo'lakli silliq sirtlar uchun ham o'rinli bo'ladi.

71-§. Stoks va Ostrogradskiy formulalari

15.10. Stoks formulasi. (S) – silliq sirt o'zining oshkor $z = z(x, y)$ tenglamasi bilan berilgan bo'lib, uning chegarasi (L) –bo'lakli silliq yopiq chiziqdan iborat bo'lsin. (S) sirtning Oy tekislikdagi proyeksiyasi (D) bo'lib, (L) egri chiziqning proyeksiyasi esa, (K) bo'lakli silliq yopiq chiziqdan iborat bo'lsin.

(S) sirtida uzluksiz va uzluksiz $P'_x(x, y, z)$, $P'_y(x, y, z)$ xususiy hosilalar ega bo'lgan $P(x, y, z)$ funksiya berilgan bo'lsin. (S) sirtning tashqi tomonini belgilaymiz (15.9-chizma) Yuqoridagi (S) sirtga qo'yilgan



15.9-chizma.

shartlarda quyidagi teorema o'rinli bo'ladi.

15.10.1. Teorema. Agar $P(x, y, z)$ funksiya (S) da uzluksiz va uzluksiz birinchi tartibli xususiy hosilalarga ega bo'lsa, u holda quyidagi formula o'rinli bo'ladi:

$$\int_{(L)} P(x, y, z) dx = \iint_{(S)} \frac{\partial P}{\partial z} dx dz - \frac{\partial P}{\partial y} dx dy.$$

Isbot. Shartga ko'ra $\int_{(L)} P(x, y, z) dx$ – egri chiziqli integral mavjud

Avvalo egri chiziq bo'yicha olingan egri chiziqli integralni (L) ning tekislikdagi proyeksiyasi (K) egri chiziq bo'yicha olingan egri chiziqli integralga aylantiramiz:

$$\int_{(L)} P dx = \int_{(K)} P \cdot \left(\frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv \right). \quad (15.10.2)$$

(15.10.2) tenglikning to'g'riligini ko'rsatish uchun (K) egri chiziqning parametrik tenglamasi:

$$u = u(t), v = v(t) \quad (\alpha \leq t \leq \beta)$$

dan foydalanamiz: (K) egri chiziq tenglamasiga ko'ra (L) egri chiziq tenglamasi $x = x(u(t), v(t))$, $y = y(u(t), v(t))$, $z = z(u(t), v(t))$ ko'rinishda bo'ladi. U holda ikala integral ham bir xil t bo'yicha olingan oddiy

$$\int_{\alpha}^{\beta} P \cdot \left(\frac{\partial x}{\partial u} \cdot \frac{du}{dt} + \frac{\partial x}{\partial v} \cdot \frac{dv}{dt} \right) dt$$

integralga keltiriladi. Endi (15.10.2) integralning o'ng tomoniga Grii formulasini qo'llaymiz:

$$\int_{(K)} P \cdot \left(\frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv \right) = \iint_{(\Delta)} \left\{ \frac{\partial}{\partial u} \left(P \frac{\partial x}{\partial v} \right) - \frac{\partial}{\partial v} \left(P \frac{\partial x}{\partial u} \right) \right\} du dv. \quad (15.10.3)$$

Keyingi tenglikning o'ng tomonidagi integral ostidagi {.....} ifodaning to'liq kengaytirilgan ko'rinishini yozamiz:

$$\begin{aligned} \{.....\} &= \left(\frac{\partial P}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial P}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial P}{\partial z} \frac{\partial z}{\partial u} \right) \cdot \frac{\partial x}{\partial v} + P \frac{\partial^2 x}{\partial u \partial v} - \\ &- \left(\frac{\partial P}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial P}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial P}{\partial z} \frac{\partial z}{\partial v} \right) \cdot \frac{\partial x}{\partial u} - P \frac{\partial^2 x}{\partial u \partial v} = \\ &= \frac{\partial P}{\partial z} \left(\frac{\partial z}{\partial u} \frac{\partial x}{\partial v} - \frac{\partial z}{\partial v} \frac{\partial x}{\partial u} \right) - \frac{\partial P}{\partial y} \left(\frac{\partial y}{\partial u} \frac{\partial x}{\partial v} - \frac{\partial y}{\partial v} \frac{\partial x}{\partial u} \right) \end{aligned}$$

(15.1.5) formulani e'tiborga olgan holda

$$\left\{ \dots \right\} = \frac{\partial P}{\partial z} B - \frac{\partial P}{\partial y} C$$

ni hosil qilamiz. U holda (15.10.3) ning o'ng tomonidagi integral

$$\iint_{(\Delta)} \left\{ \frac{\partial P}{\partial z} B - \frac{\partial P}{\partial y} C \right\} du dv$$

to'rinishga keladi. Bu integralni (15.9.2) formula yordamida sirtning belgilangan tomoni bo'yicha sirt integraliga keltiramiz:

$$\iint_{(\Delta)} \left\{ \frac{\partial P}{\partial z} B - \frac{\partial P}{\partial y} C \right\} du dv = \iint_{(S)} \frac{\partial P}{\partial z} dz dx - \frac{\partial P}{\partial y} dx dy$$

shu bilan teorema isbot bo'ladi.

Xuddi shunday (s) sirtida berilgan $Q(x, y, z)$, $R(x, y, z)$ funksiyalar tegishli shartlarni qanoatlantirganda ushbu

$$\int_{(L)} Q(x, y, z) dy = \iint_{(s)} \frac{\partial Q(x, y, z)}{\partial x} dx dy - \frac{\partial Q(x, y, z)}{\partial z} dy dz, \quad (15.10.4)$$

$$\int_{(L)} R(x, y, z) dz = \iint_{(s)} \frac{\partial R(x, y, z)}{\partial y} dy dz - \frac{\partial R(x, y, z)}{\partial x} dz dx \quad (15.10.5)$$

formulalarning o'rinli ekanligini ko'rsatish mumkin. (15.10.2), (15.10.4) va (15.10.5) formulalar hadlab qo'shish natijasida

$$\begin{aligned} \int_{(L)} P dx + Q dy + R dz &= \iint_{(s)} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy + \\ &+ \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz dx \end{aligned} \quad (15.10.6)$$

formulani hosil qilamiz. Odatda (15.10.6) formula *Stoks formulasi* deb ataladi.

Shunday qilib, Stoks formulasi fazoviy yopiq (L) egri chiziq bo'yicha olingan egri chiziqli integralni shu yopiq chiziqdan bilan chegaralangan (s) sirt bo'yicha olingan sirt integralini bog'lovchi formula bo'lib hisoblanar ekan. (15.10.6) formulani birinchi tur sirt integrali orqali ham yozish mumkin:

$$\begin{aligned} &\int_{(L)} P dx + Q dy + R dz = \\ &= \iint_{(s)} \left[\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \cos \alpha + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \cos \beta + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \cos \gamma \right] ds \end{aligned} \quad (15.10.7)$$

(15.10.7) formulani ham *Stoks formulasi* deyiladi.

15.10.8. Eslatma. Grik formulasi, Stoks formulasining xususiy holidir. Haqiqatan ham (15.10.6) formulada (s) sirt sifatida oxy tekisligidagi (D) soha olinsa, unda $z = 0$ bo'lib, (15.10.6) formuladan

$$\int_{(\kappa)} Pdx + Qdy = \iint_{(D)} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

Grin formulasi kelib chiqadi.

15.11. Stoks formulasining fazodagi egri chiziqli integrallarni hisoblashga qo'llanishi. Ochiq (V) sohada P, Q, R uzluksiz

funksiyalar berilgan bo'lib, ular uzluksiz $\frac{\partial Q}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial R}{\partial z}, \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z}, \frac{\partial R}{\partial x}$ xususiy hosilalarga ega bo'lsin. Stoks formulasi yordamida fazodagi egri chiziqli $\int_{(L)} Pdx + Qdy + Rdz$ integralning nolga teng bo'lishi uchun yetarli

va zaruriy shartni keltirib chiqarish mumkin (bunda (L), (L') sohada yotuvchi sodda chiziq):

Agar

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}; \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}; \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \quad (15.11.1)$$

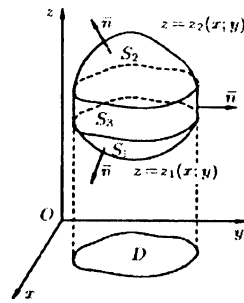
shartlar bajarilganda, Stoks formulasidan fazoviy yopiq (L) egri chiziq bo'yicha olingan egri chiziqli integralning nolga teng bo'lishi kelib chiqadi:

$$\int_{(L)} Pdx + Qdy + Rdz = 0 \quad (15.11.2)$$

Bu holda, egri chiziqli integral integrallash yo'liga bog'liq bo'lmasligi kelib chiqadi. Tekislikdagi singari (15.11.1) shart, (15.11.2) tenglikning bajarilishi uchun zaruriy va yetarli bo'lib hisoblanadi. (15.11.1) yoki (15.11.2) shartlarining bajarilishi integral ostidagi $Pdx + Qdy + Rdz$ ifodaning biror $U(x, y, z)$ funksiyaning to'liq differensial: $du = Pdx + Qdy + Rdz$ bo'lishi uchun zarur va yetarli shart bo'lib hisoblanadi

15.12. Ostroradskiy formulasi.

Yuqoridan $(S_1): z = Z(x, y)$, pastdan $(S_2): z = Z_0(x, y)$ - silliq sirtlar bilan, yon tomondan esa, yasovchilari Oz o'qiga parallel bo'lgan silindrik (S_3) sirt bilan chegaralangan (V) sohani qaraymiz. Uning xOy tekislikdagi proyeksiyasi (D) ning chegarasi bo'lakli – silliq (K) yopiq chiziq (nol yuzga ega) bo'lib, u silindrik sirtning yo'naltiruvchisi sifatida olinadi.



15.10-chizma.

(15.10-chizma) $(z_0 \leq Z)(V)$ sohada biror $R(x, y, z)$ funksiya aniqlangan bo'lib u (V) sohada va uning chegarasida uzluksiz va uzluksiz $\frac{\partial R}{\partial z}$ xususiy hosilaga ega bo'lsin. Shu shartlarda

$$\iiint_{(V)} \frac{\partial R}{\partial z} dx dy dz = \iint_{(S)} R dx dy \quad (15.12.1)$$

formula o'rinli bo'ladi, bunda $(S)-(V)$ sohani chegarasi, o'ng tomonidagi integral (S) integralning tashqi tomoni bo'yicha olinagan.

Isboti. Bu holda $\iiint_{(V)} \frac{\partial R}{\partial z} dx dy dz$ - integral mavjud bo'ladi va u

(13.19.2) formulaga ko'ra quyidagicha hisoblanadi:

$$\begin{aligned} \iiint_{(V)} \frac{\partial R}{\partial z} dx dy dz &= \iint_{(D)} dx dy \int_{z_0(x,y)}^{z(x,y)} \frac{\partial R}{\partial z} dz = \iint_{(D)} R(x, y, Z(x, y)) dx dy - \\ &- \iint_{(D)} R(x, y, z_0(x, y)) dx dy \end{aligned}$$

Tenglikning o'ng tomonidagi ikki karali integrallarni (15.8.3) formuladan foydalanib ikkinchi tur sirt integraliga keltiramiz: (15.12.1) ning o'ng tomonidagi integral (S) ning tashqi tomoni bo'yicha olinganligi e'tiborga olib,

$$\begin{aligned} \iiint_{(V)} \frac{\partial R}{\partial z} dx dy dz &= \iint_{(S_1)} R(x, y, z) dx dy - \iint_{(S_2)} R(x, y, z) dx dy = \\ &= \iint_{(S_1)} R(x, y, z) dx dy + \iint_{(S_2)} R(x, y, z) dx dy \end{aligned} \quad (15.12.2)$$

(S_1) sirt yasovchilari Oz o'qiga parallel bo'lgan silindrik, sirt bo'lganligi uchun

$$\iint_{(S_1)} R(x, y, z) dx dy = 0 \quad (15.12.3)$$

bo'ladi. (15.12.2) ning o'ng tomoniga (15.12.3) ni qo'shib

$$\iiint_{(V)} \frac{\partial R}{\partial z} dx dy dz = \iint_{(S)} R(x, y, z) dx dy$$

ni hosil qilamiz, bunda $(S)-(V)$ jismni o'rab turuvchi sirt: $(S) = (S_1) + (S_2) + (S_3)$

Xuddi shunday yo'l bilan (V) sohada aniqlangan $P(x, y, z), Q(x, y, z)$ funksiyalar $R(x, y, z)$ funksiya singari shartlarni qanoatlantirganda quyidagi

$$\iiint_{(V)} \frac{\partial P}{\partial x} dx dy dz = \iint_{(S)} P dy dz \quad (15.12.4)$$

$$\iiint_{(V)} \frac{\partial Q}{\partial y} dx dy dz = \iint_{(S)} Q dz dx \quad (15.12.5)$$

formulaning o'rinli ekanligini ko'rsatish mumkin. (15.12.1), (15.12.4) (15.12.5) formulalarni hadlab qo'shish natijasida Ostrogradskiy formulasi deb ataluvchi ushbu

$$\iiint_{(V)} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx dy dz = \iint_{(S)} P dy dz + Q dz dx + R dx dy \quad (15.12.6)$$

formulani hosil qilamiz. (15.12.6) ning o'ng tomonidagi ikkinchi tur sirt integralini birinchi tur sirt integraliga aylantirish natijasida uning ko'rinishini

$$\iiint_{(V)} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx dy dz = \iint_{(S)} (P \cos \alpha + Q \cos \beta + R \cos \gamma) ds \quad (15.12.7)$$

shaklida ham yozish mumkin. (15.12.7) formulaga ham Ostrogradskiy formulasi deyiladi, bunda $\cos \alpha, \cos \beta, \cos \gamma$ lar (S) sirtga o'tkazilgan tashqi normalning mos ravishda koordinata o'qlari bilan tashkil qilgan burchaklari. Ostrogradskiy formulasi (V) fazoviy soha bo'yicha olingan uchkarali integralni (V) sohaning chegarasi (S) sirt bo'yicha olingan sirt integrali bilan bog'laydigan formuladir.

15.13. Ostrogradskiy formulasining qo'llanishi. (15.12.6) formulada P, Q, R funksiyalarni, shunday tanlash kerakki, natijada uch karali integral ostidagi ifoda birga teng bo'lsin, ya'ni (V) jismning hajmi $V(V)$ jismni o'rab turuvchi (S) sirt bo'yicha olingan sirt integrali bilan ifodalansin:

$$V = \iiint_{(V)} P dydz + Q dzdx + R dx dy,$$

bunda

$$1) P = x, Q = 0, R = 0; 2) P = 0, Q = y, R = 0; 3) P = 0, Q = 0, R = z$$

deb tanlasak, natijada

$$V = \iiint_{(S)} x dydz = \iiint_{(S)} y dzdx = \iiint_{(S)} z dx dy$$

$$4) P = \frac{1}{3}x, Q = \frac{1}{3}y, R = \frac{1}{3}z$$

deb tanlasak

$$V = \frac{1}{3} \iiint_{(S)} x dydz + y dzdx + z dx dy$$

yoki

$$V = \frac{1}{3} \iiint_{(S)} (x \cos \alpha + y \cos \beta + z \cos \gamma) ds$$

(bunda $\cos \alpha, \cos \beta, \cos \gamma$ lar (S) sirtga o'tkazilgan tashqi $\vec{n}$ normalning yo'naltiruvchi kosinuslari)

72-§. Maydonlar nazariyasi elementlari

15.15. Skalyar va vektorli maydon. Egri chiziqli va sirt integrallarining fizik ma'nosini berish uchun maydonlar nazariyasining elementlarini qisqacha berib o'tamiz.

Skalyar maydon. Fazoda (yoki tekislikda) biror (G) soha berilgan bo'lsin. Agar har bir $M \in (G)$ nuqtaga biror qoidaga asosan $U(M)$ son mos quyilgan bo'lsa, u holda (G) sohada skalyar maydon aniqlangan deyiladi. Skalyar maydonga misol sifatida temperatura, massa, zichlik hajm va hakazolarni olish mumkin.

Agar (G) sohada koordinatalar sistemasi o'rnatilgan bo'lsa, u holda (G) sohada u skalyar maydonning berilishi $u(x, y, z)$ – skalyar funksiyaning berilishiga ekvivalent bo'ladi. Biz bundan keyin $u(x, y, z)$ funksiya hamma argumentlari bo'yicha uzluksiz hosilalarga ega deb faraz qilamiz. Agar bu xosilalar birdaniga nolga teng bo'lmasa, u holda

$$u(x, y, z) = C \quad (C = \text{const})$$

tenglama maxsus nuqtaga ega bo'lmagan biror sirtni ifodalaydi. Bu sirt bo'ylab $u(M)$ funksiya o'zgarmas qiymat qabul qiladi va uni *sirt sathi* (*chizig'i*) deb ataladi C ga har xil o'zgarmas qiymatlar qo'yish bilan berilgan skalyar maydonida sirtlar (chiziqlar) oilasini hosil qilamiz. (G) soha bu sirtlar bilan to'lg'azilgan bo'ladi, ya'ni (G) sohaning har bir nuqtasidan bitta va faqat bitta sirt sathi (*chizig'i*) o'tadi. Ravshanki, sirt sathlari (*chiziqlari*) o'zaro kesishmaydi.

Skalyar maydon koordinatalar sistemasini tanlashga bog'liq bo'lmaydi. U kattalik faqat M nuqtaning funksiya bo'ladi, u vaqtga bog'liq bo'lishi ham mumkin. Agar u t vaqtga bog'liq bo'lsa, u holda *statsionar bo'lmagan maydon* deyiladi.

Agar fazoda to'g'ri burchakli $Ox_1x_2x_3$ koordinatalar sistemasi o'rnatilgan, bo'lsa, u holda skalyar maydon $u = u(x, y, z) \quad ((x, y, z) \in (G))$ funksiya bilan ifoda qilinadi.

Vektorli maydon. Agar har bir $M \in (G)$ nuqtaga biror qoidaga ko'ra $\vec{a}(M)$ vektor mos qo'yilgan bo'lsa, u holda (G) sohada *vektorli maydon aniqlangan* deyiladi. Vektorli maydonga misol sifatida: elektr zaryadlar maydonini, kuchlar maydonini, tezliklar va tezlanishlar maydonini va hakazolarni olish mumkin. Agar (G) sohada koordinatalar sistemasi, o'rnatilgan bo'lsa, u holda $\vec{a}(M)$ vektorli maydonning berilishi $\vec{a}(M)$ vektorning mos o'qlardagi proyeksiyalari

$$a_1(x, y, z), a_2(x, y, z), a_3(x, y, z)$$

ning berilishiga ekvivalent bo'ladi.

Bundan keyin bu funksiyalarni uzluksiz xususiy hosilalarga ega deb faraz qilamiz. Vektorli maydonlar nazariyasini o'rganishda vektor *chizig'i* tushunchasi muhim rol o'ynaydi. Chiziqning har bir M

nuqtasiga o'tkazilgan o'rinmaning yo'nalishi, shu nuqtaga mos qo'yilgan $\vec{a}(M)$ vektoring yo'nalishiga mos tushsa, bunday chiziqqa vektor chizig'i deyiladi. M_0 nuqtadan o'tuvchi vektor chizig'ining tenglamasi $\vec{r} = \vec{r}(t)$, (t - parametr) ko'rinishda bo'lsin. Vektor chizig'ning ixtiyoriy nuqtasidagi $\vec{a}(M)$ vektor bilan o'rinma vektor $\vec{r}$ ning koleniarlik sharti

$$\frac{d\vec{r}}{dt} = \lambda \vec{a} \quad (15.15.1)$$

ko'rinishda bo'ladi, bunda λ - biror son.

$$\left[\frac{d\vec{r}}{dt} \cdot \vec{a} \right] = 0 \quad (15.15.2)$$

(15.15.2) ning ikkala tomonini dt ga ko'paytirish bilan uni ushbu

$$[d\vec{r} \cdot \vec{a}] = 0 \quad (15.15.3)$$

ko'rinishda yozish mumkin. (15.15.1)-(15.15.3) tenglamalarning har biri vektor chizig'ining vektor ko'rinishidagi tenglamasini ifoda qilib, u vektor chiziqlar oilasini (to'plamini) ifoda qiladi. Berilgan M_0 nuqtadan o'tuvchi aniq vektor chizig'i qo'shimcha

$$\vec{r}(t_0) = \vec{r}_0 \quad (15.15.4)$$

($\vec{r}_0$ - M nuqtaning radius vektori) shart bilan aniqlanadi.

Vektorli maydon koordinatlar sistemasini tanlashga bog'liq emas.

Har bir M nuqtadagi $\vec{a}(M)$ vektor o'zining uzunligi va yo'nalishi bilan to'liq aniqlanadi. Agar fazoda to'g'ri burchakli $Oxyz$ sistema o'rnatilgan bo'lsa, u holda $\vec{a}(M)$ vektorli maydon uchta o'zgaruvchili vektor $\vec{a}(x, y, z)$ funksiya bilan yoki uchta skalyar $P(x, y, z)$, $Q(x, y, z)$, $R(x, y, z)$ funksiyalar bilan ifoda qilinadi:

$$\vec{a}(x, y, z) = \{P(x, y, z), Q(x, y, z), R(x, y, z)\}, (x, y, z) \in (G).$$

To'g'ri burchakli koordinat sistemasida $d\vec{r} = \{dx, dy, dz\}$ bo'lganligi uchun vektor chizig'ining (15.15.3) ko'rinishidagi vektor tenglamasi ushbu

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

differensial tenglamalar sistemasiga ekvivalent bo'ladi. (15.15.4) qo'shimcha shart quyidagi shartlarga ekvivalent

$$x(t_0) = x_0, y(t_0) = y_0, z(t_0) = z_0,$$

bunda $x_0, y_0, z_0 - M_0$ nuqtaning koordinatalari.

15.15. Potensial maydon.

15.15.1. Ta'rif. Agar (G) sohada aniqlangan $\vec{a}(M)$ vektorni shu sohada berilgan biror skalyar maydoning gradiyenti:

$$\vec{a} = \text{gradu} \quad (15.15.2)$$

shaklida ifodalash mumkin bo'lsa, u holda $\vec{a}(M)$ vektorli maydonga *potensial maydon* deyiladi.

$u(M)$ funksiyaga $\vec{a}(M)$ vektorli maydoning potentsiali deyiladi.

Agar $\vec{a} = \{P, Q, R\}$ bo'lsa, u holda (15.15.2) dan

$$P = \frac{\partial u}{\partial x}, Q = \frac{\partial u}{\partial y}, R = \frac{\partial u}{\partial z} \quad (15.15.3)$$

ekanligi kelib chiqadi. Shunday qilib $\vec{a}(M)$ vektorli maydon potensial maydon bo'lishi uchun shunday $u(M)$ funksiya mavjud bo'lib, u (15.15.3) shartli qanoatlantirishi kerak ekan.

Biz yuqoridagi 15.11-bandda ko'rdikki, agar P, Q, R – uzluksiz va uzluksiz birinchi tartibli xususiy hosilalarga ega bo'lsa, u holda $Pdx + Qdy + Rdz$ ifoda biror $u(x, y, z)$ funksiyaning to'liq differensialini ifodalashi uchun

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z} \quad (15.15.4)$$

shartning bajarilishi kerak edi.

Agar $Pdx + Qdy + Rdz = du$ bo'lsa, u holda (15.15.3) ham o'rinli bo'ladi, ya'ni (15.15.4) shart berilgan vektorli maydonning potensial

maydon bo'lishlilik shartini ifodalaydi. Bu holda $u(x, y, z)$ funksiya vektorli maydonning potensiyasi deyiladi.

15.16. Vektorli maydon oqimi haqidagi masala. Faraz qilaylik fazoda biror $\vec{v}(M)$ vektorli maydon berilgan, ya'ni fazo harakatdagi suyuqlik bilan to'ldirilgan bo'lib, uning har bir $M(x, y, z)$ nuqtasidagi tezligi

$$\vec{v}(M) = P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k}$$

vektor bilan ifoda qilingan bo'lsin, bunda P, Q va R lar tezlikning koordinat o'qlaridagi proyeksiyalari bo'lsin. Bu funksiyalarni uzluksiz deb faraz qilamiz. Endi fazodagi biror (L) chiziqlik bilan chegaralangan bir birlik vaqt davomida oriyentrlangan birar (S) sirtning sirtidan oqib o'tayotgan suyuqlikning oqimi Π ni topamiz. Soddalik uchun suyuqlikning zichligi $\rho = 1$ bo'lsin deb faraz qilamiz.

(s) sirtga o'tkazilgan normalning birlik vektori

$$\vec{n} = \cos\alpha \vec{i} + \cos\beta \vec{j} + \cos\gamma \vec{k}$$

bo'lib, uning yo'naltiruvchi kosinuslari x, y, z o'zgaruvchilarning uzluksiz funksiyalari bo'lsin.

(s) sirtni yuzalari $\Delta s_1, \Delta s_2, \dots, \Delta s_n$ bo'lgan n ta bo'lakga bo'lamiz. Bu bo'laklarning har biridan $M_i(x_i, y_i, z_i)$ nuqtani tanlaymiz. Bir birlik vaqt ichida (s_i) sirt bo'lagidan o'tayotgan suyuqlik oqimi Π_i ni topamiz $\vec{n}_i = \vec{n}(M_i)$ va $\vec{v}_i = \vec{v}(M_i)$ vektorlar orasidagi burchakni φ deb belgilaymiz. Agar bu burchak o'tkir bo'lsa, ya'ni suyuqlik sirtning $\vec{n}$ normalning yo'nalishi ko'rsatilgan tomoni bo'yicha oqayotgan bo'lsa, Π_i kattalik musbat bo'ladi; agar burchak o'tmas bo'lsa, ya'ni suyuqlik sirtning teskari tomoni bo'yicha oqayotgan bo'lsa, Π_i kattalik manfiy bo'ladi. (s) sirtni istalgancha kichik qilib bo'laklarga bo'lganda sirtning (s_i) bo'lagidagi hamma nuqtalardagi suyuqlikning tezligi taqribiy $\vec{v}_i = \vec{v}(M_i)$ ga teng deb faraz qilinsa $((s_i)$ -tekis sirt), u holda Π , asosining yuzi Δs_i balandligi $\vec{v}_i$ vektorning $\vec{n}_i$ normaldagi proyeksiyasining

moduli h ga teng bo'lgan silindrning hajmiga taqribiy teng, ya'ni

$\Pi_i \approx \Delta S_i h$, bunda $h = \vec{v}_i \cdot \vec{n}_i$ vektorning $\vec{n}_i$ dagi proyeksiyasi bo'lgani uchun

$$h = \left| \vec{v}_i \right| \cdot \cos \varphi = \left| \vec{v}_i \right| \cdot \left| \vec{n}_i \right| \cdot \cos \varphi = \left(\vec{v}_i \cdot \vec{n}_i \right).$$

$\Pi_i \approx \left(\vec{v}_i \cdot \vec{n}_i \right) \Delta S_i$ bo'ladi.

Bundan

$$\Pi \approx \sum_{i=1}^n \left(\vec{v}_i \cdot \vec{n}_i \right) \Delta S_i \quad (15.16.1)$$

$\vec{v} = \{P, Q, R\}$, $\vec{v} = \{\cos \alpha, \cos \beta, \cos \gamma\}$, bo'lganligi uchun

$$\left(\vec{v} \cdot \vec{n} \right) = P \cos \alpha + Q \cos \beta + R \cos \gamma.$$

Ixtiyoriy P bo'linishga mos kelgan $(S_i) (i=1, 2, \dots, n)$ sirtlar diametrlarining eng kattasini λ_r deb belgilasak, u holda (15.16.1) dan

$$\Pi = \lim_{\lambda_r \rightarrow 0} \sum_{i=1}^n \left(\vec{v}_i \cdot \vec{n}_i \right) \Delta S_i = \iint_{(S)} \left(\vec{v} \cdot \vec{n} \right) ds$$

yoki

$$\Pi = \iint_{(S)} (P \cos \alpha + Q \cos \beta + R \cos \gamma) ds = \iint_{(S)} P dy dz + Q dz dx + R dx dy \quad (15.16.2)$$

hosil qilamiz.

Shunday qilib bir birlik vaqt ichida oriyehtlangan (s) sirtidan o'tayotgan suyuqlik oqimi sirt integralini ifoda qilargan.

Ixtiyoriy $\vec{v}(M)$ vektorli maydon uchun (15.16.2) ko'rinishdagi ikkinchi tur sirt integrali $\vec{v}(M)$ maydoning oqimi deyiladi. Agar, yuqoridagi ko'rilgan suyuqlikning tezliklari maydini o'rniga boshqa maydon qaralsa, u holda oqim albatta boshqa fizik ma'noni anglatadi.

15.17. Divergensiya. Fazoda (s) sirt bilan chegaralangan (V) soha berilgan bo'lsin. (V) sohada $\vec{a}(M) = \{P, Q, R\}$ vektorli maydon aniqlangan bo'lib, P, Q, R funksiyalar (V) da uzluksiz va uzluksiz xususiy hosilalarga ega.

15.17.1. Ta'rif. Ushbu $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$ skalyar funksiyaga

$\vec{a}(M) = P(x, y, z) \vec{i} + Q(x, y, z) \vec{j} + R(x, y, z) \vec{k}$ maydonning *divergensiyasi* deyiladi va

$$\operatorname{div} \vec{a}(M) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \quad (15.17.2)$$

kabi belgilanadi.

(15.16.2) va (15.17.2) larni e'tiborga olib Ostrogradskiy formulasini ushbu

$$\iint_{(S)} (\vec{a} \cdot \vec{n}) ds = \iiint_{(V)} \operatorname{div} \vec{a}(M) dV \quad (15.17.3)$$

ko'rinishda yozish mumkin.

Shunday qilib, (S) yopiq sirtidan o'tgan $\vec{a}(M)$ vektorli maydon oqimi $\vec{a}(M)$ maydoning divergensiyasidan (S) sirt bilan chegaralangan (V) soha bo'yicha olingan uch karali integralga teng bo'lar ekan.

Divergensiyaning yuqoridagi ta'rif formal ravishda bo'lsa ham koordinat sistemasini tanlashga bog'liq. Divergensiyaning koordinatlar sistemasini tanlashga bog'liq emasligini ko'rsatish uchun ixtiyoriy M nuqtani olib, uni (S) sirt bilan chegaralangan biror (V) soha bilan o'raymiz va (15.17.3) formulani yozamiz, so'ngra unga o'rta qiymat haqidagi teoremani qo'llab

$$\iint_{(S)} (\vec{a} \cdot \vec{n}) ds = \operatorname{div} \vec{a}(M_0) V$$

(bunda $M_0 - (V)$ sohadagi biror nuqta, $V - (V)$ sohaning hajmi) ni hosil qilamiz. Bundan

$$\operatorname{div} \vec{a}(M_0) = \frac{\iint_{(S)} (\vec{a} \cdot \vec{n}) ds}{V}.$$

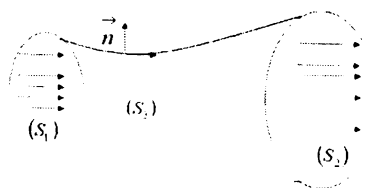
Endi (V) sohani M nuqtaga intiltiramiz, ya'ni $M_0 \rightarrow M, (V) \rightarrow 0$

$$\operatorname{div} \vec{a}(M) = \lim_{V \rightarrow 0} \frac{\iint_{(S)} (\vec{a} \cdot \vec{n}) ds}{V}. \quad (15.17.4)$$

(15.17.4) formula divergensiyaning koordinat sistemasini tanlashga bog'liq bo'lmagan ta'rifini ifodalaydi, ya'ni $\vec{a}(M)$ vektorli maydonning M nuqtadagi divergensiya shu nuqtani o'rab turgan (S) sirtidan o'tuvchi $\vec{a}(M)$ vektor oqimining (V) soha hajmiga nisbatining limitiga teng bo'lar ekan. Oqim va hajm koordinat sistemasini tanlashga bog'liq bo'lmaganligi sababli divergensiya ham koordinatlar sistemasini tanlashga bog'liq bo'lmaydi.

$\vec{a}(M)$ vektorli maydonning har bir nuqtasida $\text{div } \vec{a}(M) = 0$ bo'lsa, $\vec{a}(M)$ vektorli maydonga *solinoidal maydon*, ya'ni *turubkasimon* maydon deyiladi.

Ma'lumki, $\text{div } \vec{a}$ - divergensiya $\vec{a}(M)$ - vektorli maydon manbasining zichligini ifoda qiladi, shuning uchun $\vec{a}(M)$ maydon salenoidal bo'lganda, bu maydonning manbasi bo'lmaydi. Maydonning salenoidallik sharti



15.11-chizma.

$$\iint_{(S)} (\vec{a} \cdot \vec{n}) ds = 0 \quad (15.17.5)$$

shartga teng kuchli (ya'ni biror (V) jismni o'rovchi (S) sirtidan o'tayotgan vektor oqimi nolga teng).

Ushbu 15.11-chizmadagi vektor trubkani qaraylik. Bu trubka ixtiyoriy (S_1) va (S_2) sirtlar bilan chegaralangan bo'lsin. Trubkaning o'zining sirtini (S_3) belgilaymiz. Bu holda (15.17.5) dan

$$\iint_{(S_1)} (\vec{a} \cdot \vec{n}) ds + \iint_{(S_2)} (\vec{a} \cdot \vec{n}) ds + \iint_{(S_3)} (\vec{a} \cdot \vec{n}) ds = 0,$$

bunda normal hamma hollarda tashqariga yo'nalgan. Bunda,

$\iint_{(S_3)} (\vec{a} \cdot \vec{n}) ds = 0$, chunki (S_3) da $(\vec{a} \cdot \vec{n}) = 0$, (S_1) kesimda normalning

yo'nalishini (S_2) dagi normalning yo'nalishiga mos qilib o'zgartirsak, u holda keyingi tenglikdan

$$\iint_{(S_1)} (\vec{a} \cdot \vec{n}) dS = \iint_{(S_2)} (\vec{a} \cdot \vec{n}) dS$$

tenglikni hosil qilamiz.

Shunday qilib, bundan solenoidal maydonning quyidagi xossasiga ega bo'lamiz, vektor trubkasining ko'ndalang kesimidan o'tayotgan vektor oqimi o'zgarmas qiymat saqlaydi, uni vektor trubkasining *intevsizligi* deyiladi (15.11-chizma).

15.18'. Sirkulyatsiya. Vektorli maydonning rotori.' Biror (V)

sohada $\vec{a}(M) = P(x, y, z) \vec{i} + Q(x, y, z) \vec{j} + R(x, y, z) \vec{k}$ vektorli maydon berilgan bo'lsin.

(V) sohada (K) –silliqli yoki bo'lakli silliq chiziqli bo'ylab olingan

$$\int_{(K)} Pdx + Qdy + Rdz$$

egri chiziqli integralni qaraylik. Agar (K) – chiziqli yopiq bo'lsa u holda bu integral $\vec{a}(M)$ vektor maydonning (K) – chiziqli bo'ylab olingan *sirkulyatsiyasi* deyiladi va u $S = \oint_{(K)} Pdx + Qdy + Rdz$ kabi belgilanadi.

Agar $\vec{a}(M)$ vektor maydon kuchlar maydoni bo'lsa, u vaqtda sirkulyatsiya material nuqtani biror kuch ta'sirida (K) –chiziqli bo'ylab siljitganda bajargan ishini ifoda qiladi. Agar $\vec{a}(M)$ maydon kuchlar maydonidan boshqa tabiiy maydon bo'lsa, sirkulyatsiya boshqa fizik ma'noga ega bo'ladi.

15.18.1. Ta'rif. Ushbu

$$\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k}$$

vektorga $\vec{a}(M)$ vektorli maydonning *rotori* (*vixr-o'ram*) deyiladi va u quyidagicha belgilanadi

$$\text{rot } \vec{a}(M) = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k}.$$

$\vec{a}(M)$ vektorli maydonning *rotorini* ushbu

$$\text{rot } \vec{a}(M) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

ko'rinishda ham ifodalash mumkin.

Rotor va sirkulyatsiya tushunchalari yordamida Stoks formulasini quyidagi ko'rinishda yozamiz:

$$\oint_{(K)} \left(\vec{a}(M) \cdot d\vec{r} \right) = \iint_{(S)} [\text{rot } \vec{a}(M) \cdot \vec{n}] ds,$$

bunda $d\vec{r} = dx \vec{i} + dy \vec{j} + dz \vec{k}$.

Shunday qilib, $\vec{a}(M)$ - vektor maydonning (K) chiziq bo'yicha sirkulyatsiyasi shu vektor maydon rotorining (S) sirdant bo'yicha oqayotgan rotor oqimiga teng bo'lar ekan.

Divergensiya singari $\text{rot } \vec{a}(M)$ ham koordinat sistemasini tanlashga bog'liq bo'lmasligini ko'rsatish mumkin.

Rotor tushunchasi potensial maydon bilan chambarchas bog'liq. Yuqorida ko'rdikki vektorli maydon $\vec{a}(M) = \{P, Q, R\}$ potensial maydon bo'lishi uchun

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$$

shart bajarilishi kerak edi, bundan esa, $\vec{a}(M)$ maydon rotorining uchala koordinatasi ham nolga teng bo'lishi kerakligi kelib chiqadi, ya'ni $\vec{a}(M)$ vektorli maydon potensial maydon bo'lishi uchun

$$\text{rot } \vec{a}(M) = 0$$

bo'lishi zarur va yetarli.

15.19. Gamelton operatori. Maydonlar nazariyasining asosiy tushunchalaridan: gradiyent, divergensiya, rotor va ular ustidagi amallarni Gamelton operatori («nabla») deb ataluvchi

$$\nabla = \frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k}$$

simvolik operatori orqali ifodalash qo'llay bo'ladi ∇ vektorni $u(x, y, z)$ –skalyar funksiyaga ko'paytirish natijasida

$$\nabla u = \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) u = \vec{i} \frac{\partial u}{\partial x} + \vec{j} \frac{\partial u}{\partial y} + \vec{k} \frac{\partial u}{\partial z} = \text{gradu}$$

hosil bo'ladi. ∇ vektorni $\vec{a}(M) = \{P, Q, R\}$ vektorga skalyar ko'paytirish natijasida $\text{div} \vec{a}$ hosil bo'ladi:

$$(\nabla \cdot \vec{a}) = \frac{\partial}{\partial x} P + \frac{\partial}{\partial y} Q + \frac{\partial}{\partial z} R = \text{div} \vec{a}.$$

∇ vektorini $\vec{a}(M) = P \vec{i} + Q \vec{j} + R \vec{k}$ vektor funksiyaga vektorli ko'paytirish natijasida $\text{rot} \vec{a}$ hosil bo'ladi:

$$[\nabla \cdot \vec{a}] = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \text{rot} \vec{a}.$$

Amaliyotda ko'pincha yuqorida keltirilgan uchta operatsiyalarning ikkitadan kombinatsiyasi uchraydi:

$$1^0. \quad \text{div} \text{rot} \vec{a}(M) = 0;$$

$$2^0. \quad \text{rot} \text{gradu} = 0,$$

$$3^0. \quad \text{div} \text{gradu} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) u = \Delta u,$$

bunda $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ -Laplas operatori deb ataladi,

$$(\nabla \cdot \nabla) = \left(\frac{\partial}{\partial x} \right)^2 + \left(\frac{\partial}{\partial y} \right)^2 + \left(\frac{\partial}{\partial z} \right)^2 = \Delta.$$

73-§. Fure qatorining ta'rifi

16.1. Ba'zi asosiy tushunchalar. 1. *Davriy funksiya tushunchasi va uning asosiy xossalari.* Tabiatda va texnikada shunday xossalari uchraydiki, ular ma'lum bir t vaqt o'tgandan so'ng o'zlarining boshlang'ich holatlarini takrorlaydi. Bularga osmon jismlarining aylanma harakati, har xil asbob va mashinalarning tebranma harakati misol bo'la oladi. Bunday hodisalarni matematikada *davriy* deb ataluvchi funksiyalar yoradimida o'rganiladi.

16.1.1. Ta'rif. Agar $f(x)$ ($-\infty < x < \infty$) funksiya uchun shunday T ($T \neq 0$) son mavjud bo'lib, barcha $x \in (-\infty; \infty)$ lar uchun

$$f(x+T) = f(x)$$

tenglik o'rinli bo'lsa, u holda $f(x)$ *davriy funksiya*, T esa, uning *davri* deyiladi.

16.1.1-ta'rifdan ko'rinadiki, $f(x)$ davriy funksiyaning grafigini biror $[a, a+T]$ ($a \neq 0$) oralig'ida yasab so'ngra uni butun o'qqa davriylik qonuniga asosan davom ettirish yetarli.

1⁰. Agar T son $f(x)$ funksiyaning davri bo'lsa, u holda ixtiyoriy butun (musbat yoki manfiy) k uchun $T_1 = kT$ ham $f(x)$ funksiyaning davri bo'ladi, yani

$$f(x+T_1) = f(x).$$

2⁰. Agar T_1 va T_2 sonlar $f(x)$ funksiya uchun davr bo'lsa, $T_1 \pm T_2$ ham $f(x)$ funksiyaning davri bo'ladi.

3⁰. Agar $f(x)$ funksiya barcha $x \in (-\infty; \infty)$ lar uchun o'zgarmas bo'lsa, u davriy funksiya bo'lib, uning davri T son bo'ladi.

4⁰. Agar $f(x)$ funksiya ($f(x) \neq 0$) uzliksiz davriy funksiya bolsa, u vaqtda u yeng kichik musbat τ davrga ega bo'ladi va u funksiyaning davri deb qabul qilinadi.

2. Funksiyani davriy davom yettirish. $f(x)$ funksiya $(a, b]$ intervalda aniqlangan bo'lib, bu funksiya yordamida yordamchi quyidagi

$f^*(x) = f(x - (b-a)m), x \in (a + m(b-a), b + m(b-a))$
 $(m = 0, \pm 1, \pm 2, \pm 3, \dots)$ funksiyani tuzamiz.

Ravshanki, $f^*(x)$ funksiya $(-\infty; \infty)$ kesmada davriy funksiya bo'lib, uning davri $T_0 = b - a$ ga teng bo'ladi. Bu bajarilgan jarayonni funksiyani *davriy davom ettirish* deyiladi.

Agar $f(x)$ funksiya $[a, b]$ da uzliksiz bo'lib va

$$f(a+0) = \lim_{x \rightarrow a+0} f(x) = f(b)$$

bo'lsa, u holda davom yettirilgan $f^*(x)$ funksiya $(-\infty; \infty)$ da uzliksiz bo'ladi.

Agar $f(x)$ funksiya $[a, b]$ oraliqda berilgan bo'lsa, uni davriy davom ettirish ham yuqoridagi singari bo'ladi:

$$f^*(x) = f(x - (b-a)m), x \in [a + m(b-a), b + m(b-a)) \quad (m = 0, \pm 1, \pm 2, \pm 3, \dots)$$

Agar $f(x)$ funksiya (a, b) intervalda berilgan bo'lsa, uni $(-\infty; \infty) / \{a + m(b-a); m = 0, \pm 1, \pm 2, \pm 3, \dots\} = X^*$ to'plamga davriy davom ettirish mumkin:

$$f^*(x) = f(x - (b-a)m), x \in (a + m(b-a), b + m(b-a)) \quad (m = 0, \pm 1, \pm 2, \pm 3, \dots)$$

Agar $f(x)$ funksiya $[a, b]$ da berilgan bo'lsa, uni $(-\infty; \infty)$ ga umuman aytganda, ikki xil davriy davom ettirish mumkin:

$$f^*(x) = f(x - (b-a)m), x \in (a + m(b-a), b + m(b-a)) \quad (m = 0, \pm 1, \pm 2, \pm 3, \dots)$$

$$f^{**}(x) = f(x - (b-a)m), x \in [a + m(b-a), b + m(b-a)) \quad (m = 0, \pm 1, \pm 2, \dots).$$

5⁰. 1) agar $f_1(x), f_2(x), f_3(x), f_4(x), \dots, f_n(x)$ funksiyalarning har biri T davrga ega bo'lsa, u holda $(-\infty; \infty)$ da $F(x) = \sum_{k=1}^n f_k(x)$ funksiya ham T davrga ega bo'ladi.

2) agar $f_1(x), f_2(x), f_3(x), f_4(x), \dots, f_n(x)$ funksiyalarning har biri T davrga ega bo'lsa, u holda $F(x) = \prod_{k=1}^n f_k(x)$ funksiya ham T davrga ega bo'ladi.

3) agar $f_1(x)$ va $f_2(x)$ funksiyalar bir xil T davrga ega bo'lsa, u holda $\frac{f_1(x)}{f_2(x)}$ yoki $\frac{f_2(x)}{f_1(x)}$ funksiyalar ham T davrga ega bo'ladi.

16.1.2. Eslatma. Bu xossadan, $f_1(x)$ va $f_2(x)$ funksiyalar uchun eng kichik T davr bo'linma funksiya uchun ham eng kichik davr bo'ladi degan natija kelib chiqmasligini qayd qilamiz. Masalan, $\sin x$ va $\cos x$ funksiyalarning har biri 2π davrga ega bo'lgani uchun $\operatorname{tg} x$ va $\operatorname{ctg} x$ uchun ham 2π davr bo'ladi, lekin keiyingi ikki funksiya uchun 2π dan kichik π son ham davr bo'ladi.

4) agar $f_1(x)$ va $f_2(x)$ funksiyalar mos ravishda o'lchovdosh, ya'ni

$$\frac{T_1}{T_2} = \frac{n_1}{n_2} \quad (16.1.3)$$

(bunda n_1 va n_2 –biror butun sonlar) bo'lgan T_1 va T_2 davrlarga ega bo'lsa, u holda $T = n_2 T_1 = n_1 T_2$ son har ikki funksiya uchun davr bo'ladi. O'lchovdosh, ya'ni (16.1.3) ni qanoatlantiruvchi T_1 va T_2 davrlarga ega bo'lgan ikkita funksiya yig'indisi, ayirmasi va ko'paytmasi davriy funksiya bo'lib, $T = n_2 T_1 = n_1 T_2$ ular uchun davr bo'ladi.

Masalan, $\sin x$ va $\operatorname{tg} x$ funksiya uchun davrlari mos ravishda $T_1 = 2\pi$ va $T_2 = \pi$ bo'lib, $\frac{T_1}{T_2} = \frac{2}{1}$ yoki $\frac{T_2}{T_1} = \frac{1}{2}$ bo'lgani uchun $T = 2\pi$ har ikkalasi uchun davr bo'ladi, undan tashqari, 2π son $\sin x + \operatorname{tg} x$, $\operatorname{tg} x \cdot \sin x$, $\cos x$, $\sec x$ funksiyalar uchun ham davr bo'ladi.

5⁰. Agar T davrga ega bo'lgan $f(x)$ funksiya o'zining aniqlanish sohasida differinsiallanuvchi bo'lsa, u holda uning $f'(x)$ hosilasi ham T davrga ega bo'ladi.

6⁰. Agar $f(x)$ funksiya T davrga ega va integrallanuvchi bo'lib, $\int_0^T f(x) dx = 0$ bo'lsa, u holda $F(x) = \int_0^x f(t) dt$ funksiya ham T davrga ega bo'ladi.

7⁰. Agar $f(x)$ funksiya davrli integrallanuvchi bo'lsa, u holda $\forall a \in R$ uchun ushbu

$$\int_a^{a+T} f'(x)dx = \int_0^T f'(x)dx$$

tenglik o'rinli bo'ladi.

16.2. Garmonikalar superpozitsiyasi. Berilgan funksiyani Fure trigonimetrik qatoriga yoyish. Fizika va texnika fanlarida uchraydigan eng sodda davriy funksiya moddiy nuqtaning biror F kuch ta'siridagi tebranma harakatini ifodalovchi ushbu

$$G(t) = A \sin(\omega t + \varphi) \quad (16.2.1)$$

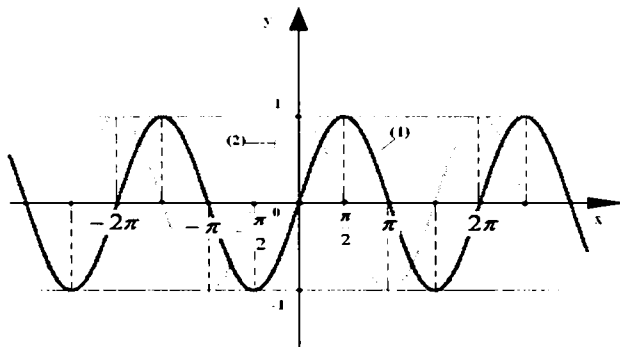
funksiyadir. Odatda bu funksiyani *garmonika* deb yuritiladi. Bu funksiya uchun $T = \frac{2\pi}{\omega}$ son davr bo'ladi.

Haqiqatan ham,

$$A \sin\left[\omega\left(t + \frac{2\pi}{\omega}\right) + \varphi\right] = A \sin[(\omega t + \varphi) + 2\pi] = A \sin(\omega t + \varphi), \quad G(t + T) = G(t)$$

bu funksiyaning eng katta qiymati $|G(t)| = A > 0$ bo'lib, u tebranuvchi moddiy nuqtaning boshlang'ich holatidan, eng katta uzoqlashishni ifodalaydi. Bu uzoqlashish mexanikada tebranish *amplitudasi* deyiladi. $\omega = \frac{2\pi}{T}$ son 2π vaqt ichidagi tebranishlar sonini ifodalaydi va u vaqtda *chastota* deb ataldi, φ son moddiy nuqtaning ($t=0$ da) boshlang'ich $G(0) = A \sin \varphi$ holatini aniqlovchi miqdor bo'lgani uchun uni *boshlang'ich fazo* deyiladi.

(16.2.1) dan xususiy holda: 1) $\varphi = 0, A = 1, \omega = 1$ bo'lsa, $G(t) = \cos t$ funksiya hosil bo'ladi, uning grafigi odatda kosinusoida deb ataladi (16.1-chizma).



16.1-chizma. (1) – $y = \sin x$, (2) – $y = \cos x$.

Endi ushbu

$$A_k \sin\left(\frac{2\pi k}{T}x + \varphi_k\right), \quad k = 1, 2, \dots, -\infty < x < \infty \quad (16.2.2)$$

garmonikalar ketma-ketligini qaraylik, bunda $T_k = \frac{T}{k}$ son k garmonikaning davri, $T = kT_k$ son esa, (16.2.2) garmonikalar ketma-ketligining umumiy davri, $\lambda_k = \frac{2\pi k}{T}$ ($k = 1, 2, \dots$) k -garmonikanin chastotasi. (16.2.2) garmonikalar ketma-ketligining chastotasi $\frac{2\pi}{T}$ song karrali bo'ladi. (16.2.2) dan

$$f_N(x) = A_0 + \sum_{k=1}^N A_k \sin\left(\frac{2\pi k}{T}x + \varphi_k\right) \quad (16.2.3)$$

ifodani tuzamiz. Bu funksiya ham T davrga ega, chunki T so garmonikalar ketma ketligining umumiy davri, A_0 ni 3^0 -ga asosan, davrga ega bo'lgan funksiya deb qarash mumkin.

Xuddi shunday yaqinlashuvchi ushbu

$$f(x) = A_0 + \sum_{k=1}^{\infty} A_k \sin\left(\frac{2\pi k}{T}x + \varphi_k\right) \quad (16.2.4)$$

qatorning yig'indisi ham T davrga ega bo'lgan davriy funksiya bo'ladi.

Haqiqatan ham, (16.2.4) qator yaqinlashuvchi bo'lsa, u yig'indig ega bo'ladi, uni biz $f(x)$ deb belgilaylik

$$f(x+T) = A_0 + \sum_{k=1}^{\infty} A_k \sin\left[\frac{2\pi k}{T}(x+T) + \varphi_k\right] =$$

$$= A_0 + \sum_{k=1}^{\infty} A_k \sin\left[\left(\frac{2\pi k}{T}x + \varphi_k\right) + 2\pi k\right] = A_0 + \sum_{k=1}^{\infty} A_k \sin\left(\frac{2\pi k}{T}x + \varphi_k\right) = f(x).$$

Agar ushbu

$$A_k \sin\left(\frac{2\pi k}{T}x + \varphi_k\right) = A_k \sin \varphi_k \cdot \cos \frac{2\pi k}{T}x + A_k \cos \varphi_k \cdot \sin \frac{2\pi k}{T}x$$

tenglikda $a_k = A_k \sin \varphi_k$, $b_k = A_k \cos \varphi_k$, $T = 2l$, $A_0 = \frac{a_0}{2}$ deb olsak, u holda (16.2.3) va (16.2.4) ning ko'rinishi quyidagicha bo'ladi:

$$f_N(x) = \frac{a_0}{2} + \sum_{k=1}^N \left(a_k \cos \frac{k\pi}{l}x + b_k \sin \frac{k\pi}{l}x\right),$$

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos \frac{k\pi}{l}x + b_k \sin \frac{k\pi}{l}x\right). \quad (16.2.5)$$

Yuqorida takidlaganimizdek, bu ikki tenglikning o'ng tomonidagi yig'indilarning har biri $2l$ davrlidir. Odatda (16.2.5) qatorga *trigonometrik qator* deyiladi.

Fure qatori nazariyasida ham funksional qatorlar nazariyasidagi singari quyidagi uchta masala muhim ahamiyatga ega:

1) berilgan $f(x)$ funksiyani $2l$ davrga ega bo'lgan davriy funksiyalardan tuzilgan (16.2.5) ko'rinishdagi cheksiz yig'indi, ya'ni qator orqali ifodalaymiz;

2) agar (16.2.5) qator berilgan bo'lsa, u x ning qaysi kesmadagi qymatlari uchun qanday funksiyaga yaqinlashadi yoki qanday funksiyaning yoyilmasi bo'ladi?

3) agar (16.2.5) tenglik o'rinli bo'lsa, qatorning koeffitsiyentlari a_0, a_k, b_k lar qanday topiladi?

Qo'yilgan masalani yechishdan avval, kelgusida muhim ahamiyatga ega bo'lgan trigonometrik fuunksiyalar sinus va kosinuslardan tuzilgan asosiy trigonometrik funksiyalar sistemasi deb ataluvchi sistemaning ortogonallik xossasi bilan tanishamiz.

Asosiy trigonometrik sistema va uning ortogonalligi

Ravshanki (16.2.5) qatorning hadlari

$$\frac{1}{2}, \cos \frac{\pi x}{l}, \sin \frac{\pi x}{l}, \cos \frac{2\pi x}{l}, \sin \frac{2\pi x}{l}, \dots, \cos \frac{k\pi x}{l}, \sin \frac{k\pi x}{l}, \dots \quad (16.2.6)$$

sistema elementlari orqali tuzilgan. Odatda (16.2.6) sistema asosiy *trigongometrik funksiyalar sistemasi* deyiladi. (16.2.5) asosiy trigonometrik sistema $[-l, l]$ kesmada ortogonal bo'lgan sistemadir, ya'ni uning o'zaro teng bo'lmagan ikkita xar hil hadi ko'paytmasidan $[-l, l]$ kesma bo'yicha olingan integral nolga teng; har birining kvadrataridan $[-l, l]$ kesma bo'yicha olingan integral noldan farqli;

$$1) \left. \begin{aligned} \int_{-l}^l \frac{1}{2} \cos \frac{k\pi x}{l} dx &= \frac{1}{2} \frac{1}{k\pi} \sin \frac{k\pi x}{l} \Big|_{x=-l}^{x=l} = 0 \\ \int_{-l}^l \frac{1}{2} \sin \frac{k\pi x}{l} dx &= \frac{1}{2} \frac{1}{k\pi} \cos \frac{k\pi x}{l} \Big|_{x=-l}^{x=l} = \\ &= -\int_{-l}^l \frac{1}{2} \cos \frac{k\pi x}{l} dx = -\frac{1}{2} \frac{l}{k\pi} [(-1)^k - (-1)^k] = 0 \end{aligned} \right\} \quad (16.2.7)$$

2)

$$\int_{-l}^l \frac{1}{2} \cos \frac{k\pi x}{l} \cos \frac{n\pi x}{l} dx = \frac{1}{2} \int_{-l}^l \left[\cos \frac{(k+n)\pi x}{l} + \cos \frac{(k-n)\pi x}{l} \right] dx = 0, n \neq k. \quad (16.2.8)$$

Xuddi shunday,

$$\left. \begin{aligned} \int_{-l}^l \sin \frac{k\pi x}{l} \sin \frac{n\pi x}{l} dx &= 0, n \neq k \\ \int_{-l}^l \sin \frac{k\pi x}{l} \cos \frac{n\pi x}{l} dx &= 0, n \leq k \end{aligned} \right\} \quad (16.2.9)$$

$$3) \left. \begin{aligned} \int_{-l}^l \cos^2 \frac{k\pi x}{l} dx &= \int_{-l}^l \frac{1 + \cos^2 \frac{k\pi x}{l}}{2} dx = l, \\ \int_{-l}^l \sin^2 \frac{k\pi x}{l} dx &= \int_{-l}^l \frac{1 - \cos^2 \frac{k\pi x}{l}}{2} dx = l, \\ \int_{-l}^l \left(\frac{1}{2}\right)^2 dx &= \frac{1}{2}. \end{aligned} \right\} \quad (16.2.10)$$

Endi (16.2.5) yoyilmaning koeffetsiyentlarini topish masalasi bilan shug'ullanamiz.

Agar (16.2.5) qator $[-l, l]$ kesmada $f(x)$ ga tekis yoki o'rtacha yaqinlashuvchi bo'lsa, u holda uni hadma-had integrallash mumkin bo'ladi. Bu tasdiq, (16.2.5) ni biror integrallanuvchi funksiyaga ko'paytirgan holda ham o'rinli bo'ladi. (16.2.6) sistemaning ortagonalligini e'tiborga olib, (16.2.5) ning ikkala tomonini $[-l, l]$ kesmada hadma-had integrallab, a_0 ko'effetsiyentni topamiz:

$$\int_{-l}^l f(x) dx = \frac{a_0}{2} \int_{-l}^l dx + \sum_{k=1}^{\infty} \left[a_k \int_{-l}^l \cos \frac{k\pi}{l} x dx + b_k \int_{-l}^l \sin \frac{k\pi}{l} x dx \right] = a_0 \cdot l,$$

$$a_0 = \frac{1}{l} \int_{-l}^l f(x) dx. \quad (16.2.11)$$

$\cos \frac{k\pi}{l} x$ ning oldidagi ko'effetsiyent a_k ni topish uchun (16.2.5) ning ikkala tomonini $\cos \frac{n\pi}{l} x$ ga ko'paytirib, uni x bo'yicha $[-l, l]$ kesmada integrallaymiz ((16.2.7)-(16.2.10) larni e'tiborga olgan holda):

$$\begin{aligned} \int_{-l}^l f(x) \cos \frac{n\pi}{l} x dx &= \frac{a_0}{2} \int_{-l}^l \cos \frac{n\pi}{l} x dx + \\ &+ \sum_{k=1}^{\infty} a_k \int_{-l}^l \cos \frac{k\pi}{l} \cos \frac{n\pi}{l} x dx + \sum_{k=1}^{\infty} b_k \int_{-l}^l \sin \frac{k\pi}{l} x \cos \frac{n\pi}{l} x dx = \\ &= a_n \int_{-l}^l \cos^2 \frac{n\pi}{l} x dx = a_n l, \quad a_n = \frac{1}{l} \int_{-l}^l f(x) \cos \frac{n\pi}{l} x dx \end{aligned} \quad (16.2.12)$$

ni topamiz.

Xuddi shunday $\sin \frac{k\pi}{l} x$ ning oldidagi ko'effetsiyent b_k ni topish uchun ((16.2.7)-(16.2.10) larni e'tiborga olgan holda) (16.2.5) ning ikkala tomonini $\sin \frac{n\pi}{l} x$ ga ko'paytirib, x bo'yicha $[-l, l]$ kesmada integrallab,

$$b_k = \frac{1}{l} \int_{-l}^l f(x) \sin \frac{n\pi}{l} x dx \quad (16.2.13)$$

ni topamiz.

16.2.14. Ta'rif. Ko'effetsiyentlari (16.2.11), (16.2.12) va (16.2.13) formulalar orqali topiladigan ushbu

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos \frac{k\pi}{l} x + b_k \sin \frac{k\pi}{l} x \right) \quad (16.2.15)$$

trigonometrik qatorga $f(x)$ funksiyaning Fure qatori, a_0, a_k, b_k koefitsientlarni esa uning *Fure koefitsiyentlari* deyiladi.

(16.2.11), (16.2.12) va (16.2.13) integrallarning mavjud bo'lishi uchun $f(x)$ funksiyaning $[-l, l]$ kesmada integrallanuvchi bo'lishi yetarli. Shuning uchun har bir $[-l, l]$ kesmada integrallanuvchi $f(x)$ funksiyaga koefitsiyentlari (16.2.11), (16.2.12) va (16.2.13) formulalar bilan aniqlanadigan (16.2.15) trigonometrik qatorni mos qo'yish mumkin:

$$f(x) \approx \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos \frac{k\pi}{l} x + b_k \sin \frac{k\pi}{l} x \right). \quad (16.2.16)$$

Umuman olganda, $f(x)$ funksiyadan integrallanuvchanligidan tashqari boshqa shart talab qilinmasa (16.2.16) da tenglik ishorasini qo'yib bo'lmaydi.

Agarda $n \rightarrow \infty$ da $\int_a^b |f_n(x) - f(x)| dx \rightarrow 0$ bo'lsa, $\{f_n(x)\}$ ketma-ketlik

$[a, b]$ kesmada $f(x)$ funksiyaga *o'rtacha yaqinlashuvchi* deyiladi.

Endi $f(x)$ ga qanday yetarli shart qo'yilganda (16.2.16) da tenglik ishorasini qo'yish mumkin degan masala bilan shug'ullanamiz. Bu masalani qarashdan avval biror berilgan $f(x)$ funksiyaning (16.2.10) ko'rinishdagi trigonometrik qatorini tuzishga doir bir nechta misollarni qaraymiz.

16.2.17. Misol. Ushbu $f(x) = \begin{cases} 0, & -\pi < x < 0, \\ x, & 0 \leq x \leq \pi \end{cases}$ 2π davrga ega bo'lgan

funksiyani Fure qatoriga yoying.

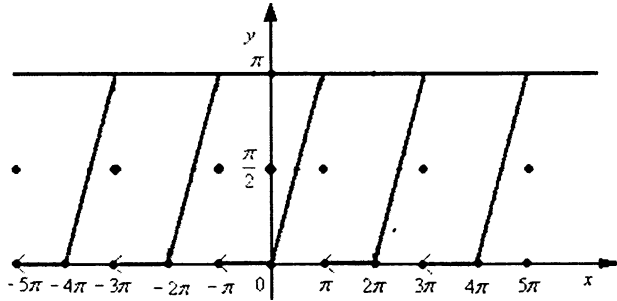
Yechilishi. Berilgan funksiyaning Fure koefitsiyentlarini topamiz:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_0^{\pi} x dx = \frac{1}{\pi} \cdot \frac{x^2}{2} \Big|_0^{\pi} = \frac{\pi}{2},$$

$$a_n = \frac{1}{\pi} \int_0^{\pi} x \cos nx dx = \left| \begin{matrix} u = x, dv = \cos nx dx \\ du = dx, v = \frac{1}{n} \sin nx \end{matrix} \right| = \frac{1}{\pi} \left(\frac{x}{n} \sin nx \Big|_0^{\pi} - \int_0^{\pi} \frac{1}{n} \sin nx dx \right) =$$

$$\left. \frac{1}{\pi} \frac{1}{n^2} \cos nx \right|_0^\pi = \frac{1}{\pi n^2} ((-1)^n - 1), \quad (n \in \mathbb{N}),$$

$$b_n = \frac{1}{\pi} \int_0^\pi x \sin nx dx = \frac{1}{\pi} \left(-\frac{x}{n} \cos nx \right|_0^\pi - \frac{1}{n^2} \sin nx \Big|_0^\pi \right) = -\frac{1}{\pi} \frac{\pi}{n} \cos n\pi = \frac{(-1)^{n-1}}{n}, \quad (n \in \mathbb{N}).$$



16.2-chizma.

Topilgan koeffitsiyentlarni (16.2.15) qatorga qo'yib

$$\frac{\pi}{4} + \sum_{n=1}^{\infty} \left(-\frac{2}{\pi(2n-1)^2} \cos((2n-1)x) + \frac{(-1)^{n-1}}{n} \sin nx \right)$$

topamiz. Bu qator $x \neq (2n-1)\pi$ nuqtalarda 2π davrlı funksiyaga yaqinlashadi. $x = (2n-1)\pi$ nuqtalarda qatorning yig'indisi $\frac{\pi+0}{2} = \frac{\pi}{2}$ ga teng bo'ladi (16.2-chizma).

16.2.18. Misol. $[0, 2\pi]$ kesmada aniqlangan $f(x) = x^2$ funksiyaning trigonometrik Fure qatori tuzilsin.

Yechilishi. (16.2.11), (16.2.12) va (16.2.13) formulalardan foydalanib tirgonometrik qatorning a_0, a_n, b_n koeffitsiyentlarini aniqlaymiz:

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_0^{2\pi} x^2 dx = \frac{1}{\pi} \frac{x^3}{3} \Big|_0^{2\pi} = \frac{8\pi^2}{3}; \\ a_n &= \frac{1}{\pi} \int_0^{2\pi} x^2 \cos nx dx = \frac{1}{\pi} \left[\frac{x^2 \sin nx}{n} \Big|_0^{2\pi} - \frac{1}{n} \int_0^{2\pi} \cos nx dx \right] = \\ &= -\frac{2}{\pi n} \int_0^{2\pi} x \sin nx dx = \frac{2}{n\pi} \left[\frac{x \cos nx}{n} \Big|_0^{2\pi} - \frac{1}{n} \int_0^{2\pi} \cos nx dx \right] = \end{aligned}$$

$$\begin{aligned}
&= \frac{2}{\pi i} \left[\frac{2\pi}{n} - \frac{\sin nx}{n^2} \right]_0^{2\pi} = \frac{4}{n^2} \quad (n=1,2,3,\dots); \\
b_n &= \frac{1}{\pi} \int_0^{2\pi} x^2 \sin nx dx = \frac{1}{\pi} \left[-\frac{x^2 \cos nx}{n} \right]_0^{2\pi} + \frac{2}{n} \int_0^{2\pi} x \cos nx dx = \\
&= \frac{1}{\pi} \left[-\frac{4\pi^2}{n} + \frac{2}{n} \left(\frac{x \sin nx}{n} \right) \right]_0^{2\pi} - \frac{1}{n} \int_0^{2\pi} \sin nx dx = \\
&= \frac{1}{\pi} \left[-\frac{4\pi^2}{n} + \frac{2}{n^2} \cos nx \right]_0^{2\pi} = -\frac{4\pi}{n}, \quad (n=1,2,3,\dots)
\end{aligned}$$

topilgan koeffitsiyentlarning qiymatlarini (16.2.5) ga qo'yib, ushbu

$$\frac{4\pi^2}{3} + 4 \sum_{n=1}^{\infty} \left(\frac{\cos nx}{n^2} - \pi \frac{\sin nx}{n} \right)$$

qatorni hosil qilamiz. Bunda $x=0$ desak, ushbu

$$\frac{4\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

sonli qator hosil bo'ladi. Ravshanki bu qator yaqinlashuvchi bo'lib, noldan farqli yig'indiga ega. Ikkinchi tomondan $f(0)=0$ Bundan

$$f(0) \neq \frac{4\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Agar $x=\pi$ desak, $f(\pi)=\pi^2$ bo'lib, qator

$$\frac{4\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

ko'rinishga keladi. $x=2\pi$ desak, $f(2\pi)=4\pi^2$ bo'lib, qator esa, ya'na

$$\frac{4\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

ko'rinishda bo'ladi.

16.2.19. Misol. $[0, 2\pi]$ kesmada aniqlangan $f(x) = \frac{\pi-x}{2}$

funksiyaning Fure qatori tuzilsin.

Yechilishi. Bu funksiya yordamida qatorning koeffitsiyentlarini topamiz.

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} \frac{\pi-x}{2} dx = -\frac{(\pi-x)^2}{4\pi} \Big|_0^{2\pi} = -\frac{\pi}{4} + \frac{\pi}{4} = 0;$$

$$\begin{aligned}
 a_n &= \frac{1}{\pi} \int_0^{2\pi} \frac{\pi-x}{2} \cos nx dx = \frac{1}{\pi} \left[\frac{\pi-x}{2} \cdot \frac{\sin nx}{n} \right]_0^{2\pi} + \frac{1}{2\pi} \int_0^{2\pi} \sin nx dx = \\
 &= -\frac{1}{2\pi n^2} \cos nx \Big|_0^{2\pi} = \frac{1}{n}, \\
 b_n &= \frac{1}{\pi} \int_0^{2\pi} \frac{\pi-x}{2} \sin nx dx = -\frac{1}{\pi} \left[-\frac{\pi-x}{2} \cdot \frac{\sin nx}{n} \right]_0^{2\pi} - \frac{1}{2\pi} \int_0^{2\pi} \cos nx dx = \\
 &= \frac{1}{n} - \frac{1}{2\pi n} \sin nx \Big|_0^{2\pi} = \frac{1}{n}.
 \end{aligned}$$

Shunday qilib, qator $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ ko'rinishda bo'ladi. Bundan

a) $x=0$ deb olsak, $f(0) = \frac{\pi}{2}$ bo'lib, qator nolga aylanadi.

Demak, $f(0) = \sum_{n=1}^{\infty} \frac{\sin 0}{n} = 0$, ya'ni $x=0$ da qator $f(x)$ ga yaqinlashadi;

b) $x=\pi$ bo'lsa, $f(x)=0$ va qator nolga aylanadi, ya'ni $f(\pi)=0$ va $\sum_{n=1}^{\infty} \frac{\sin n\pi}{n} = 0$ ya'ni $f(\pi)=0$.

Demak, $f(\pi) = \sum_{n=1}^{\infty} \frac{\sin n\pi}{n}$, ya'ni $x=\pi$ da qator $f(x)$ ga yaqinlashadi;

c) $x=2\pi$ da $f(2\pi) = -\frac{\pi}{2}; \sum_{n=1}^{\infty} \frac{\sin 2n\pi}{n} = 0; f(2\pi) \neq \sum_{n=1}^{\infty} \frac{\sin 2\pi n}{n}$. Demak, tuzilgan qator $x=2\pi$ da $f(x)$ ga yaqinlashmaydi.

Biz yuqorida ikki misolda ko'rdikki, x ning funksiya ning aniqlanish oralig'iga tegishli istalgan qiymatida qator yig'indisi bilan $f(x)$ funksiya qiymati bir-birga teng bo'lavermaydi. Qatorning berilgan funksiya ga yaqinlashadigan oralig'iga tegishli x lar uchungina qator yig'indisi bilan $f(x)$ ning qiymati o'zaro teng bo'ladi. Shuning uchun qatorning $f(x)$ ga yaqinlashish oralig'ini aniqlamasdan turib, funksiya bilan qator orasiga tenglik belgisini qo'yish mumkin emas. Shunga ko'ra hozircha biz qator bilan funksiya orasiga ($\approx$) belgisini qo'yamiz:

$$f(x) \approx \frac{a_0}{2} + \sum_{k=1}^x (a_k \cos \frac{k\pi}{l} x + b_k \sin \frac{k\pi}{l} x)$$

Yuqoridagi ko'rilgan misollarni quyidagicha yozish mumkin: $x \in [0, 2\pi]$ uchun

$$1) x^2 \approx \frac{4\pi^2}{3} + 4 \sum_{n=1}^x \left(\frac{\cos nx}{n^2} - \pi \frac{\sin nx}{n} \right), \quad 2) \frac{\pi - x}{2} \approx \sum_{n=1}^x \frac{\sin nx}{n}.$$

16.3. Toq va juft funksiyalarni Fure qatoriga yoyish. $f(x)$ funksiya $[-l, l]$ kesmada aniqlangan bo'lsin. $f(x)$ funksiya $\forall x \in [-l, l]$ uchun $f(-x) = f(x)$ shartni qanoatlantirsa, u holda $f(x)$ *juft* funksiya deyiladi, agar $\forall x \in [-l, l]$ uchun $f(-x) = -f(x)$ tenglik o'rinli bo'lsa, $f(x)$ *toq* funksiya deyiladi. Bu ta'riflardan ko'rinadiki, juft funksiyaning grafigi esa, ordinata o'qiga nisbatan simmetrik bo'ladi, toq funksiyaning grafigi esa, koordinata boshiga nisbatan simmetrik bo'ladi.

Agar $f(x)$ funksiya $[-l, l]$ kesmada berilgan ixtiyoriy funksiya bo'lsa, u holda

$$f_1(x) = \frac{f(x) + f(-x)}{2}, \quad f_2(x) = \frac{f(x) - f(-x)}{2}$$

ko'rinishda tuzilgan funksiyalarning birinchisi juft, ikkinchisi esa, toq funksiya bo'ladi.

$[-l, l]$ kesmada berilgan harqanday $f(x)$ funksiyani toq va juft funksiyalar yig'indisi shaklida tasvirlash mumkin, ya'ni

$$f(x) = f_1(x) + f_2(x), \quad x \in [-l, l].$$

Agar $f(x)$ funksiya $[-l, l]$ da integrallanuvchi bo'lsa, u holda

$$\int_{-l}^l f(x) dx = \int_{-l}^0 f(x) dx + \int_0^l f(x) dx = \int_0^l [f(x) + f(-x)] dx, \quad (16.3.1)$$

bunda $\int_{-l}^0 f(x) dx = -\int_{-l}^0 f(-x) dx + \int_0^l f(-x) dx$ (16.3.1) dan

$$\int_{-l}^l f(x) dx = \begin{cases} 2 \int_0^l f(x) dx, & \text{agar } f(x) \text{ - juft funksiya bo'lsa,} \\ 0, & \text{agar } f(x) \text{ - toq funksiya bo'lsa} \end{cases}$$

$$\frac{1}{2}, \cos \frac{\pi x}{l}, \cos \frac{2\pi x}{l}, \dots, \cos \frac{k\pi x}{l}, \dots \text{-juft, } \sin \frac{\pi x}{l}, \sin \frac{2\pi x}{l}, \dots, \sin \frac{k\pi x}{l}, \dots$$

-toq funksiyalar.

Agar $f(x)$ $[-l, l]$ kesmada juft funksiya bo'lsa, u holda uning Fure qatori

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos \frac{k\pi}{l} x$$

ko'rinishda bo'ladi.

Agarda $f(x)$ $[-l, l]$ kesmada toq funksiya bo'lsa, uning Fure qatori

$$\sum_{k=1}^{\infty} b_k \sin \frac{k\pi}{l} x$$

ko'rinishda bo'ladi.

Haqiqatán ham, agar $f(x)$ juft funksiya bo'lsa, u holda $f(x) \cos \frac{k\pi}{l} x$ -ko'paytma juft, $f(x) \sin \frac{k\pi}{l} x$ - ko'paytma toq funksiya bo'ladi. Shuning uchun

$$\left. \begin{aligned} a_0 &= \frac{1}{l} \int_{-l}^l f(\xi) d\xi = \frac{2}{l} \int_0^l f(\xi) d\xi, \\ a_k &= \frac{1}{l} \int_{-l}^l f(\xi) \cos \frac{k\pi}{l} \xi d\xi = \frac{2}{l} \int_0^l f(\xi) \cos \frac{k\pi}{l} \xi d\xi \\ b_k &= \frac{1}{l} \int_{-l}^l f(\xi) \sin \frac{k\pi}{l} \xi d\xi = 0 \end{aligned} \right\}$$

Agar $f(x)$ -toq funksiya bo'lsa, u holda $f(x) \cos \frac{k\pi}{l} x$ -toq, $f(x) \sin \frac{k\pi}{l} x$ -juft funksiya bo'ladi. Shuning uchun

$$\begin{aligned} a_0 &= \frac{1}{l} \int_{-l}^l f(\xi) d\xi = 0, \quad a_k = \frac{1}{l} \int_{-l}^l f(\xi) \cos \frac{k\pi}{l} \xi d\xi = 0, \\ b_k &= \frac{1}{l} \int_{-l}^l f(\xi) \sin \frac{k\pi}{l} \xi d\xi = \frac{2}{l} \int_0^l f(\xi) \sin \frac{k\pi}{l} \xi d\xi. \end{aligned}$$

16.3.2. Misol. $f(x) = x$ funksiya $[-l, l]$ oraliqda Fure trigonometrik qatoriga yoyilsin.

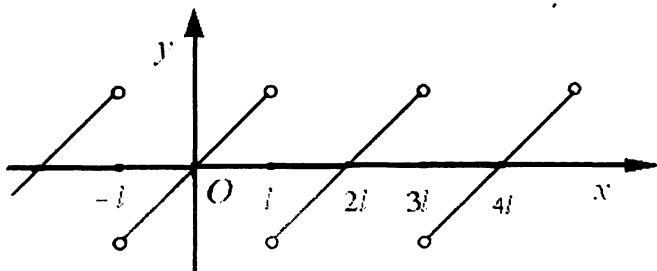
Yechilishi. $f(x) = x$ toq funksiya bo'lgani uchun, $a_0 = 0, a_n = 0$,

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi}{l} x dx =$$

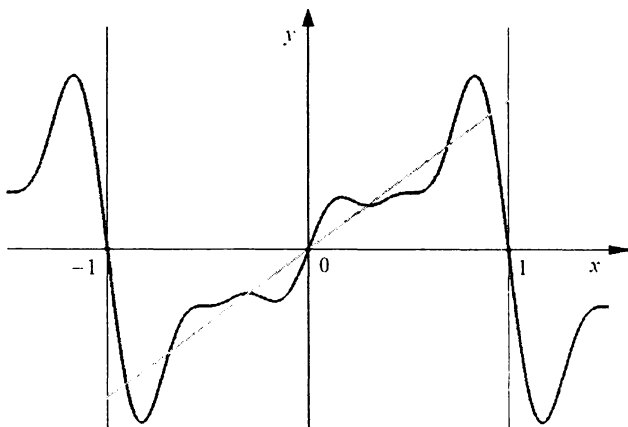
$$= \frac{2}{l} \left\{ \left(-\frac{xl}{n\pi} \cos \frac{n\pi}{l} x \right) \Big|_{x=0}^{x=l} + \frac{1}{n\pi} \int_0^l \cos \frac{n\pi}{l} x dx \right\} = \frac{2l}{n\pi} (-1)^{n+1}.$$

Yuqoridagi asosiy teoremaga asosan,

$$S(x) = \frac{2l}{n\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin \frac{k\pi x}{l} = \begin{cases} \frac{f(x-0) + f(x+0)}{2} = x, -l < x < l \\ \frac{f(-l+0) + f(l-0)}{2} = \frac{-l+l}{2} = 0, x = \pm l \end{cases}$$



16.3-chizma.



16.4-chizma.

$f(x)$ va $S(x)$ ning grafigi 16.3-chizmada berilgan. $S(x) - 2l$ davrli funksiya. $-l < x < l$ da $S(x) = x$. $x = (2k+1)l$ ($k = 0, \pm 1, \pm 2, \dots$) nuqtalarda uzilishga ega bo'lib, bu nuqtalardagi qiymati nolga teng, chunki bu nuqtalardagi o'ng va chap limitik qiymatlari nolga teng. 16.4-chizmada $S_5(x) = \frac{2l}{\pi} \sum_{k=1}^5 \frac{(-1)^{k+1}}{k} \sin \frac{k\pi x}{l}$ ning grafigi berilgan.

16.3.3. Misol. Ushbu $f(x) = x^2$ ni $[-l, l]$ da Fure qatoriga yoyilsin.

Yechilishi. $f(x) = x^2$ funksiya juft funksiya bo'lgani uchun

$$b_n = 0, a_0 = \frac{2}{l} \int_0^l x^2 dx = \frac{2}{3} l^2, a_n = \frac{2}{l} \int_0^l x^2 \cos \frac{n\pi x}{l} dx, n = 1, 2, \dots$$

Asosiy teoreмага asosan,

$$S(x) = \frac{l^2}{3} + \frac{4l^2}{\pi^2} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos \frac{k\pi x}{l} = x^2, -l \leq x \leq l, \quad (16.3.4)$$

$S(x)$ funksiya $x = (2k+1)l$ ($k = 0, \pm 1, \pm 2, \dots$) nuqtalarda ham uzluksiz. $f(l-0) = f(-l+0) = l^2$.

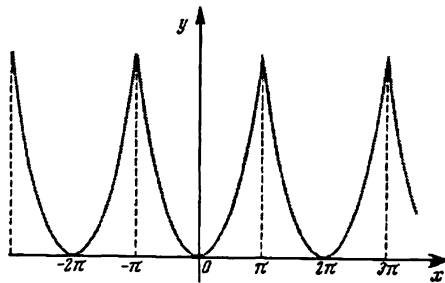
Agar yoyilma $[-\pi, \pi]$ kesmada olib borilgan bo'lsa, ya'ni $l = \pi$, u holda (16.3.4) tenglik

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos kx \quad (16.3.5)$$

ko'rinishda bo'ladi (16.5-chizma). $x = 0$ bo'lganda (16.3.5) dan

$$\frac{\pi^2}{12} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$$

ni hosil qilamiz.



16.5-chizma.

16.4. Funksiyani $[-\pi, \pi]$ kesmada Fure qatoriga yoyish. Agar $[-\pi, \pi]$ kesmada berilgan $f(x)$ funksiyani trigonometrik Fure qatoriga yoyish talab qilingan bo'lsa, u holda (16.2.11), (16.2.12) va (16.2.13) formulalarda $l = \pi$ deb, ushbu Fure qatori ko'effitsiyentlari va qatoriga

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\xi) d\xi, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\xi) \cos \frac{k\pi}{l} \xi d\xi, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\xi) \sin \frac{k\pi}{l} \xi d\xi,$$

$$f(x) \approx \frac{a_0}{2} + \sum_{k=1}^n (a_k \cos \frac{k\pi}{l} x + b_k \sin \frac{k\pi}{l} x) \quad (16.4.1)$$

ega bo'lamiz. $f(x)$ funksiya umumiy holda $[-\pi, \pi]$ da berilganda (16.4.1) ko'rinishga $x' = \frac{\pi x}{l}, \varphi(x') = f(\frac{l x'}{\pi}), -\pi \leq x' \leq \pi$ almashtirish orqali keladi.

16.5. Trigonometrik Fure qatorining yaqinlashishi to'g'risida asosiy teorema. Biz endi berilgan funksiya uchun tuzilgan Fure qatorining o'sha funksiyaga yaqinlashishiga oid masalani o'rganamiz. Buning uchun matematik analizning yuqorida ko'rilgan ba'zi tushunchalarini eslab o'tamiz.

Bo'lakli uzluksiz, bo'lakli silliq funksiya.

16.5.1. Ta'rif. Agar $f(x)$ funksiya $[a, b]$ kesmaning chekli sondagi nuqtalaridan tashqari hamma nuqtalarida uzluksiz bo'lib, chekli sondagi nuqtalarida 1-tur uzilishga ega bo'lsa, u holda $f(x)$ funksiya $[a, b]$ da *bo'lakli uzluksiz* deyiladi.

16.5.1-ta'rifdan ko'rinadiki, bunday funksiya $[a, b]$ kesmaga qarashli ixtiyoriy x_0 nuqtada chekli o'ng va chap (agar x_0 uzluksizlik nuqtasi bo'lsa o'zaro teng) limitlarga ega, ya'ni

$$\lim_{x \rightarrow x_0-0} f(x) = f(x_{0-0}), \quad \lim_{x \rightarrow x_0+0} f(x) = f(x_{0+0})$$

Shuningdek, chetki nuqtalarida

$$\lim_{x \rightarrow a+0} f(x) = f(a+0), \quad \lim_{x \rightarrow b-0} f(x) = f(b-0)$$

ega bo'ladi. Masalan, $f_1(x) = \begin{cases} 3, & -1 \leq x < 0, \\ x^2, & 0 \leq x < 2, \\ x^2 + 1, & 2 \leq x \leq 3 \end{cases}$ funksiya uchta uzluksiz

bo'lakdan iborat bo'lib, $x=0$ va $x=2$ nuqtalar $f_1(x)$ uchun birinchi tur uzilish nuqtalari bo'lib,

$$\lim_{x \rightarrow -0} f_1(x) = 3, \lim_{x \rightarrow +0} f_1(x) = 0, \lim_{x \rightarrow 2-0} f_1(x) = 4, \lim_{x \rightarrow 2+0} f_1(x) = 5.$$

Bundan tashqari, $[-1, 3]$ kesmaning chetlarida $\lim_{x \rightarrow -1+0} f_1(x) = 3$,

$\lim_{x \rightarrow 3-0} f_1(x) = 10$, $[-1, 3]$ kesmaning qolgan barcha nuqtalarida

$\lim_{x \rightarrow x_0} f_1(x) = f(x_0)$ bo'ladi.

16.5.2. Ta'rif. Agar $[a, b]$ da bo'lakli uzluksiz $f(x)$ funksiya shu kesmaning chekli sondagi nuqtalaridan boshqa barcha nuqtalarida uzluksiz $f'(x)$ hosilaga ega bo'lib, shu kesmaning faqat chekli sondagi nuqtalarida chekli o'ng va chap hosilalarga ega bo'lsa, u holda $f(x)$ funksiya $[a, b]$ da *bo'lakli silliq* deyiladi.

Bu 16.5.2-ta'rifdan ravshanki, $[a, b]$ kesmaga qarashli ixtiyoriy x_0 nuqtada

$$\lim_{x \rightarrow x_0-0} f'(x) = f'(x_0 - 0), \lim_{x \rightarrow x_0+0} f'(x) = f'(x_0 + 0)$$

chekli $f'(x_0 - 0) = f'(x_0 + 0)$ ikki yoqlama yoki chekli sondagi nuqtalarda $f'(x_0 - 0) \neq f'(x_0 + 0)$ bo'lishi zarur. Bundan tashqari chekli

$\lim_{x \rightarrow a+0} f'(x) = f'(a + 0)$, $\lim_{x \rightarrow b-0} f'(x) = f'(b - 0)$ hosilalarag ega bo'ladi.

Yuqoridagi ko'rilgan $f_1(x)$ bo'lakli silliq funksiyadir.

$$f_1'(x) = \begin{cases} 0, & -1 \leq x < 0 \\ 2x, & 0 \leq x < 2 \\ 2x, & 2 \leq x \leq 3 \end{cases}$$

16.5.3. Eslatma. Agar $f(x)$ funksiya $[a, b]$ da bo'lakli silliq funksiya bo'lsa, u holda bu kesmani har doim chekli shunday k ta $[a_0, a_1], [a_1, a_2], [a_2, a_3], \dots, [a_{k-1}, a_k]$ kesmalarga ajratish mumkinki $(a = a_0 < a_1 < a_2 < a_3 < \dots < a_k = b)$, $f(x)$ va $f'(x)$ funksiyalar istalgan $[a_i, a_{i+1}]$ ($i = 0, 1, 2, \dots, k-1$) kesmaning ichki nuqtalarida uzluksiz bo'lib, x

o'ngdan a_i ga chapadan a_{i+1} ga intilganda $f(x)$ va $f'(x)$ mos ravishda quyidagi chekli $f(a_i+0), f'(a_i+0)$ va $f(a_{i+1}-0), f'(a_{i+1}-0)$ limitlarga intiladi.

Agar

$f(a_i) = f(a_i+0), f(a_{i+1}) = f(a_{i+1}-0)$ va $f'(a_i) = f'(a_i+0), f'(a_{i+1}) = f'(a_{i+1}-0)$ deb qabul qilsak, u holda $f(x)$ va $f'(x)$ funksiyalar $[a_i, a_{i+1}]$ ($i = 0, 1, 2, \dots, k-1$) kesmalarning har birida uzliksiz bo'ladi. $y = f(x)$ funksiya, $[a, b]$ da uzliksiz bo'lib, uning $f'(x)$ hosilasi o'sha kesmada bo'lakli silliq bo'lishi ham mumkin. Masalan,

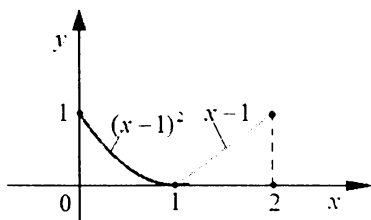
$$f(x) = \begin{cases} (x-1)^2, & 0 \leq x < 1, \\ x-1, & 1 \leq x \leq 2 \end{cases}$$

funksiya $[0, 2]$ da uzliksiz (16.6-chizma), uning

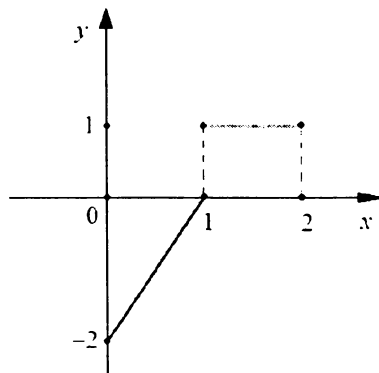
$$f'(x) = \begin{cases} 2(x-1), & 0 \leq x < 1 \\ 1, & 1 \leq x \leq 2 \end{cases}$$

hosilasi esa, ikkita silliq bo'lakdan iborat.

hosilasi esa, ikkita silliq bo'lakdan iborat (16.7-chizma)



16.6-chizma.



16.7 -chizma.

16.6. Fure trigonometrik qatorining yaqinlashishiga doir asosiy teorema (Dirixle teoremasi)

16.6.1. Teorema. Agar $f(x)$ funksiya $[-l, l]$ kesmada bo'lakli silliq bo'lsa, u holda bu funksiyaning

$$S(x) \approx \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos \frac{k\pi}{l} x + b_k \sin \frac{k\pi}{l} x)$$

fure trigonometrik qatori $[-l, l]$ kesmaga qarashli istalgan x uchun yaqinlashuvchi va uning yig'indisi $S(x)$ uchun quyidagilar o'rinalidir:

1) barcha $x \in (-l, l)$ uchun $S(x) = \frac{f(x-0) + f(x+0)}{2}$ bo'lib, agar x nuqta $f(x)$ ning uzluksizlik nuqtasi bo'lsa, u holda $f(x-0) = f(x+0) = f(x)$, demak, $S(x) = f(x)$ bo'ladi;

2) kesmaning chegaraviy nuqtalarida esa, qator yig'indisi ushbu $S(-l) = S(l) = \frac{f(-l+0) + f(+l-0)}{2}$ tenglik bilan aniqlanadi.

Bu 16.6.1-teoremani isbotlash uchun zarur bo'lgan asosiy lemmani isbot qilamiz.

16.6.2-lemma (Asosiy). Agar $f(x)$ funksiya $[a, b]$ kesmada bo'lakli silliq bo'lsa, u holda

$$\left. \begin{aligned} \lim_{\alpha \rightarrow 0} \int_a^b f(x) \sin \alpha x dx &= 0, \\ \lim_{\alpha \rightarrow 0} \int_a^b f(x) \cos \alpha x dx &= 0 \end{aligned} \right\} \quad (16.6.3)$$

bo'ladi.

Isboti. $[a, b]$ kesmani $(a = x_0 < x_1 < x_2 < \dots < x_n = b)$ (n -chekli) nuqtalar yordamida n ta shunday $[x_i, x_{i+1}]$, $i = \overline{1, n-1}$ kesmalarga ajrataylik. $f(x)$ va $f'(x)$ funksiyalar $[x_i, x_{i+1}]$ larning ichki nuqtalarida uzluksiz bo'lib, kesmaning chegaraviy nuqtalarida mos ravishda quyidagi chekli o'ng va chap limitlarga ega bo'lsin:

$$\lim_{x \rightarrow x_i + 0} f(x) = f(x_i + 0), \quad \lim_{x \rightarrow x_{i+1} - 0} f(x) = f(x_{i+1} - 0),$$

$$\lim_{x \rightarrow x_i + 0} f'(x) = f'(x_i + 0), \quad \lim_{x \rightarrow x_{i+1} - 0} f'(x) = f'(x_{i+1} - 0).$$

$f(x)\sin \alpha x$ funksiya $\forall \alpha$ uchun $[a, b]$ da integrallanuvchi bo'lgani uchun $[x_i, x_{i+1}]$ ($i = 0, 1, 2, \dots, n-1$) larning har birida ham integrallanuvchi bo'lib,

$$\int_a^b f(x)\sin \alpha x dx = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)\sin \alpha x dx$$

bo'ladi. Lemmani isbot qilish uchun

$$\lim_{\alpha \rightarrow 0} \int_{x_i}^{x_{i+1}} f(x)\sin \alpha x dx = 0 \quad (16.6.4)$$

($0 \leq i \leq n$) ni isbot qilish yetarli. Yuqoridagi 16.5.3-eslatmani e'tiborga olgan holda, $f(x)$ va $f'(x)$ funksiyalar $[x_i, x_{i+1}]$ kesmada uzluksiz bo'ladi.

Demak, ular kesmada chegaralangan. (16.6.4) dagi integralni bo'laklab integrallaymiz:

$$\int_{x_i}^{x_{i+1}} f(x)\sin \alpha x dx = -\frac{f(x)\cos \alpha x}{\alpha} \Big|_{x=x_i+0}^{x=x_{i+1}-0} + \frac{1}{\alpha} \int_{x_i}^{x_{i+1}} f'(x)\cos \alpha x dx. \quad (16.6.5)$$

$f(x)$ va $f'(x)$ funksiyalar $[a, b]$ da chegaralangan bo'lganligi uchun shunday M va M' o'zgarmas sonlar mavjud bo'lib, $|f(x)| \leq M$ va $|f'(x)| \leq M'$ ($x \in [a, b]$) tengsizliklar o'rinli bo'ladi. U holda (16.6.5) dan

$$\left| \int_{x_i}^{x_{i+1}} f(x)\sin \alpha x dx \right| \leq \frac{2M}{\alpha} + \frac{M'(a_{i+1} - a_i)}{\alpha}. \quad (16.6.6)$$

(16.6.6) dan $\alpha \rightarrow \infty$ da (16.6.4)ning o'rinli ekanligi kelib chiqadi. Shu bilan lemma isbot bo'ladi. Xuddi shunday (16.6.3) dagi ikkinchi tenglikni ham isbotlash mumkin.

16.6.7. Eslatma. Lemmani yuqorida ko'rilgan funksiyalar sinfidan kengroq sinfdagi funksiyalar uchun ham isbotlash mumkin. Masalan, $[a, b]$ da absolyut integrallanuvchi, ya'ni $\int_a^b |f'(x)| dx < \infty$ shartni qanoatlantiruvchi funksiyalar sinfi uchun ham lemma o'rinlidir.

16.6.8. Eslatma. Agar $f(x)$ funksiya $2l$ davrlı bo'lib, uzunligi $2l$ ga teng bo'lgan kesmada bo'lakli silliq bo'lsa, u holda uning Fure ko'effitsiyentlari, ya'ni a_n va b_n lar n chesizlikga intilganda nolga intiladi, ya'ni

$$\lim_{n \rightarrow \infty} a_n = 0, \lim_{n \rightarrow \infty} b_n = 0. \quad (16.6.9)$$

Haqiqatan ham (16.2.12) va (16.2.13) formulalarga ko'ra,

$$a_n = \frac{1}{l} \int_{-l}^l f(x) \cos \frac{n\pi}{l} x dx, \quad b_n = \frac{1}{l} \int_{-l}^l f(x) \sin \frac{n\pi}{l} x dx$$

larda $\alpha = \frac{n\pi}{l}$ deb olsak, $n \rightarrow \infty$ da $\alpha \rightarrow \infty$. U holda asosiy lemmaga asosan, (16.6.9) ning o'rinli ekanligi kelib chiqadi.

Asosiy 16.6.1-teoremaning isboti.

Avvalo $f(x)$ funksiyani $2l$ davrlı deb, butun OX o'qiga davriy davom yettiramiz. So'ngra $\forall x \in (-\infty, \infty)$ uchun

$$\lim_{n \rightarrow \infty} S_n(x) = \frac{f(x-0) + f(x+0)}{2}$$

ekanligini isbotlaymiz, bunda

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^n (a_k \cos \frac{k\pi}{l} x + b_k \sin \frac{k\pi}{l} x) \quad (16.6.10)$$

$f(x)$ funksiya Fure qatorining $[-l, l]$ kesmadagi qisman yig'indisi. (16.6.10) ga a_0, a_k, b_k larning (16.2.11), (16.2.12), (16.2.13) lardagi ifodalarni olib kelib qo'yamiz:

$$S_n(x) = \frac{1}{2l} \int_{-l}^l f(\xi) d\xi + \frac{1}{l} \sum_{k=1}^n \int_{-l}^l f(\xi) \left[\cos \frac{k\pi\xi}{l} x \cos \frac{k\pi}{l} x + \sin \frac{k\pi\xi}{l} x \sin \frac{k\pi}{l} x \right] d\xi = \quad (16.6.11)$$

$$= \frac{1}{l} \int_{-l}^l f(\xi) \left[\frac{1}{2} + \sum_{k=1}^n \cos \frac{k\pi(\xi-x)}{l} \right] d\xi = \frac{1}{l} \int_{-l-x}^{l-x} f(x+z) \left[\frac{1}{2} + \sum_{k=1}^n \cos \frac{k\pi z}{l} \right] dz.$$

$$\sigma_n(z) = \frac{1}{2} + \sum_{k=1}^n \cos \frac{k\pi z}{l} \quad (16.6.12)$$

deb belgilab, uning ko'rinishini quyidagi holga keltiramiz: (16.6.12)

ning ikkala tomonini $2 \sin \frac{\pi \cdot z}{2l}$ ga ko'paytiramiz, natijada

$$\begin{aligned}
2\sigma_n(z) \sin \frac{\pi z}{2l} &= \sin \frac{\pi z}{2l} + \sum_{k=1}^n 2 \sin \frac{\pi z}{2l} \sin \frac{k\pi z}{l} = \\
&= \sin \frac{\pi z}{2l} + \sum_{k=1}^n \left[\sin\left(k + \frac{1}{2}\right) \frac{\pi z}{l} - \sin\left(k - \frac{1}{2}\right) \frac{\pi z}{l} \right] = \sin\left(n + \frac{1}{2}\right) \frac{\pi z}{l},
\end{aligned}$$

bundan

$$\sigma_n(z) = \frac{\sin\left(n + \frac{1}{2}\right) \frac{\pi z}{l}}{2 \sin \frac{\pi z}{2l}} = \frac{1}{2} + \sum_{k=1}^n \cos \frac{k\pi z}{l}. \quad (16.6.13)$$

(16.6.13) ni (16.6.11) ga qo'yish natijasida Fure qatori qisman yig'indismi uchun:

$$S_n(x) = \frac{1}{l} \int_{-l-x}^{l-x} f(x+z) \frac{\sin\left(n + \frac{1}{2}\right) \frac{\pi z}{l}}{2 \sin \frac{\pi z}{2l}} dz$$

ifodani hosil qilamiz. $f(x)$ funksiya $[-l, l]$ kesmadan tashqariga $2l$ davr bilan davriy davom yettirilgan davriy funksiya. (16.6.13) ga asosan,

$\frac{\sin\left(n + \frac{1}{2}\right) \frac{\pi z}{l}}{2 \sin \frac{\pi z}{2l}}$ funksiya ham $2l$ davrli funksiya, u holda

$f(x+z) \frac{\sin\left(n + \frac{1}{2}\right) \frac{\pi z}{l}}{2 \sin \frac{\pi z}{2l}} dz$ funksiya ham $2l$ davrli funksiya. Shuning uchun bu ko'paytmadan z bo'yicha $2l$ uzunlikka ega bo'lgan ixtiyoriy iraliq bo'yicha olingan integralning qiymati bir xil bo'ladi, ya'ni

$$S_n(x) = \frac{1}{l} \int_{-l}^l f(x+z) \frac{\sin\left(n + \frac{1}{2}\right) \frac{\pi z}{l}}{2 \sin \frac{\pi z}{2l}} dz. \quad (16.6.14)$$

(16.6.14) ning o'ng tomonidagi integralga *Dirixle integrali* deyiladi.

(16.6.13) ni $\frac{1}{l}$ ga ko'paytirib, z bo'yicha $-l$ dan l gacha integrallaymiz, natijada

$$\frac{1}{l} \int_{-l}^l \frac{\sin(n + \frac{1}{2}) \frac{\pi z}{l}}{2 \sin \frac{\pi z}{2l}} dz = \frac{1}{2l} \int_{-l}^l dz = 1$$

ga ega bo'lamiz, chunki $\int_{-l}^l \cos \frac{k\pi}{l} z dz = 0, k = 1, 2, \dots$. $\frac{\sin(n + \frac{1}{2}) \frac{\pi z}{l}}{2 \sin \frac{\pi z}{2l}}$ funksiya

juf funksiya bo'lganligi uchun

$$\frac{1}{l} \int_{-l}^0 \frac{\sin(n + \frac{1}{2}) \frac{\pi z}{l}}{2 \sin \frac{\pi z}{2l}} dz = \frac{1}{2}, \quad \frac{1}{l} \int_0^l \frac{\sin(n + \frac{1}{2}) \frac{\pi z}{l}}{2 \sin \frac{\pi z}{2l}} dz = \frac{1}{2} \quad (16.6.15)$$

(16.6.15) tengliklarning birinchisini $f(x-0)$ ga, ikkinchisini $f(x+0)$ ga ko'paytirib qo'shamiz, natijada

$$\begin{aligned} \frac{f(x-0) + f(x+0)}{2} = \\ = \frac{1}{l} \int_{-l}^0 f(x-0) \frac{\sin(n + \frac{1}{2}) \frac{\pi z}{l}}{2 \sin \frac{\pi z}{2l}} dz + \frac{1}{l} \int_0^l f(x+0) \frac{\sin(n + \frac{1}{2}) \frac{\pi z}{l}}{2 \sin \frac{\pi z}{2l}} dz \end{aligned} \quad (16.6.16)$$

(16.6.14) dan (16.6.16) ni ayiramiz:

$$\begin{aligned} S_n(x) - \frac{f(x-0) + f(x+0)}{2} = \frac{1}{l} \int_{-l}^0 [f(x+z) - f(x-0)] \frac{\sin(n + \frac{1}{2}) \frac{\pi z}{l}}{2 \sin \frac{\pi z}{2l}} dz + \\ + \frac{1}{l} \int_0^l [f(x+z) - f(x+0)] \frac{\sin(n + \frac{1}{2}) \frac{\pi z}{l}}{2 \sin \frac{\pi z}{2l}} dz. \end{aligned} \quad (16.6.17)$$

Endi (16.6.17) ning o'ng tomondagi ikkala integralning ham $n \rightarrow \infty$ da nolga intilishini ko'rsatamiz. Masalan, ikkinchisining nolga intilishini ko'rsatamiz, birinchisining ham nolga intilishi shunday ko'rsatiladi. Buning uchun baholanayotgan integralni I_n deb belgilab uni quyidagicha yozib olamiz:

$$I_n = \frac{1}{l} \int_0^\delta \frac{f(x+z) - f(x+0)}{z} \cdot \frac{\frac{\pi}{2l}}{\sin \frac{\pi}{2l}} \sin(n + \frac{1}{2}) \frac{\pi}{2l} dz +$$

$$+ \frac{1}{l} \int_\delta^l \frac{f(x+z) - f(x+0)}{\sin \frac{\pi}{2l}} \cdot \sin(n + \frac{1}{2}) \frac{\pi}{2l} dz = I'_n + I''_n \quad (16.6.18)$$

($0 < \delta < l$) bo'lsin. $\varepsilon > 0$ berilgan. Berilgan ε ga ko'ra δ ni tanlash bilan (16.6.18) dagi birinchi integralni hamma $n = 1, 2, \dots$ uchun $\frac{\varepsilon}{2}$ dan kichik yekanligini ko'rsatamiz. Shartga ko'ra $\lim_{z \rightarrow 0+0} \frac{f(x+z) - f(x+0)}{z} = f'(x+0)$ ($0 < z < \delta$) bo'lgani uchun

$$\left| \frac{f(x+z) - f(x+0)}{z} \right| < |f'(x+0)| + 1.$$

$\frac{\frac{\pi}{2l}}{\sin \frac{\pi}{2l}} \rightarrow 1$ bo'lagini uchun, istalgancha kichik $\delta > 0$ larda $\forall z \in (0, 2\delta)$ va

$n = 1, 2, \dots$ uchun $1 < \frac{\frac{\pi}{2l}}{\sin \frac{\pi}{2l}} < 2$ bo'ladi. $\forall z$ va $n = 1, 2, \dots$ uchun

$\left| \sin(n + \frac{1}{2}) \frac{\pi}{l} \right| \leq 1$. Shunday qilib, hamma z va $n = 1, 2, \dots$ uchun

$$|I'_n| \leq \frac{1}{\pi} \int_0^\delta \left| \frac{f'(x+z) - f'(x+0)}{2} \right| \cdot \left| \frac{\frac{\pi}{2l}}{\sin \frac{\pi}{2l}} \right| \cdot \left| \sin(n + \frac{1}{2}) \frac{\pi}{l} \right| dz \leq \frac{2\delta}{\pi} [|f'(x+0)| + 1].$$

Berilgan $\varepsilon > 0$ son uchun shunday istalgancha kichik $\delta > 0$ tanlash mumkinki,

$$|I'_n| \leq \frac{\varepsilon}{2}. \quad (16.6.19)$$

bo'ladi. Endi tanlangan $\delta > 0$ ni belgilab (16.6.18) ning ikkinchi integralini qaraymiz:

$$I_n'' = \frac{1}{l} \int_{\delta}^l \frac{f'(x+z) - f'(x+0)}{\sin \frac{\pi z}{2l}} \cdot \sin\left(n + \frac{1}{2}\right) \frac{\pi z}{2l} dz.$$

$\frac{f(x+z) - f(x+0)}{\sin \frac{\pi z}{2l}}$ funksiya $z \in [\delta, l]$ ($\delta > 0$) kesmada bo'lakli silliq

funksiya, u holda asosiy lemmaga asosan $\lim_{n \rightarrow \infty} I_n'' = 0$ bo'ladi.

Demak, istalgan $\varepsilon > 0$ uchun shunday $N(\varepsilon)$ ko'rsatish mumkinki, barcha $n > N(\varepsilon)$ lar uchun

$$|I_n''| \leq \frac{\varepsilon}{2} \quad (16.6.20)$$

(16.6.19) va (16.6.20) larni hisobga olgan holda,

$$|I_n| \leq |I_n'| + |I_n''| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Shunday qilib, $\lim_{n \rightarrow \infty} I_n = 0$ yeknaligi kelib chiqadi, xuddi shunga o'xshash $n \rightarrow \infty$ da (16.6.17) ning birinchi integralining nolga intilishi ko'rsatiladi. Nihoyat, (16.6.17) dagi ikki integralning nolga intilishidan

$$\lim_{n \rightarrow \infty} S_n(x) = \frac{f(x-0) + f(x+0)}{2} \quad (16.6.21)$$

kelib chiqadi. Ta'kidlaymizki, $f(x)$ funksiya $[-l, l]$ kesmadan tashqariga ($2l$ davrli) davriy davom yettirilgan bo'lganligi uchun

$$f(l+0) = f(-l+0), \quad f(-l-0) = f(l-0)$$

bo'ladi. (16.6.17) da avvalo $x = -l$, so'ngra $x = l$ desak, u holda (16.6.21) ni ye'tiborga olgan holda

$$\lim_{n \rightarrow \infty} S_n(-l) = \lim_{n \rightarrow \infty} S_n(l) = \frac{f(-l+0) + f(l-0)}{2}. \blacksquare$$

16.7. $[0, l]$ da berilgan funksiyaning faqat kosinuslar yoki sinuslar bo'yicha Fure qatoriga yoyish. $f(x)$ funksiya $[0, l]$ da aniqlangan bo'lib, u shu kesmada bo'lakli uzluksiz va bo'lakli silliq bo'lsin. Uni $[-l, 0]$ kesmaqa har xil davom yettirish mumkin, xususiyl holda: 1) juft; 2) toq davom ettirish mumkin.

1) -holda $[-\pi, \pi]$ da juft funksiya hosil bo'ladi. Uning uchun

$$a_0 = \frac{2}{l} \int_0^l f(\xi) d\xi, \quad a_k = \frac{2}{l} \int_0^l f(\xi) \cos \frac{k\pi}{l} \xi d\xi, \quad b_k = 0 \quad (16.7.1)$$

bo'ladi, $[-\pi, \pi]$ dagi Fure qatori

$$f(x) \approx \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos \frac{k\pi}{l} \xi \quad (16.7.2)$$

ko'rinishda bo'ladi.

2)-holda $[-\pi, \pi]$ da toq funksiya hosil bo'lib, uning Fure ko'effitsiyentalari,

$$a_0 = 0, \quad a_k = 0, \quad b_k = \frac{2}{l} \int_0^l f(\xi) \sin \frac{k\pi}{l} \xi d\xi. \quad (16.7.3)$$

Fure qatori esa,

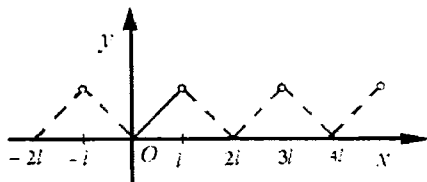
$$f(x) \approx \sum_{k=1}^{\infty} b_k \sin \frac{k\pi}{l} \xi \quad (16.7.4)$$

ko'rinishda bo'ladi. $(0, l)$ kesmada har ikkala (16.7.2) va (16.7.4) qatorlar $f(x)$ ga yaqinlashadi ($f(x)$ ning uzluksizlik nuqtalarida).

Shunday qilib, $0 \leq x \leq l$ kesmada berilgan har bir bo'lakli silliq $f(x)$ funksiyani, hoxish bo'yicha faqat kosinuslar bo'yicha ko'effitsiyentalari (16.7.1) formula bilan aniqlanadigan (16.7.2) qatorga yoyish yoki faqat sinus bo'yicha ko'effitsiyentalari (16.7.3) formula bilan aniqlanadigan (16.7.4) qatorga yoyish mumkin ekan.

16.7.5. Misol. $f(x) = x$ funksiyani $[0, l]$ oraliqda sinuslar bo'yicha Fure qatoriga yoying.

Yechilishi. Shartga ko'ra berilgan funksiyani, toqlik qonuniga ko'ra, $[-l, 0]$ ga davriy davom ettiramiz, ya'ni $[0, l]$ oraliqda $f(x) = x$ ga teng bo'lgan $2l$ davrli toq davriy funksiyani Fure qatoriga yoyamiz (16.8-chizma). Hosil bo'lgan funksiya uchun



16.8-chizma.

$$a_k = 0 \quad (k = 1, 2, \dots), \quad b_k = \frac{2}{l} \int_0^l x \sin \frac{k\pi x}{l} dx = -\frac{2l}{k} (-1)^k, \quad b_n = \frac{2l}{k} (-1)^{k+1}.$$

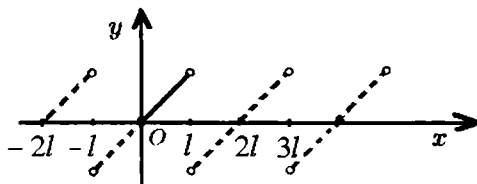
Shunday qilib $f(x) = x$ funksiyani $[0, l]$ oraliqda sinuslar bo'yicha yoyilmasi

$$S(x) = \frac{2l}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin \frac{k\pi x}{l} = \begin{cases} \frac{f(x-0) + f(x+0)}{2} = x, & -l < x < l \\ \frac{f(-l+0) + f(l-0)}{2} = \frac{-l+l}{2} = 0, & x = \pm l \end{cases}$$

ko'rinishda bo'ladi.

16.7.6. Misol. Ushbu $f(x) = x$ funksiyani $[0, l]$ oraliqda kosinuslar buyicha Fure qatoriga yoying.

Yechilishi. Masalaning shartiga ko'ra, $[0, l]$ da berilgan $f(x) = x$ funksiyani, juftlik qonuniga ko'ra, $[-l, l]$ ga davriy davom ettiramiz, ya'ni $[0, l]$ oraliqda $f(x) = x$ ga teng bo'lgan $2l$ davrli juft davriy funksiyani Fure qatoriga yoyamiz. Bu funksiya uchun



16.9-chizma.

$$b_n = 0 \quad (n=1, 2, \dots), \quad a_0 = \frac{2}{l} \int_0^l x dx = l,$$

$$a_n = \frac{2}{l} \int_0^l x \cos \frac{k\pi x}{l} dx = \frac{2l}{\pi^2 n^2} [(-1)^n - 1] = \begin{cases} 0, & n\text{-juft bo'lganda} \\ -\frac{4l}{\pi^2 n^2}, & n\text{-toq bo'lganda} \end{cases}$$

Shunday qilib,

$$x = \frac{l}{2} - \frac{4l}{\pi^2} \left\{ \cos \frac{\pi x}{l} + \frac{1}{3^2} \cos \frac{3\pi x}{l} + \frac{1}{5^2} \cos \frac{5\pi x}{l} + \dots \right\}.$$

ko'rinishda bo'ladi (16.9-chizma).

16.8. Ortogonal sistemalar bo'yicha Fure qatori. Bessel tengsizligi. $\varphi(x)$ va $\psi(x)$ funksiyalar $[a, b]$ da berilgan va oraliqda ular integrallanuvchi bo'lsin.

16.8.1. Ta'rif. Agar

$$\int_a^b \varphi(x) \psi(x) dx = 0$$

bo'lsa, u holda $\varphi(x)$ va $\psi(x)$ funksiyalar $[a, b]$ da *ortogonal* deyiladi.

$\varphi_1(x), \varphi_2(x), \varphi_3(x), \dots, \varphi_n(x), \dots$ funksiyalar sistemasi $[a, b]$ kesmada integrallanuvchi funksiyalar bo'lsin.

16.8.2. Ta'rif. Agar

$$\int_a^b \varphi_i(x) \varphi_k(x) dx = \begin{cases} 0, & \text{agar } i \neq k \text{ bo'lganda,} \\ > 0, & \text{agar } i = k \text{ bo'lganda} \end{cases} \quad (16.8.3)$$

bo'lsa, u holda bu funksiyalar sistemasi $[a, b]$ kesmada *ortogonal sistema* deyiladi.

Misollar:

$$1) \frac{1}{2}, \cos \frac{\pi x}{l}, \sin \frac{\pi x}{l}, \cos \frac{2\pi x}{l}, \sin \frac{2\pi x}{l}, \dots, \cos \frac{k\pi x}{l}, \sin \frac{k\pi x}{l}, \dots \quad (16.8.4)$$

sistema $[-l, l]$ da ortogonal;

$$2) \frac{1}{2}, \cos \frac{\pi x}{l}, \cos \frac{2\pi x}{l}, \dots, \cos \frac{k\pi x}{l}, \dots \\ \sin \frac{\pi x}{l}, \sin \frac{2\pi x}{l}, \dots, \sin \frac{k\pi x}{l}, \dots \quad (16.8.5)$$

sistemalarning har biri $[0, l]$ da ortogonal.

16.8.6-ta'rif. Agar $\varphi(x)$ $[a, b]$ da integrallanuvchi bo'lsa, u holda

$$\|\varphi\| = \left(\int_a^b \varphi^2(x) dx \right)^{1/2}$$

musbat songa $\varphi(x)$ funksiyaning *normasi* deyiladi.

Norma belgisi bo'yicha (16.8.3) ni quyidagicha yozish mumkin:

$$\int_a^b \varphi_i(x) \varphi_k(x) dx = \begin{cases} 0, & i \neq k \text{ bo'lsa,} \\ \|\varphi_k\|^2, & i = k \text{ bo'lsa.} \end{cases}$$

(16.8.4) asosiy trigonometrik funksiyalarning $[-l, l]$ dagi normasi

$$\left\| \frac{1}{2} \right\| = \sqrt{\frac{l}{2}}, \quad \left\| \cos \frac{k\pi x}{l} \right\| = \sqrt{l}, \quad \left\| \sin \frac{k\pi x}{l} \right\| = \sqrt{l}$$

$k = 1, 2, \dots$ (16.8.5) sistemadagi funksiyalarning $[0, l]$ dagi normasi:

$$\left\| \frac{1}{2} \right\| = \sqrt{\frac{l}{2}}, \quad \left\| \cos \frac{k\pi x}{l} \right\| = \sqrt{\frac{l}{2}}, \quad \left\| \sin \frac{k\pi x}{l} \right\| = \sqrt{\frac{l}{2}}$$

$f(x)$ funksiya $[a, b]$ da integrallanuvchi bo'lsin va

$$f(x) = \sum_{k=1}^{\infty} a_k \varphi_k(x) \quad (16.8.7)$$

tenglik o'rinli bo'lsin, bunda a_k -o'zgarmas son, $\varphi_k(x)$ $[a,b]$ kesmada ortogonal bo'lgan (16.8.3) dagi funksiyalar sistemasi. Hozircha faraz qilaylik (16.8.7) qator $[a,b]$ kesmada o'zining yig'indisiga tekis yoki o'rtacha yaqinlashuvchi bo'lsin. Bu holda qatorni hadma-had integrallash mumkin bo'ladi. (16.8.7) tenglikni $\varphi_n(x)$ ga ko'paytirib, x bo'yicha a dan b gacha integrallaymiz:

$$\int_a^b f(x)\varphi_n(x)dx = \sum_{k=1}^{\infty} a_k \int_a^b \varphi_k(x)\varphi_n(x)dx \quad (16.8.8)$$

(16.8.8) ning o'ng tomonidagi integrallar $k \neq n$ bo'lganda nolga aylanadi (ortogonal bo'lganligi uchun).

Shunday qilib, (16.8.8) dan

$$\int_a^b f(x)\varphi_n(x)dx = a_n \int_a^b \varphi_n^2(x)dx = a_n \|\varphi_n\|^2, \quad a_n = \frac{1}{\|\varphi_n\|^2} \int_a^b f(x)\varphi_n(x)dx \quad (16.8.9)$$

ni topamiz. (16.8.9) formula bilan aniqlanadigan a_n son $f(x)$ funksiyaning ortogonal sistemalar bo'yicha Fure ko'effetsiyenti, ko'effetsiyenti (16.8.9) formula bilan topiladigan (16.8.7) qatorni esa, $f(x)$ funksiyaning (16.8.3) ortogonal sistemalar bo'yicha Fure qatori deyiladi. a_n ko'effetsiyentni (16.8.9) formula bilan topishda $f(x)$ dan uning integralanuvchi yekanligini talab qilish yetarli (ta'kidlaymizki $\varphi_k(x)$ ($k=1,2,\dots$) funksiyalar $[a,b]$ da integrallanuvchi). U holda $[a,b]$ da integrallanuvchi $f(x)$ funksiyaga (16.8.3) ortogonal sistema bo'yicha (16.8.7) Fure qatorini mos qo'yish mumkin bo'lar ekan:

$$f(x) \sim \sum_{k=1}^{\infty} a_k \varphi_k(x), \quad (16.8.10)$$

bunda ko'effetsiyent a_n (16.8.9) formula bilan topiladi. (16.8.10) da tenglik ishorasini qo'yish uchun $f(x)$ ga qo'yiladigan yetarli shart $\{\varphi_k(x)\}$ ortogonal sistemasining xossalriga bog'liq bo'ladi. Xususiyl holda ortogonal sistemaning o'miga asosiy trigonometrik funksiyalar sistemasi olinganda $f(x)$ funksiyaga qo'yiladigan yetarli shart yuqorida isbot qilingan 16.6.1- asosiy teoremda keltirilgan. Ortogonal sistemalar uchun yuqoridagi asosiy teoremda o'xshagan teoremani isbot qilish uchun maxsus izlanishni talab etiladi.

16.9. Eng kichik kvadratik chetlanish. Bessel ayniyati. Bessel tengsizlik. (16.8.3) ortogonal funksiyalar sistemasining belgilangan

$$\varphi_1(x), \varphi_2(x), \varphi_3(x), \dots, \varphi_n(x)$$

bo'lagining chiziqli kombinatsiayasi

$$\sum_{k=1}^n \alpha_k \varphi_k(x) \quad (16.9.1)$$

ni qaraymiz, bunda a_k – ixtiyoriy sonli koeffitsiyent ($k = 1, 2, \dots, n$). (16.9.1) ko'phadga (16.8.3) ortogonal sistemalar bo'yicha n -tartibli ko'phad deyiladi. $f(x)$ funksiya $[a, b]$ kesmada integtrallanuvchi bo'lsin. Quyidagi masalani yechish talab qilinadi: a_k ($k = 1, 2, \dots, n$) ning qanday qiymatida berilagn ortogonal sistemalar bo'yicha n -tartibli (16.9.1) ko'phad $[a, b]$ kesmada $f(x)$ funksiyadan eng kichik kvadratik chetlanishga ega bo'ladi, ya'ni $a_1, a_2, \dots, a_n$ larning qanday qiymatida

$$\rho^2(f, \sum_{k=1}^n \alpha_k \varphi_k) = \int_a^b [f(x) - \sum_{k=1}^n \alpha_k \varphi_k(x)]^2 dx = \left\| f - \sum_{k=1}^n \alpha_k \varphi_k \right\|^2$$

qiymat o'zining eng kichik absolyut minimumiga erishadi?

$$\begin{aligned} \rho^2(f, \sum_{k=1}^n \alpha_k \varphi_k) &= \int_a^b f^2(x) dx - 2 \sum_{k=1}^n \alpha_k \int_a^b f(x) \varphi_k(x) dx + \\ &+ \int_a^b (\sum_{k=1}^n \alpha_k \varphi_k(x))^2 dx = \int_a^b f^2(x) dx - 2 \sum_{k=1}^n \alpha_k \|\varphi_k\|^2 + \sum_{k=1}^n \alpha_k^2 \|\varphi_k\|^2, \end{aligned} \quad (16.9.2)$$

bunda $\int_a^b f(x) \varphi_k(x) dx = a_k \|\varphi_k\|^2$. (16.9.2) dan

$$\rho^2(f, \sum_{k=1}^n \alpha_k \varphi_k) = \int_a^b f^2(x) dx - \sum_{k=1}^n \alpha_k^2 \|\varphi_k\|^2 + \sum_{k=1}^n (\alpha_k - a_k)^2 \|\varphi_k\|^2. \quad (16.9.3)$$

(16.9.3) ning o'ng tomonidan keyingi yig'indi a_k ga bog'liq. Yig'indi musbat bo'lganligi uchun u o'zining aniq quyi chegarasiga $a_k = a_n$

bo'lganda yerishadi. Shuning uchun $\rho^2(f, \sum_{k=1}^n \alpha_k \varphi_k)$ -kvadratik chetlanishning absolyut minimumi:

$$\min \rho^2(f, \sum_{k=1}^n \alpha_k \varphi_k) = \int_a^b [f(x) - \sum_{k=1}^n \alpha_k \varphi_k(x)]^2 dx =$$

$$= \int_a^b f^2(x) dx - \sum_{k=1}^n a_k^2 \|\varphi_k\|^2. \quad (16.9.4)$$

Ushbu

$$\sum_{k=1}^n \alpha_k \varphi_k(x)$$

ko'phad (bunda a_k $f(x)$ funksiyaning $\{\varphi_k(x)\}$ ortogonal sistemalar bo'yicha Fure koeffitsienti) $f(x)$ funksiyaning $\{\varphi_k(x)\}$ ortogonal sistemalar bo'yicha Fure ko'phadi deyiladi.

Shunday qilib, $[a, b]$ da $\{\varphi_k(x)\}$ ortogonal sistemalar bo'yicha hamma n -darajali ko'phadlar ichida $f(x)$ dan yeng kichik kvadratik chetlanishga ega bo'ladigani Fure ko'phadi bo'lib hisoblanar ekan. (16.9.4) tenglikni *Bessel ayniyati* deyiladi. (16.9.4) ning o'ng tomoni manfiy emas, shuning uchun undan

$$\sum_{k=1}^n a_k^2 \|\varphi_k\|^2 \leq \int_a^b f^2(x) dx \quad (16.9.5)$$

tengsizlikni hosil qilamiz. Bu tengsizlik hamma $n \geq 1$ lar uchun o'rinli bo'ladi. (16.9.5) tengsizlikning o'ng tomoni n ga bog'liq emas. Chap tomonidagi yig'indi yuqoridan chegaralangan bo'lganligi uchun n ning o'sishi bilan u kamaymaydi. (16.9.5) tengsizlikda $n \rightarrow \infty$ da limitga o'tib Bessel tengsizligi deb ataluvchi

$$\sum_{n=1}^{\infty} a_n^2 \|\varphi_n\|^2 \leq \int_a^b f^2(x) dx \quad (16.9.6)$$

tengsizlikni hosil qilamiz.

Xususiyl holda, $\{\varphi_k(x)\}$ ortogonal sistemaning o'rniga asosiy trigonometrik funksiyalar sistemasi olinganda (16.9.6) Bessel tengsizligining ko'rinishi ushbu

$$\frac{a_0^2}{2} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2) \leq \frac{1}{l} \int_{-l}^l f^2(x) dx \quad (16.9.7)$$

ko'rinishga keladi. (16.9.6) Bessel tengsizligini $[a, b]$ kesmada ixtiyoriy integrallanuvchi $f(x)$ uchun keltirib chiqarildi.

19.9.8. Ta'rif. Agar

$$\int_a^b f(x)dx \text{ va } \int_a^b f^2(x)dx$$

integrallar oddiy yoki xosmas ma'nosida mavjud bo'lsa, $f(x)$ funksiya $[a,b]$ oraliqda *kvadrati bo'yicha integrallanuvchi* deyiladi.

(16.9.6) Bessel tengsizligi (16.9.7 ham) ixtiyoriy kvadrati bo'yicha integrallanuvchi $f(x)$ funksiya uchun ham o'rinli bo'ladi. Bundan tashqari (16.9.6) Bessel tengsizligi $\{\varphi_k(x)\}$ ortogonal funksiyalari sistemasi kvadrati bo'yicha integrallanuvchi bo'lgan holda ham o'rinli bo'ladi.

Haqiqatdan ham, $\int_a^b f^2(x)dx$, $\int_a^b \varphi_k^2(x)dx$ integrallarning yaqinlashuvchanligidan va ushbu

$$|f(x)\varphi_k(x)| \leq \frac{f^2(x) + \varphi_k^2(x)}{2}, \quad |\varphi_i(x)\varphi_k(x)| \leq \frac{\varphi_i^2(x) + \varphi_k^2(x)}{2}$$

tengsizliklardan

$$\int_a^b f(x)\varphi_k(x)dx \text{ va } \int_a^b \varphi_i(x)\varphi_k(x)dx$$

integrallarning absolyut yaqinlashuvchanligi kelib chiqadi. Asosiy trigonometrik funksiyalar sistemasi uchun Bessey tengsizligi (16.9.7) dan

$$\frac{a_0^2}{2} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2)$$

qatorining yaqinlashuvchiligi va

$$\left. \begin{aligned} \lim_{k \rightarrow \infty} a_k &= \frac{1}{l} \lim_{k \rightarrow \infty} \int_a^b f(x) \cos \frac{k\pi x}{l} dx = 0, \\ \lim_{k \rightarrow \infty} b_k &= \frac{1}{l} \lim_{k \rightarrow \infty} \int_a^b f(x) \sin \frac{k\pi x}{l} dx = 0 \end{aligned} \right\} \quad (16.9.8)$$

ekanligi kelib chiqadi. (16.9.8) munosabat 16.6.2-lemma (asosiy) ning xususiy holidir (16.6.7-eslatmaga qarang).

16.10. Funksiyaning sillqlik darajasi bilan uning trigonometrik qatorlarining yaqinlashishi orsidagi bog'lanish. Avvalo funksiya trigonometrik qatorlarining tekis yaqinlashish shartini topishni, so'ngra $f(x)$ funksiya sillqlik darajasi bilan uning Furi

koeffitsiyentlari a_k va b_k larning $k \rightarrow \infty$ da kamayish tezligi orasidagi bog'lanishni va bu qator yaqinlashish tezligini baholash bilan shug'ullanamiz.

a) Trigonometrik Fure qatorining tekis yaqinlashish sharti.

Trigonometrik Fure qatorining tekis yaqinlashish shartini ba'zi bir qo'shimcha zaruriy shartni qanoatlantiruvchi uzluksiz va bo'lakli silliq funksiyalar uchun isbotlaymiz.

16.10.1. Teorema. Agar $[-l, l]$ kesmada uzluksiz va bo'lakli silliq $f(x)$ funksiya kesmaning uchlarida teng qiymatlarni (ya'ni $f(-l) = f(l)$) qabul qilsa, u holda uning trigonometrik Fure qatori

$$S(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos \frac{k\pi x}{l} + b_k \sin \frac{k\pi x}{l}), \quad (16.10.2)$$

$$a_0 = \frac{1}{l} \int_{-l}^l f(\xi) d\xi, \quad a_n = \frac{1}{l} \int_{-l}^l f(\xi) \cos \frac{n\pi \xi}{l} d\xi, \quad b_n = \frac{1}{l} \int_{-l}^l f(\xi) \sin \frac{n\pi \xi}{l} d\xi$$

$[-l, l]$ kesmada tekis yaqinlashuvchi va kesmaning har bir nuqtasida $S(x) = f(x)$ bo'ladi.

Isboti. (16.10.2) qatorning tekis yaqinlashishini ko'rsatish uchun unga majorant bo'lgan

$$\frac{|a_0|}{2} + \sum_{k=1}^{\infty} (|a_k| + |b_k|)$$

qatorning yoki

$$\sum_{k=1}^{\infty} (|a_k| + |b_k|)$$

qatorning yaqinlashishini ko'rsatish yetarli. U holda Veyershtrass alomatiga asosan, (16.10.2) qator butun Ox o'qida tekis yaqinlashuvchi bo'ladi, chunki $\left| a_k \cos \frac{k\pi x}{l} \right| \leq |a_k|$, $\left| b_k \sin \frac{k\pi x}{l} \right| \leq |b_k|$, $\forall x \in (-\infty, \infty)$.

$f'(x)$ funksiyaning Fure koeffitsiyentlarini a'_k va b'_k lar bilan belgilaymiz:

$$a'_k = \frac{1}{l} \int_{-l}^l f'(x) \cos \frac{k\pi x}{l} dx, \quad b'_k = \frac{1}{l} \int_{-l}^l f'(x) \sin \frac{k\pi x}{l} dx.$$

$f(x)$ funksiya Fure qatori koeffitsiyentlarini bo'laklab integrallaymiz:

$$a_k = \frac{1}{l} \int_{-l}^l f(x) \cos \frac{k\pi x}{l} dx = \frac{1}{l} \cdot \frac{l}{k\pi} f(x) \sin \frac{k\pi x}{l} \Big|_{-l}^l - \frac{1}{l} \cdot \frac{l}{k\pi} \int_{-l}^l f'(x) \sin \frac{k\pi x}{l} dx = -\frac{1}{\pi} \cdot \frac{b'_k}{k};$$

$$b_k = \frac{1}{l} \int_{-l}^l f(x) \sin \frac{k\pi x}{l} dx = -\frac{1}{l} \cdot \frac{l}{k\pi} f(x) \cos \frac{k\pi x}{l} \Big|_{-l}^l + \frac{1}{l} \cdot \frac{l}{k\pi} \int_{-l}^l f'(x) \cos \frac{k\pi x}{l} dx = -\frac{1}{\pi} \cdot \frac{a'_k}{k}.$$

chunki

$$\begin{aligned} \sin \frac{k\pi x}{l} \Big|_{-l}^l &= 0, \cos \frac{k\pi x}{l} f(x) \Big|_{-l}^l = \\ &= f(l)(-1)^k - f(-l)(-1)^k = (-1)^k (f(l) - f(-l)) = 0, \end{aligned}$$

shartga ko'ra, $f(-l) = f(l)$.

Shunday qilib,

$$|a_k| + |b_k| \leq \frac{l}{\pi} \left(\frac{|a'_k|}{k} + \frac{|b'_k|}{k} \right). \quad (16.10.3)$$

$f'(x)$ funksiy $[-l, l]$ oraliqda bo'lakli silliq funksiya bo'lganligi uchun u shu oraliqda integrallanuvchi va u uchun Bessel tengsizligi bajariladi:

$$\frac{a_0'^2}{2} + \sum_{k=1}^{\infty} (a_k'^2 + b_k'^2) \leq \frac{1}{l} \int_{-l}^l f'^2(x) dx.$$

Demak,

$$\frac{a_0'^2}{2} + \sum_{k=1}^{\infty} (a_k'^2 + b_k'^2) \quad (16.10.4)$$

qator yaqinlashuvchi. Ushbu

$$0 \leq \left(|a'_k| - \frac{1}{k} \right)^2 = a_k'^2 - \frac{2|a'_k|}{k} + \frac{1}{k^2}, \quad 0 \leq \left(|b'_k| - \frac{1}{k} \right)^2 = b_k'^2 - \frac{2|b'_k|}{k} + \frac{1}{k^2}$$

tengsizliklardan

$$\frac{|a'_k|}{k} \leq \frac{1}{2} \left(a_k'^2 + \frac{1}{k^2} \right), \quad \frac{|b'_k|}{k} \leq \frac{1}{2} \left(b_k'^2 + \frac{1}{k^2} \right)$$

tengsizliklarni hosil qilamiz. Bulardan

$$\frac{|a'_k|}{k} + \frac{|b'_k|}{k} \leq \frac{1}{2} (b_k'^2 + a_k'^2) + \frac{1}{k^2}. \quad (16.10.5)$$

(16.10.4) va $\sum_{k=1}^{\infty} \frac{1}{k^2}$ qatorlarning yaqinlashuvchiligidan

$$\sum_{k=1}^{\infty} \left\{ \frac{1}{2} (a_k'^2 + b_k'^2) + \frac{1}{k^2} \right\}$$

qatorining yaqinlashuvchanligi kelib chiqadi.

Shunday qilib, taqqoslash alomati va (16.10.3), (16.10.5) tengsizliklarga asosan,

$$\frac{|a_0|}{2} + \sum_{k=1}^{\infty} (|a_k| + |b_k|)$$

qator yaqinlashuvchi bo'ladi. Bundan $f(x)$ funksiya trigonometrik Fure qatorining butun x o'qida $S(x)$ ga tekis yaqinlashuvchiligi kelib chiqadi.

$[-l, l]$ da $S(x) = f(x)$ tengligi 16.6.1-teoremadan kelib chiqadi, chunki 16.6.1-teoremaning hamma shartlari bu yerda bajariladi.

16.10.5. Eslatma. $f(x)$ funksiyaning $[-l, l]$ oraliqning uchlaridagi qiymatlarining tengligi $f(x)$ funksiya (16.10.2) trigonometrik Fure qatorining oraliqning chetlarida $f(x)$ ga yaqinlashish uchun zaruriy shart bo'ladi.

Haqiqatdan ham, agar bu qator yig'indisi $S(x)$ uchun

$$S(-l) = f(-l), \quad S(l) = f(l)$$

shart bajarilsa, u holda $S(-l) = S(l)$ shartga asosan,

$$f(-l) = f(l) \quad (16.10.6)$$

tenglik bajariladi.

Demak, (16.10.6) tenglik (16.10.2) Fure qatorining $[-l, l]$ da $f(x)$ ga tekis yaqinlashishi uchun zaruriy shart bo'lib hisoblanadi.

b) Funksiyaning sillqlik darajasi bilan uning trigonometrik Fure qatorining yaqinlashish tezligi orasidagi bog'lanish.

16.10.7. Teorima. Agar $f(x), f'(x), \dots, f^{(m)}(x)$ ($m \geq 0$) funksiyalar $[-l, l]$ oraliqda uzluksiz va orliqning chetki nuqtalarida teng, ya'ni

$$f(-l) = f(l), f'(-l) = f'(l), f''(-l) = f''(l), \dots, f^{(m)}(-l) = f^{(m)}(l)$$

qiymatlarni qabul qilsa, hamda $f^{(m-1)}(x)$ funksiya $[-l, l]$ da bo'lakli uzluksiz bo'lsa, u holda $f(x)$ funksiya Fure koeffitsiyentlari uchun

$$a_k = o\left(\frac{1}{k^{m+1}}\right), \quad b_k = o\left(\frac{1}{k^{m+1}}\right)$$

munosabatlar bajariladi va

$$\sum_{k=1}^{\infty} k^{\nu} (|a_k| + |b_k|) \quad (16.10.8)$$

$\nu = 0, 1, 2, \dots, m$ qator yaqinlashuvchi bo'ladi.

Isboti. $f(x)$ funksiya Fure koeffitsiyentlarini ketma-ket $m+1$ marta bo'laklab integrallaymiz:

$$\begin{aligned} a_k &= \frac{1}{l} \int_{-l}^l f(\xi) \cos \frac{k\pi\xi}{l} d\xi = \frac{1}{l} \cdot \frac{l}{k\pi} f(\xi) \sin \frac{k\pi\xi}{l} \Big|_{-l}^l - \frac{1}{l} \cdot \frac{l}{k\pi} \int_{-l}^l f'(\xi) \sin \frac{k\pi\xi}{l} d\xi = \\ &= -\frac{1}{l} \cdot \frac{l}{k\pi} \int_{-l}^l f'(\xi) \sin \frac{k\pi\xi}{l} d\xi = \\ &= \frac{l}{k^2\pi^2} f'(\xi) \cos \frac{k\pi\xi}{l} \Big|_{-l}^l - \frac{l}{k^2\pi^2} \int_{-l}^l f''(\xi) \cos \frac{k\pi\xi}{l} d\xi = \\ &= -\frac{l}{k^2\pi^2} \int_{-l}^l f''(\xi) \cos \frac{k\pi\xi}{l} d\xi = \dots = \pm \frac{l^{m+1}}{k^{m+1}\pi^{m+1}} \int_{-l}^l f^{(m+1)}(\xi) \left\{ \begin{array}{l} \sin \frac{k\pi\xi}{l} \\ \cos \frac{k\pi\xi}{l} \end{array} \right\} d\xi. \quad (16.10.9) \end{aligned}$$

Teoremaning shartlaridan, kosinusning juftligidan, sinusning $[-l, l]$ oralig'ining chetlrida nolga aylanishidan foydalanib

$$f^{(s)}(\xi) \cos \frac{k\pi\xi}{l} \Big|_{-l}^l = 0, \quad f^{(s)}(\xi) \sin \frac{k\pi\xi}{l} \Big|_{-l}^l = 0, \quad 0 \leq s \leq m,$$

ekanligini topamiz.

Xuddi shunday

$$b_k = \pm \frac{l^{m+1}}{k^{m+1}\pi^{m+1}} \cdot \frac{1}{l} \int_{-l}^l f^{(m+1)}(\xi) \left\{ \begin{array}{l} \sin \frac{k\pi\xi}{l} \\ \cos \frac{k\pi\xi}{l} \end{array} \right\} d\xi. \quad (16.10.10)$$

(16.10.9) va (16.10.10) tengliklardan

$$|a_k| + |b_k| \leq \frac{l^{m+1}}{\pi^{m+1}} \left(\frac{|a_k^{m+1}|}{k^{m+1}} + \frac{|b_k^{m+1}|}{k^{m+1}} \right), \quad (16.10.11)$$

bunda a_k^{m+1} va b_k^{m+1} lar $f^{(m+1)}(x)$ funksiya Fure koeffitsiyentlari. a_k^{m+1} , b_k^{m+1} lar $k \rightarrow +\infty$ da nolga intiladi, shuning uchun $a_k = o\left(\frac{1}{k^{m+1}}\right)$, $b_k = o\left(\frac{1}{k^{m+1}}\right)$ (16.10.11) dan

$$k^m(|a_k| + |b_k|) \leq \frac{l^{m+1}}{\pi^{m+1}} \left(\frac{|a_k^{m+1}|}{k} + \frac{|b_k^{m+1}|}{k} \right) \leq \frac{l^{m+1}}{\pi^{m+1}} \left\{ |a_k^{(m+1)}|^2 + |b_k^{(m+1)}|^2 + \frac{1}{k^2} \right\},$$

chunki

$$\frac{|a_k^{m+1}|}{k} \leq \frac{1}{2} \left(|a_k^{(m+1)}| + \frac{1}{k^2} \right), \quad \frac{|b_k^{m+1}|}{k} \leq \frac{1}{2} \left(|b_k^{(m+1)}| + \frac{1}{k^2} \right).$$

Shuning uchun Bessel tengsizligiga asosan, (16.10.11) dan

$$\frac{|a_0^{(m+1)}|^2}{2} + \sum_{k=1}^{\infty} \left(|a_k^{(m+1)}|^2 + |b_k^{(m+1)}|^2 \right) \leq \frac{1}{l} \int_{-l}^l |f^{(m+1)}(\xi)|^2 d\xi$$

Shunday qilib, $\sum_{k=1}^{\infty} \frac{1}{k^2}$ qatorning yaqinlashuvchanligidan

$\sum_{k=1}^{\infty} k^m (|a_k| + |b_k|)$ qatorning yaqinlashuvchiligi kelib chiqadi, bundan esa (16.10.8) qatorning hammasining yaqinlashuvchiligi kelib chiqadi. ■

16.11. Ortogonal sistemalarning to‘liqligi va yopiqligi. $[a, b]$

kesmada hamma bo‘lakli-uzluksiz bo‘lgan funksiyalar sinfini $Q[a, b]$ deb belgilaymiz. $[a, b]$ kesmada

$$\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x), \dots \quad (16.11.1)$$

ortogonal funksiyalar sistemi berilgan bo‘lsin.

16.11.2. Ta’rif. Agar $\forall f(x) \in Q[a, b]$ funksiyalarning (16.11.1) ortogonal sistemalar bo‘yicha

$$\sum_{k=1}^{\infty} c_k \varphi_k(x) \quad (16.11.3)$$

$$\text{bunda} \left(c_k = \frac{1}{\|\varphi_k\|^2} \int_a^b f(x) \varphi_k(x) dx, \quad \|\varphi_k\| = \left(\int_a^b \varphi_k^2(x) dx \right)^{1/2} \right).$$

Fure qatori $[a, b]$ kesmada $f(x)$ funksiyaga o‘rtacha yaqinlashsa, ya’ni

$$\rho^2(f, \sum_{k=1}^{\infty} c_k \varphi_k) = \int_a^b \left[f(x) - \sum_{k=1}^{\infty} c_k \varphi_k(x) \right]^2 dx \xrightarrow{n \rightarrow \infty} 0$$

bo‘lsa, u holda (16.11.1) ortogonal sistema $Q[a, b]$ fazoda to‘liq deyiladi.

Bu holda (16.11.1) sistema $Q[a, b]$ fazoning bazisini tashkil qiladi deb yuritiladi, chunki (16.11.1) sistema to‘liq bo‘lganda $\forall f(x) \in Q[a, b]$ uchun

$$f(x) = \sum_{k=1}^{\infty} c_k \varphi_k(x), \quad c_k = \frac{1}{\|\varphi_k\|^2} \int_a^b f(x) \varphi_k(x) dx$$

tenglik o'rinli. Bessel ayniyatiga asosan,

$$\rho^2(f, \sum_{k=1}^n c_k \varphi_k) = \int_a^b \left[f(x) - \sum_{k=1}^n c_k \varphi_k(x) \right]^2 dx = \int_a^b f^2(x) dx - \sum_{k=1}^n c_k^2 \|\varphi_k\|^2.$$

Bunda $n \rightarrow \infty$ limitga o'tish orqali

$$\lim_{n \rightarrow \infty} \rho^2(f, \sum_{k=1}^n c_k \varphi_k) = \int_a^b f^2(x) dx - \sum_{k=1}^{\infty} c_k^2 \|\varphi_k\|^2,$$

bundan $\lim_{n \rightarrow \infty} \rho^2(f, \sum_{k=1}^n c_k \varphi_k) = 0$ bu esa,

$$\sum_{k=1}^{\infty} c_k^2 \|\varphi_k\|^2 = \int_a^b f^2(x) dx \quad (16.11.4)$$

ga teng kuchli. (16.11.4) tenglikga *Parseval tengligi* deyiladi.

Shunday qilib, (16.11.1) ortogonal sistema $Q[a,b]$ fazoda to'liq bo'lishi uchun $\forall f(x) \in Q[a,b]$ funksiyalar uchun (16.11.4) Parseval tengligining bajarilishi zarur va yetarlidir.

To'liq ortogonal sistemlarning xossalari.

Agar $[a,b]$ da ortogonal bo'lgan, (16.11.1) sistemaning hamma elementlariga $[a,b]$ da ortogonal bo'lgan $\forall f(x) \in Q[a,b]$ funksiy $Q[a,b]$ fazoning nolini tashkil etsa, ya'ni $f(x)$ funksiyaning uzluksizlik nuqtalarida nolga teng bo'lib, qolgan chekli sondagi nuqtalarida noldan farqli bo'lsa, u holda $[a,b]$ da ortogonal bo'lgan, (16.11.1) ortogonal sistema $Q[a,b]$ fazoda yopiq deyiladi.

16.11.5. Teorema. Agar $[a,b]$ da ortogonal bo'lgan (16.11.1) sistema $Q[a,b]$ da to'liq bo'lsa, u holda u $Q[a,b]$ da yopiq bo'ladi.

Isboti. $f(x)$ bo'lakli uzluksiz funksiya $[a,b]$ da (16.11.1) ortogonal bo'lgan sistemaning hamma elementlariga ortogonal, ya'ni

$$\int_a^b f(x) \varphi_k(x) dx = 0, \quad k = 1, 2, \dots$$

bo'lsin. U holda $f(x)$ funksiyaning (16.11.1) ortogonal sistemalar bo'yicha Fure koeffitsiyenti

$$c_k = \frac{1}{\|\varphi_k\|^2} \int_a^b f(x) \varphi_k(x) dx = 0, \quad k = 1, 2, \dots \quad (16.11.6)$$

bo'ladi. (16.11.1) sistemaning $Q[a, b]$ da to'liq bo'lgani uchun, $\forall f(x) \in Q[a, b]$ funksiya uchun Parsevi tengligi, ya'ni

$$\int_a^b f^2(x) dx = \sum_{k=1}^{\infty} c_k^2 \|\varphi_k\|^2 \quad (16.11.7)$$

tenglik bajariladi. Unda (16.11.6) va (16.11.7) dan

$$\int_a^b f^2(x) dx = 0 \quad (16.11.8)$$

ekanligi kelib chiqadi.

Faraz qilaylik, $f(x_0) \neq 0$ bo'lsin, bunda x_0 nuqta $f(x)$ ning uzliksizlik nuqtasi. x_0 nuqtani $f(x)$ funksiya uzliksiz bo'lgan $[a', b']$ ($[a', b'] \subset [a, b]$) oraliq bilan o'raymiz. $f^2(x)$ $[a', b']$ da uzluksiz va manfiy bo'lmagan, ya'ni $f(x_0) > 0$ funksiya bo'lganligi uchun $\int_{a'}^{b'} f^2(x) dx > 0$

bo'ladi. U holda albatta $\int_a^b f^2(x) dx > 0$ bo'ladi. Bu esa, (16.11.8) ga qarama-qarshilik tug'diradi.

Bundan esa, $f'(x)$ funksiyaning $[a, b]$ dagi uzluksizlik nuqtalarida $f(x) \equiv 0$ ekanligi kelib chiqadi. ■

16.11.9-teorema. Agar ikkita $f(x), g(x) \in Q[a, b]$ funksiyalar $[a, b]$ da to'liq ortogonal sistemalar bo'yicha bir xil Fure qatoriga ega bo'lsa, u holda ular $Q[a, b]$ ning elementi sifatida bir-biriga teng bo'ladi, ya'ni ular $[a, b]$ ning chikli sondagi nuqtalaridagina bir-biridan farq qiladi.

Isboti. $\psi(x) = (f(x) - g(x)) \in Q[a, b]$ funksiya (16.11.1) ortogonal sistemaning har bir elemetiga ortogonal bo'ladi.

Haqiqatdan ham,

$$\begin{aligned} \int_a^b \psi(x) \varphi_k(x) dx &= \int_a^b f(x) \varphi_k(x) dx - \int_a^b g(x) \varphi_k(x) dx = \\ &= c_k^f \|\varphi_k\|^2 - c_k^g \|\varphi_k\|^2 = (c_k^f - c_k^g) \|\varphi_k\|^2, \quad k = 1, 2, \dots \end{aligned} \quad (16.11.10)$$

bunda $c_k^f - f(x)$ funksiyaning Fure koeffitsienti, $c_k^g - g(x)$ funksiyaning Fure koeffitsiyenti. Shartga ko'ra bu funksiyalarning Fure qatori bir xil, ya'ni $c_k^f = c_k^g$, $k = 1, 2, \dots$ bo'lganligi uchun (16.11.10) dan

$$\int_a^b \psi(x) \varphi_k(x) dx = 0, \quad k = 1, 2, \dots$$

ekanligi kelib chiqadi. U holda 16.11.5-teoremaga asosan, $\psi(x) = f(x) - g(x)$ ayirma $[a, b]$ oraliqning $\psi(x)$ funksiya uzliksiz bo'ladigan nuqtalarda aynan nolga teng bo'ladi. $[a, b]$ ning chikli sondagi nuqtalarida noldan farqli bo'lishi mumkin. ■

16.11.11. Teorem. Agar (16.11.1) ortogonal sistema $Q[a, b]$ da to'liq bo'lsa, u holda $Q[a, b]$ dagi ixtiyoriy ikkita $f(x)$ va $g(x)$ ($f(x) \in Q[a, b]$, $g(x) \in Q[a, b]$) funksiyalar uchun umumlashgan Parseval tengligi o'rinli bo'ladi, ya'ni

$$\int_a^b f(x)g(x)dx = \sum_{k=1}^{\infty} c_k^f c_k^g \|\varphi_k\|^2, \quad (16.11.12)$$

bunda c_k^f va c_k^g - mos ravishda $f(x)$ va $g(x)$ funksiyalarning ortogonal sistema bo'yicha Fure koeffitsiyentlari.

Isboti. $f(x), g(x) \in Q[a, b]$ bo'lganligi uchun $f(x) \pm g(x) \in Q[a, b]$ bo'ladi. $f(x) + g(x)$ va $f(x) - g(x)$ funksiyalar uchun Parseval tengligini yozamiz:

$$\begin{aligned} \int_a^b (f(x) + g(x))^2 dx &= \sum_{k=1}^{\infty} (c_k^f + c_k^g)^2 \|\varphi_k\|^2, \\ \int_a^b (f(x) - g(x))^2 dx &= \sum_{k=1}^{\infty} (c_k^f - c_k^g)^2 \|\varphi_k\|^2. \end{aligned}$$

Bulardan,

$$\begin{aligned} \int_a^b (f^2(x) + 2f(x)g(x) + g^2(x))dx &= \sum_{k=1}^{\infty} ((c_k^f)^2 + 2c_k^f c_k^g + (c_k^g)^2) \|\varphi_k\|^2, \\ \int_a^b (f^2(x) - 2f(x)g(x) + g^2(x))dx &= \sum_{k=1}^{\infty} ((c_k^f)^2 - 2c_k^f c_k^g + (c_k^g)^2) \|\varphi_k\|^2. \end{aligned}$$

$$\begin{aligned}
& \int_a^b f^2(x)dx + 2 \int_a^b f(x)g(x)dx + \int_a^b g^2(x)dx = \\
& = \sum_{k=1}^{\infty} (c_k^f)^2 \|\varphi_k\|^2 + 2 \sum_{k=1}^{\infty} c_k^f c_k^g \|\varphi_k\|^2 + \sum_{k=1}^{\infty} (c_k^g)^2 \|\varphi_k\|^2, \\
& \int_a^b f^2(x)dx - 2 \int_a^b f(x)g(x)dx + \int_a^b g^2(x)dx = \\
& = \sum_{k=1}^{\infty} (c_k^f)^2 \|\varphi_k\|^2 - 2 \sum_{k=1}^{\infty} c_k^f c_k^g \|\varphi_k\|^2 + \sum_{k=1}^{\infty} (c_k^g)^2 \|\varphi_k\|^2.
\end{aligned}$$

Yuqoridagi tenglikdan pastdagi tenglikni ayiramiz:

$$4 \int_a^b f(x)g(x)dx = 4 \sum_{k=1}^{\infty} c_k^f c_k^g \|\varphi_k\|^2, \quad \int_a^b f(x)g(x)dx = \sum_{k=1}^{\infty} c_k^f c_k^g \|\varphi_k\|^2 \quad \text{..}\blacksquare$$

16.11.13. Teorema. Agar $f(x) \in Q[a,b]$ bo'lib, (16.11.1) ortogonal sistema $Q[a,b]$ da to'liq bo'lsa, u holda $f(x)$ funksiyning (16.11.1) ortogonal sistemalar bo'yicha Fure qatorini hadma-had integrallash mumkin:

$$\int_{x_0}^x f(\xi)d\xi = \sum_{k=1}^{n_1} c_k \int_{x_0}^x \varphi_k(\xi)d\xi \quad (16.11.14)$$

(x_0, x -lar, $[a,b]$ dan olingn ixtiyoriy nuqtalar). (16.11.12) qator x bo'yicha $[a,b]$ tekis yaqinlashuvchi bo'ladi.

Isboti. Shartga ko'ra (16.11.1) ortogonal sistema $Q[a,b]$ da to'liq bo'lganligi uchun $f(x) \in Q[a,b]$ ning ortogonal sistema bo'yicha Fure qatori $[a,b]$ $f(x)$ ga o'rtacha yaqinlashuvchi funksional qatorlarni hadma-had integrallash hqidagi 11.9.1- teoremaga asosan, $f(x)$ funksiyaning (16.11.1) ortogonal sistemlr bo'yicha (16.11.3) Fure qatorini hadma-had integrallash mumkin, ya'ni (16.11.14) formula urinli bo'ladi.■

16.11.15. Eslatma. Xususi holda $[a,b]$ da (16.11.1) ortogonal sistema, asosiy trigonometrik sisemadan iborat, ya'ni

$$\frac{1}{2}, \cos \frac{\pi x}{l}, \sin \frac{\pi x}{l}, \dots, \cos \frac{k\pi x}{l}, \sin \frac{k\pi x}{l}, \dots$$

bo'lsa, bu asosiy trigonometrik sistema $Q[-l,l]$ da to'liq bo'ladi.

16.11.16. Misol. Ushbu $f(x) = \operatorname{sgn}(\sin x)$ funksiyani Fure qatoriga yoying va yoyilmadan foydalanib $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2}$ qatorning yig'indisini toping; Parseval formulasini yozing; birinchi ikkita qisman yig'indisining grafigini chizing.

Yechilishi. Berilgan funksiya juft emas, 2π davrga ega. Berilgan funksiya $[-\pi, \pi]$ oraliqning $x = -\pi, x = 0, x = \pi$ nuqtalaridan tashqari, hamma joyida uzluksiz, ko'rsatilgan nuqtalarda 1- tur uzilishga ega. $x = -\pi, x = 0, x = \pi$ nuqtalardan tashqari $[-\pi, \pi]$ oraliqning hamma nuqtalarida hosilaga ega va bu hosila chegaralangan. Shuning uchun, berilgan funksiyaning Fure qatori hamma x larda yaqinlashuvchi (funksiyaning uzluksizlik nuqtalarida qatorning yig'indisi funksiyaning bu nuqtalardagi qiymatiga teng bo'ladi, $x = -\pi, x = 0, x = \pi$ nuqtalarda esa, funksiyaning o'ng va chap limitik qiymatlarining o'rta arifmetigiga teng).

Endi Fure koefitsiyentlarini topamiz:

$$a_n = 0 \quad (n = 1, 2, \dots), \quad b_n = \frac{2}{\pi} \int_0^{\pi} 1 \sin nx dx = -\frac{2}{n\pi} \cos nx \Big|_0^{\pi}, \quad \cos n\pi = (-1)^n$$

bo'lgani uchun, $b_n = \frac{2}{\pi} \frac{1 - (-1)^n}{n}$ ($n = 1, 2, \dots$), bundan n - juft bo'lganda

$b_n = 0$, n -toq bo'lganda esa, $b_n = \frac{4}{n\pi}$ ga teng bo'ladi. Shunday qilib $\forall x$

lar uchun $\operatorname{sgn}(\sin x) = \frac{\pi}{4} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}$. Bunda, $x = \frac{\pi}{2}$ bo'lganda

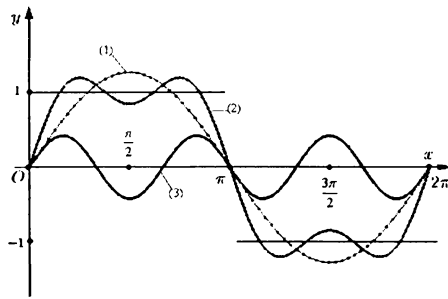
$$1 = \frac{\pi}{4} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}, \quad \frac{4}{\pi} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}.$$

$x \neq 0$ larda $(\operatorname{sgn}(\sin x))^2 = 1$ bo'lgani uchun, Parseval formulasiga asosan,

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x) dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2), \quad \frac{1}{\pi} \int_{-\pi}^{\pi} 1 dx = \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2},$$

bundan $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$. Birinchi ikkita qisman yig'indisining grafigi

16.10- chizmada tasvirlangan.



16.10-chizma. (1) - $y = \frac{4}{\pi} \sin x$, (2) - $y = \frac{4}{3\pi} \sin 3x$, (3) - $y = \frac{4}{\pi} (\sin x + \frac{1}{3} \sin 3x)$.

16.11.17. Misol. Ushbu $y = x$ funksiyani (3;5) oraliqda Fure qatoriga yoying.

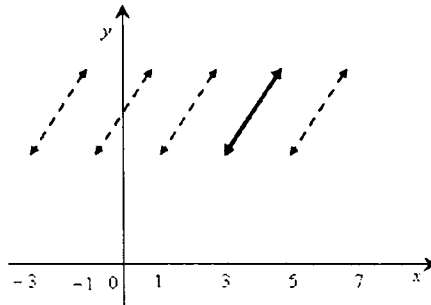
Yechilishi. $y = x$, $3 < x < 5$ funksiyani butun o'qqa, davri 2 ga teng davriy funksiya sifatida davriy davom ettiramiz. Bu holda, $2l = 5 - 3 = 2$, $l = 1$

$$a_0 = \int_3^5 x dx = \frac{x^2}{2} \Big|_3^5 = 8, a_n = \int_3^5 x \cos nx dx = 0, b_n = \int_3^5 x \sin nx dx = \frac{2(-1)^{n+1}}{\pi n}, \quad (n \in \mathbb{N}).$$

Funksiyaning uzluksizlik nuqtalarida ushbu

$$f(x) = 4 + \sum_{n=1}^{\infty} \frac{2}{\pi n} (-1)^{n+1} \sin n\pi x$$

yoyilma o'rinni bo'ladi (16.11-chizma).



16.11-chizma.

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